A Linear Singularity Analysis by the Continuous Shearlet Transform

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Abstract

Several authors have related the linear singularities of two-variable functions to the decays of their parabolic transforms. Kutyniok and Labate computed the decay rates of the continuous shearlet transform of certain point, linear and corner singularities. Modelling the linear singularity of a function by two (non-parallel) directional Hölder exponents, Lakhonchai, Sampo and Sumetkijakan obtained decay estimates of its shearlet transform. They found that the decay estimates for shearlets whose major axes are close to the singularity line are proportional to the Hölder exponent in the direction of the singularity line. Likewise, the decay estimates for the other shearlets are proportional to the Hölder exponent in a non-parallel direction. In this paper, we improve Lakhonchai et al.'s result by narrowing the gap between the decay estimate and the Hölder exponent in the case of the non-parallel direction. As a result, we obtained the decay rates of a linear singularity model reminiscent to that of Kutyniok et al.

Key words and phrases: Directional Hölder regularity, Shearlet transforms, Parabolic scaling, Direction of singularity, Linear singularity

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1 Introduction

Almost as soon as wavelet transform became popular, various transforms have been studied with respect to their abilities to detect edges. This direction of research is mainly due to the characterizations of point singularity by the continuous wavelet transform, discovered independently by Jaffard [19] and Holschneider and Tchamitchian [18] in the early 1990's. These transforms included the Smith transform [33], curvelet transform [2, 3] and, the most recently introduced, the shearlet transform [25].

Due partly to their success in edge detection, an attempt to use the Smith and continuous curvelet transforms to characterize pointwise and uniform Hölder regularities has been reported [32]. The more interesting result of their study concerned how a set of directional and local Hölder regularities yielded the decay estimates of the transforms. Though the converse is not at all plausible, they managed to prove that these decay estimates imply pointwise regularity of certain exponents.

Based on a beautiful theory of wavelets with composite dilations [16, 17], the shearlet system is defined [10] in the time domain by translations, parabolic dilations, and shearing of a single admissible function, and have a simple reconstruction formula. Kutyniok and Labate [25] computed the decay rates of continuous shearlet transforms when the distributions possess point, linear, polygonal or circular singularities. Independently, Lakhonchai, Sampo and Sumetkijakan [27] gave estimates of the continuous shearlet transforms of functions satisfying a set of directional Hölder regularities and so which models a linear singularity.

In this paper, we first give an improved version of the theorem of Lakhonchai et al. (Theorem 5.3 in [27]) with a weaker condition. The same argument works just as well in relaxing the $C^{2\alpha+1+\varepsilon}$ -condition of Sampo and Sumetkijakan (Theorem 8 of [32]) where continuous curvelet and Hart Smith transforms are used. In light of the analysis of linear singularities of a linear delta distribution in [25], we present an analysis of the linear singularities of our model. As a corollary, in the direction of the singularity line, the decay rate of the continuous shearlet transform corresponds precisely to the Hölder exponent in the horizontal direction at the same point, while the decay rate of the transform in other directions corresponds to the Hölder exponent in the direction of the singularity line.

2 Basics of Hölder Regularity and Shearlets

Points in the time domain are denoted by column vectors $\mathbf{x} = (x_1, \dots, x_d)^T$ while points in the frequency domain are denoted by row vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$, such that $\boldsymbol{\xi}\mathbf{x} = \sum_{i=1}^d \xi_i x_i$. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is defined by $\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi}\mathbf{x}} d\mathbf{x}$, $\boldsymbol{\xi} \in \mathbb{R}^d$ and the Fourier transform of an L^2 -function is defined in the limit. So the Plancherel's formula becomes $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ and the Fourier inversion formula is $f(\mathbf{x}) = \int_{\widehat{\mathbb{R}^d}} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi}\mathbf{x}} d\boldsymbol{\xi}$.

Pointwise, uniform, and directional Hölder regularities of functions of several variables are defined as follows.

