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ON SOME ASPECTS OF THE MASS DIPOLE PROBLEM OF A SPECIAL WEYL SOLUTION IN THE EINSTEIN-MAXWELL THEORY

By

E. SCHMUTZER and J. NOTTROTT

INSTITUTE FOR THEORETICAL PHYSICS OF THE UNIVERSITY OF JENA, JENA, DDR

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The exterior solution of the Einstein—Maxwell equations, describing all static axisymmetric problems, has been found by WEYL [1]. This so-called Weyl solution has been specialized by BONNOR [2] with respect to a point singularity interpreted as a magnetic dipole, which in certain cases induces a mass dipole. In this paper a special calculus of delta functions [3] is used to obtain information about the structure of the singular part of T_4^4 of the point source of the field.

§ 1. Field equations and the dipole solution

Let us use the metric

$$ds^2 = e^{\lambda(z,r)}\,(dz^2 + dr^2) + e^{-\varrho(z,r)}\,r^2\,d\vartheta^2 - e^