Definition 1. Let $\alpha \in (0, \infty) \setminus \mathbb{N}$ and $\mathbf{u} \in \mathbb{R}^d$. The function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *pointwise Hölder regular with exponent α at $\mathbf{u}$* , denoted by $f \in C^\alpha(\mathbf{u})$, if there exists a polynomial $P = P_{\mathbf{u}}$ of a degree less than α and a constant $C = C_{\mathbf{u}}$ such that for all $\mathbf{x}$ in a neighborhood of $\mathbf{u}$

$$|f(\mathbf{x}) - P(\mathbf{x} - \mathbf{u})| \leq C \|\mathbf{x} - \mathbf{u}\|^\alpha. \quad (1)$$

Let Ω be an open subset of $\mathbb{R}^d$. If there exists a uniform constant C so that for all $\mathbf{u} \in \Omega$ there is a polynomial $P_{\mathbf{u}}$ of a degree less than α such that (1) holds for all $\mathbf{x} \in \Omega$, then we say that f is *uniformly Hölder regular with exponent α on Ω* or $f \in C^\alpha(\Omega)$.

For a fixed unit vector $\mathbf{v} \in \mathbb{R}^d$, f is said to be *pointwise Hölder regular with exponent α at $\mathbf{u}$ in the direction $\mathbf{v}$* , denoted by $f \in C^\alpha(\mathbf{u}; \mathbf{v})$, if there exist a constant $C = C_{\mathbf{u}, \mathbf{v}}$ and a polynomial $P = P_{\mathbf{u}, \mathbf{v}}$ of a degree less than α such that (1) holds for all $\mathbf{x}$ in an open line segment that contains the point $\mathbf{u}$ and is parallel to $\mathbf{v}$.

Let Ω_1 be a subset of $\mathbb{R}^d$ and Ω_2 be an open neighborhood of Ω_1 . Then, f is said to be in $C^\alpha(\Omega_1, \Omega_2; \mathbf{v})$ if there exists a constant $C = C_{\mathbf{v}}$ so that for all $\mathbf{u} \in \Omega_1$ there is a polynomial $P = P_{\mathbf{u}, \mathbf{v}}$ of a degree less than α such that (1) holds for all $\mathbf{x} \in \Omega_2$ on the line passing through $\mathbf{u}$ and parallel to $\mathbf{v}$. If $\Omega_1 = \Omega_2$, then we denote $C^\alpha(\Omega_1, \Omega_2; \mathbf{v})$ by $C^\alpha(\Omega_1; \mathbf{v})$.

2.1 Continuous Shearlet Transform

Following Kutyniok and Labate [25], we choose the following definition of the continuous shearlet transform.

Definition 2. Given ψ_1 and $\psi_2 \in L^2(\mathbb{R})$, let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\boldsymbol{\xi}) = \hat{\psi}_1(\xi_1) \hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \quad \text{for } \boldsymbol{\xi} = (\xi_1, \xi_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}. \quad (2)$$

Then ψ is called a *continuous shearlet function* if:

- i. $\psi_1 \in L^2(\mathbb{R})$ satisfies the admissibility condition, that is

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty,$$

and $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ with $\text{supp } \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$; and

- ii. $\|\psi\|_2 = 1$, and $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ with $\text{supp } \hat{\psi}_2 \subset [-1, 1]$ and $\hat{\psi}_2 > 0$ on $(-1, 1)$.

A *continuous shearlet system* is the set of functions generated by ψ , namely,

$$\left\{ \psi_{ast} = a^{-\frac{3}{4}} \psi(M_{as}^{-1}(\cdot - \mathbf{t})) : a \in (0, 1), s \in [-2, 2], \mathbf{t} \in \mathbb{R}^2 \right\},$$

where $M_{as} = B_s D_a$, B_s is the *shear matrix* $\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$ and D_a is the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$. The *continuous shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is then defined by

$$\mathcal{SH}_\psi f(a, s, \mathbf{t}) = \langle f, \psi_{ast} \rangle, \quad \text{for } a \in (0, 1), s \in [-2, 2] \text{ and } \mathbf{t} \in \mathbb{R}^2.$$

Let $E \subset \widehat{\mathbb{R}^2}$ be given by $E = \left\{ (\xi_1, \xi_2) \in \widehat{\mathbb{R}^2} : |\xi_1| \geq 2 \text{ and } \left| \frac{\xi_2}{\xi_1} \right| \leq 1 \right\}$ and define $L^2(E)^\sim = \{f \in L^2(\mathbb{R}^2) : \text{supp } \hat{f} \subset E\}$. Then, there is a reconstruction formula for functions in this proper subspace.

Theorem 2.1. *Let $\psi \in L^2(\mathbb{R}^2)$ be a shearlet function. Then, for all $f \in L^2(E)^\sim$,*

$$f = \int_{\mathbb{R}^2} \int_{-2}^2 \int_0^1 \langle \psi_{ast}, f \rangle \psi_{ast} \frac{da}{a^3} ds dt \quad \text{in } L^2(E)^\sim. \quad (3)$$

However, it is still possible to obtain a reproducing formula for all $f \in L^2(\mathbb{R}^2)$ by defining a vertical shearlet function $\psi^{(v)}$ by

$$\widehat{\psi^{(v)}}(\boldsymbol{\xi}) = \widehat{\psi^{(v)}}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_2) \hat{\psi}_2\left(\frac{\xi_1}{\xi_2}\right) \quad (4)$$

where $\hat{\psi}_1, \hat{\psi}_2$ are defined as in Definition 2 above. The shearlets $\psi_{ast}^{(v)}$ are defined by $\psi_{ast}^{(v)} = a^{-\frac{3}{4}} \psi\left((M_{as}^{(v)})^{-1}(\cdot - \mathbf{t})\right)$, where $M_{as}^{(v)} = B_s^{(v)} D_a^{(v)}$ with $B_s^{(v)} = B_s^T$ and $D_a^{(v)} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix}$. Then $\{\psi_{ast}^{(v)}\}$ is the continuous shearlet system for $L^2(E^{(v)})^\sim$ where $E^{(v)} = \left\{ (\xi_1, \xi_2) \in \widehat{\mathbb{R}^2} : |\xi_2| \geq 2 \text{ and } \left| \frac{\xi_2}{\xi_1} \right| > 1 \right\}$, and the associated vertical continuous shearlet transform is

$$\mathcal{SH}_\psi^{(v)} f(a, s, \mathbf{t}) = \left\langle f, \psi_{ast}^{(v)} \right\rangle.$$

A corresponding reconstruction formula holds for $f \in L^2(E^{(v)})^\sim$. As a result, all functions whose frequency support is contained in $\widehat{\mathbb{R}^2} \setminus (-2, 2)^2$ can be reconstructed from the respective horizontal and vertical continuous shearlet transforms. Since bandlimited functions are necessarily C^∞ , these continuous shearlet transforms are sufficient for regularity analysis purposes. Moreover, as ψ_{ast} and $\psi_{ast}^{(v)}$ have essentially the same decay properties and complimentary vanishing directional moments, we shall only focus on ψ_{ast} , but note that analogous statements for $\psi_{ast}^{(v)}$ can be obtained in a similar fashion.

2.2 Properties of Shearlet Functions

Definition 3. The function f of two variables is said to have an L -order vanishing directional moments along the direction $\mathbf{v} = (v_1, v_2)^T \neq \mathbf{0}$ if

$$\int_{\mathbb{R}} b^n f(b\mathbf{v} + \mathbf{w}) db = 0, \quad \text{for all } \mathbf{w} \in \mathbb{R}^2 \text{ and } 0 \leq n < L.$$

The following decay and vanishing directional moments of shearlet functions, taken from [27], are used heavily in our regularity analysis and so are shown below.

Lemma 2.2. *For all $a \in (0, 1)$, $s \in [-2, 2]$ and $\mathbf{t} \in \mathbb{R}^2$, the following is true.*

1. *The shearlet functions ψ_{ast} have vanishing directional moments of any order $L < \infty$ along any direction $\mathbf{v} = (v_1, v_2)^T$ satisfying $|v_2 s + v_1| > |v_2| \sqrt{a}$.*
2. *For each $N = 1, 2, \dots$, there is a constant C_N independent of a, s and $\mathbf{t}$ such that*

$$|\partial^\nu \psi_{ast}(\mathbf{x})| \leq \frac{C_N a^{-3/4-|\nu|} (\sqrt{a} + |s|)^{\nu_2}}{1 + \|\mathbf{D}_{1/a} \mathbf{B}_{-s}(\mathbf{x} - \mathbf{t})\|^{2N}} \quad (5)$$

for all $\mathbf{x} \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$.

3 The Main Results

In this section, for any given $L > 0$, $s_0 \in [-2, 2]$ and $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, let $\Gamma_{\mathbf{u}}$ denote the vertical line passing through $\mathbf{u}$ and let $\Gamma_{\mathbf{u}, s_0}$ denote the line passing through $\mathbf{u}$ with slope $-\frac{1}{s_0}$. Observe that we may write $\Gamma_{\mathbf{u}} = \Gamma_{\mathbf{u}, 0}$ so that $(x_1, x_2) \in \Gamma_{\mathbf{u}, s_0}$ if and only if $x_1 = -s_0(x_2 - u_2) + u_1$. Recall that for a given subset N of $\mathbb{R}^2$, $N(L)$ denotes the L -neighborhood of N , i.e. the set of all points whose distance to N is less than L .

For the purpose of comparison, we quote Theorem 5.3 from [27].

Theorem 3.1. *Let $u_1 \in \mathbb{R}$, $\alpha \in (0, 1]$, $L > 0$, and $\varepsilon > 0$. Suppose f is a bounded function in $C^\alpha(\Gamma_{(u_1, 0)}, \Gamma_{(u_1, 0)}(L); (1, 0))$ and $C^{2\alpha+1+\varepsilon}(\Gamma_{(u_1, 0)}(L); (0, 1))$. Then there exist $C < \infty$ and a fixed scale $a_0 < 1$ such that if $0 < a < a_0$, $-2 \leq s \leq 2$, and $\mathbf{t} = (t_1, t_2) \in \Gamma_{(u_1, 0)}(r)$ with $r < L/2$, then*

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\alpha+\frac{3}{4}} \left(1 + \left|\frac{t_1 - u_1}{a}\right|^\alpha\right) & \text{if } |s| \leq \sqrt{a}, \\ Ca^{\alpha+\frac{5}{4}} & \text{if } |s| \geq \sqrt{a}. \end{cases}$$

In this theorem, the two main assumptions are (i) that the “local” directional Hölder regularity of f on the line $x = u_1$ in the perpendicular direction is at least α and (ii) that its directional Hölder regularity on a neighborhood of the line $x = u_1$ in the parallel direction is at least $2\alpha + 1 + \varepsilon$. The first C^α regularity assumption in the perpendicular direction implies a decay estimate of the order $\alpha + \frac{3}{4}$ for small values of the shear-scale ratio ($|s| \leq a$), while the $C^{2\alpha+1+\varepsilon}$ regularity assumption in the parallel direction implies a decay estimate of the order $\alpha + \frac{5}{4}$ for a large shear-scale ratio ($|s| \geq a$). This means that the $C^{\beta+\varepsilon}$ regularity in the parallel direction would only imply a decay estimate of the order $(\beta - 1)/2$. Our investigation shows that the decay estimate resulting from the regularity assumption in the parallel direction can be improved for larger values of $|s|$. Even though there is no improvement in the other half of the Theorem,

corresponding to the regularity in the perpendicular direction, we include its statement here in item 1 for completeness. Its proof, which can be found in [27], is naturally omitted.

Theorem 3.2. *Let $0 < \alpha \leq 1$, $\beta > 0$ and $0 < \varepsilon < 1$ be such that $(1 + \varepsilon)\beta \notin \mathbb{N}$. Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded and let $u_1 \in \mathbb{R}$ and $L > 0$ be given.*

1. *If $f \in C^\alpha(\Gamma_{(u_1,0)}, \Gamma_{(u_1,0)}(L); (1,0))$ then there exists a constant $C < \infty$ such that for all $a \in (0,1)$, $\mathbf{t} \in \mathbb{R}^2$ and $|s| \leq \sqrt{a}$,*

$$|\langle \psi_{ast}, f \rangle| \leq Ca^{\frac{3}{4}} (a^\alpha + |t_1 - u_1|^\alpha).$$

2. *If $f \in C^{(1+\varepsilon)\beta}(\Gamma_{(u_1,0)}(L); (0,1))$ then there is a constant $C < \infty$ and a fixed coarse scale $a_0 \in (0,1)$ such that for all $a \in (0, a_0)$, $\mathbf{t} \in \Gamma_{(u_1,0)}(L/2)$ and $|s| \geq \sqrt{a}$,*

$$|\langle \psi_{ast}, f \rangle| \leq Ca^{\frac{\beta}{2} + \frac{3}{4}}.$$

3. *If $f \in C^{(1+\varepsilon)\beta}(\Gamma_{(u_1,0)}(L); (0,1))$ and $0 < \gamma < \frac{\varepsilon}{1+\varepsilon} < \frac{1}{2}$ then there is a constant $C < \infty$ and a fixed coarse scale $a_0 \in (0,1)$ such that for all $a \in (0, a_0)$, $\mathbf{t} \in \Gamma_{(u_1,0)}(L/2)$ and $|s| \geq a^\gamma$,*

$$|\langle \psi_{ast}, f \rangle| \leq Ca^{\beta + \frac{3}{4}}.$$

Proof. To prove suppositions 2-3 above., we assume without loss of generality that $u_1 = 0$. Let $c \in (0, \frac{1}{2})$ to be chosen later depending on which statement we want to prove and $R_a = [-a^{-c}, a^{-c}]^2$, then $D_a R_a = [-a^{1-c}, a^{1-c}] \times [-a^{\frac{1}{2}-c}, a^{\frac{1}{2}-c}]$, and hence $B_s D_a R_a$, shrinks to $\{\mathbf{0}\}$ while $R_a \rightarrow \mathbb{R}^2$ as $a \rightarrow 0^+$. For each $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, let $\mathbf{v}_\mathbf{x}$ denote the vector $(x_1, \frac{x_1}{s})$ which makes $\mathbf{x} - \mathbf{v}_\mathbf{x} = (0, x_2 + \frac{x_1}{s})$. Observe that $x_2 + \frac{x_1}{s} = \frac{w_1}{s}$ if and only if $\mathbf{x}$ is on the line $\mathbf{x} = B_s(w_1, x_2)^T$. That is, for a fixed $w_1 \in \mathbb{R}$, the sheared line $B_s \ell_{w_1}$ of the vertical line $\ell_{w_1} = \{(w_1, x_2)^T : x_2 \in \mathbb{R}\}$ is equal to the set of points $\mathbf{x}$ satisfying $x_2 + \frac{x_1}{s} = \frac{w_1}{s}$. Consequently, for each $\mathbf{x} \in B_s D_a R_a = B_s([-a^{1-c}, a^{1-c}] \times [-a^{\frac{1}{2}-c}, a^{\frac{1}{2}-c}])$, $\mathbf{x} \in B_s \ell_{w_1}$ for some $|w_1| \leq a^{1-c}$ and so $\|\mathbf{x} - \mathbf{v}_\mathbf{x}\| = |x_2 + \frac{x_1}{s}| = |\frac{w_1}{s}| \leq \frac{a^{1-c}}{|s|}$.

Let $a_0 < 1$ be so small that $B_s D_a R_a \subseteq \Gamma_{\mathbf{0}}(L/2)$ whenever $|s| < 2$. Let $\mathbf{t} \in \Gamma_{\mathbf{0}}(L/2)$ and $0 < a < a_0$. Then $\tau_{-\mathbf{t}} f \in C^{(1+\varepsilon)\beta}(\Gamma_{\mathbf{0}}(L/2); (0,1))$. Hence, for each $\mathbf{x} \in B_s D_a R_a \subseteq \Gamma_{\mathbf{0}}(L/2)$, since $\mathbf{v}_\mathbf{x} \in \Gamma_{\mathbf{0}}(L/2)$, there exists a polynomial $P_{\mathbf{v}_\mathbf{x}}$ of a degree less than $(1 + \varepsilon)\beta$ such that

$$|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_\mathbf{x}}(\mathbf{x} - \mathbf{v}_\mathbf{x})| \leq C \|\mathbf{x} - \mathbf{v}_\mathbf{x}\|^{(1+\varepsilon)\beta} \leq C \frac{a^{(1-c)(1+\varepsilon)\beta}}{|s|^{(1+\varepsilon)\beta}}.$$

If $|s| \geq a^{1/2}$ then $|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)| \leq C \frac{a^{(1-c)(1+\varepsilon)\beta}}{a^{(1+\varepsilon)\beta/2}} = Ca^{(1/2-c)(1+\varepsilon)\beta}$. Choosing $c = \frac{\varepsilon}{2(1+\varepsilon)}$, we have $|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)| \leq Ca^{\beta/2}$. While if we instead assume $|s| \geq a^\gamma$ then $|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)| \leq Ca^{(1-c-\gamma)(1+\varepsilon)\beta}$. Choosing $c = \frac{\varepsilon}{1+\varepsilon} - \gamma$, we have $|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)| \leq Ca^\beta$.

This inequality on $B_s D_a R_a$ for $|s| \geq a^\gamma$ yields the estimate:

$$\begin{aligned} \left| \int_{B_s D_a R_a} (f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)) \psi_{as\mathbf{0}}(\mathbf{x}) d\mathbf{x} \right| &\leq C \int_{B_s D_a R_a} \frac{a^\beta a^{-\frac{3}{4}}}{1 + \|D_{\frac{1}{a}} B_{-s} \mathbf{x}\|^{2N}} d\mathbf{x} \\ &\leq Ca^\beta \int_{R_a} \frac{a^{\frac{3}{4}}}{1 + \|\mathbf{y}\|^{2N}} d\mathbf{y} \\ &\leq Ca^{\beta + \frac{3}{4}}, \end{aligned} \tag{6}$$

where Lemma 2.2 is used in the inequality (6). If we only assume $|s| \geq a^{1/2}$, similar arguments yield $\left| \int_{B_s D_a R_a} (f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)) \psi_{as\mathbf{0}}(\mathbf{x}) d\mathbf{x} \right| \leq Ca^{\frac{\beta}{2} + \frac{3}{4}}$.

By the decay estimate of $\psi_{as\mathbf{0}}$ (Lemma 2.2) and a change of variable ($\mathbf{x} = B_s D_a \mathbf{y}$), if M is an upper bound of $|f(\cdot + \mathbf{t})|$ then the integral on $\mathbb{R}^2 \setminus B_s D_a R_a$ can be bounded by

$$\begin{aligned} &\left| \int_{\mathbb{R}^2 \setminus B_s D_a R_a} (f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)) \psi_{as\mathbf{0}}(\mathbf{x}) d\mathbf{x} \right| \\ &\leq Ca^{-\frac{3}{4}} \int_{\mathbb{R}^2 \setminus B_s D_a R_a} \frac{|f(\mathbf{x} + \mathbf{t}) - P_{\mathbf{v}_x}(\mathbf{x} - \mathbf{v}_x)|}{1 + \|D_{\frac{1}{a}} B_{-s} \mathbf{x}\|^{2N}} d\mathbf{x} \\ &\leq Ca^{\frac{3}{4}} \int_{\mathbb{R}^2 \setminus R_a} \frac{M + P_{\mathbf{y}'}(C' \|\mathbf{y}\|)}{1 + \|\mathbf{y}\|^{2N}} d\mathbf{y} \quad (\text{for some } C' > 0) \\ &= Ca^{\frac{3}{4} + c(2N-1 - \text{degree } P_{\mathbf{y}'})} = Ca^K \end{aligned} \tag{7}$$

where $\mathbf{y}' = \mathbf{v}_{B_s D_a \mathbf{y}}$ and K can be chosen arbitrarily large as c is fixed and N is arbitrary. $\square$

The singularity lines $\Gamma_{\mathbf{u}}$ investigated in Theorem 3.2 are always vertical. When one wants to consider singularity along a non-vertical line $\Gamma_{\mathbf{u}, s_0}$, the following lemma from [27, Lemmas 5.1 & 5.2] is essential.

Lemma 3.3. *Let $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, $s_0 \in [-2, 2]$, $L > 0$ and f be a bounded function. Denote $\mathbf{u}' = (u_1 + s_0 u_2, 0)$.*

1. *If $f \in C^\alpha(\Gamma_{\mathbf{u}, s_0}, \Gamma_{\mathbf{u}, s_0}(L); (1, 0))$ then $f \circ B_{s_0} \in C^\alpha(\Gamma_{\mathbf{u}'}, \Gamma_{\mathbf{u}'}(L); (1, 0))$.*
2. *If $f \in C^\alpha(\Gamma_{\mathbf{u}, s_0}(L); B_{s_0}(0, 1))$ then $f \circ B_{s_0} \in C^\alpha(\Gamma_{\mathbf{u}'}(L); (0, 1))$.*

Applying the lemma, Theorem 3.2 can be generalized to the case with a sheared singularity line, which can be shown by applying the theorem to $f \circ B_{s_0}$ and using the fact that $\langle \psi_{ast}, f \rangle = \langle \psi_{a(s-s_0)B_{-s_0}t}, f \circ B_{s_0} \rangle$.

Theorem 3.4. *Let $0 < \alpha \leq 1$, $\beta > 0$ and $0 < \varepsilon < 1$ be such that $(1 + \varepsilon)\beta \notin \mathbb{N}$. Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded and let $s_0 \in [-2, 2]$, $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ and $L > 0$ be given.*

1. *If $f \in C^\alpha(\Gamma_{\mathbf{u}, s_0}, \Gamma_{\mathbf{u}, s_0}(L); (1, 0))$ then there exists a constant $C < \infty$ such that for all $a \in (0, 1)$, $\mathbf{t} \in \mathbb{R}^2$ and $|s - s_0| \leq \sqrt{a}$,*

$$|\langle \psi_{ast}, f \rangle| \leq Ca^{\frac{3}{4}} (a^\alpha + |t_1 + s_0 t_2 - u_1 - s_0 u_2|^\alpha).$$

2. *If $f \in C^{(1+\varepsilon)\beta}(\Gamma_{\mathbf{u}, s_0}(L); B_{s_0}(0, 1))$ and $0 < \gamma < \frac{\varepsilon}{1+\varepsilon} < \frac{1}{2}$ then there are a constant $C < \infty$ and a fixed coarse scale $a_0 \in (0, 1)$ such that for all $a \in (0, a_0)$ and $\mathbf{t} \in \Gamma_{\mathbf{u}, s_0}(L/2)$,*

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\frac{\beta}{2} + \frac{3}{4}} & \text{if } |s - s_0| \geq \sqrt{a}, \\ Ca^{\beta + \frac{3}{4}} & \text{if } |s - s_0| \geq a^\gamma. \end{cases}$$

As a result of the theorem, we put together a set of directional regularity conditions to model a linear singularity situation, a similar case of which is considered in [25].

Corollary 3.5. *Let $0 < \alpha \leq 1$, $\mathbf{u} \in \mathbb{R}^d$, $s_0 \in [-2, 2]$ and $N, \varepsilon, \gamma > 0$ such that $(1 + \varepsilon)N \notin \mathbb{N}$ and $\gamma < \frac{\varepsilon}{1+\varepsilon}$. Suppose f is a bounded function in $C^\alpha(\Gamma_{\mathbf{u}, s_0}, \Gamma_{\mathbf{u}, s_0}(L); (1, 0))$, $C^N(\Gamma_{\mathbf{u}, s_0}(L) \setminus \Gamma_{\mathbf{u}, s_0}; (1, 0))$ and $C^{(1+\varepsilon)N}(\Gamma_{\mathbf{u}, s_0}(L); B_{s_0}(0, 1))$ for some $L > 0$. Then there is a constant $C > 0$ and $a_0 < 1$ such that for all $0 < a < a_0$, $\mathbf{t} \in \Gamma_{\mathbf{u}, s_0}(L/2)$ and $s \in [-2, 2]$*

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} Ca^{\alpha + \frac{3}{4}} & \text{if } s = s_0 \text{ and } \mathbf{t} \in \Gamma_{\mathbf{u}, s_0} \\ Ca^{N + \frac{3}{4}} & \text{if } s = s_0 \text{ and } \mathbf{t} \notin \Gamma_{\mathbf{u}, s_0} \\ Ca^{N + \frac{3}{4}} & \text{if } |s - s_0| \geq a^\gamma \end{cases}$$

Consequently, for $s = s_0$ and $\mathbf{t} \in \Gamma_{\mathbf{u}, s_0}$, $\langle \psi_{ast}, f \rangle = O(a^{\alpha + \frac{3}{4}})$ as $a \rightarrow 0^+$ while for all other cases $\langle \psi_{ast}, f \rangle = O(a^{N + \frac{3}{4}})$ as $a \rightarrow 0^+$.

Proof. Let $\mathbf{t} \in \Gamma_{\mathbf{u}, s_0}(L/2)$. If $s = s_0$ and $\mathbf{t}$ is on the line $\Gamma_{\mathbf{u}, s_0}$ then $|s - s_0| \leq \sqrt{a}$ and $s_0(t_2 - u_2) + (t_1 - u_1) = 0$. So Theorem 3.4 (1.) gives $|\langle \psi_{ast}, f \rangle| \leq Ca^{\alpha + \frac{3}{4}}$ for all $0 < a < 1$.

Let $s = s_0$ and $\mathbf{t} \notin \Gamma_{\mathbf{u}, s_0}$. In the light of Theorem 3.4 (1.) on the line $\Gamma_{\mathbf{t}, s_0}$, the assumption that $f \in C^N(\Gamma_{\mathbf{u}, s_0}(L) \setminus \Gamma_{\mathbf{u}, s_0}; (1, 0))$ implies that $|\langle \psi_{ast}, f \rangle| \leq Ca^{N + \frac{3}{4}}$ for all $0 < a < 1$.

If $|s - s_0| \geq a^\gamma$ then, by Theorem 3.4 (2.), $|\langle \psi_{ast}, f \rangle| \leq Ca^{N+\frac{3}{4}}$ for all $0 < a < a_0$.

It is now left to show that $\langle \psi_{ast}, f \rangle = O(a^{N+\frac{3}{4}})$ as $a \rightarrow 0^+$ when $s \neq s_0$. In such a case, it is clear that if $0 < a \leq |s - s_0|^{1/\gamma}$ then the shearlet transform is bounded by $Ca^{N+\frac{3}{4}}$. $\square$

In [9], Grohs characterized the wavefront set, defined via decays of Fourier transform, of a tempered distribution in terms of the decay rates of its continuous shearlet transform. Though the possible singularity in their results is far more general than ours, the singularity in our different setting is based on lower Hölder exponents.

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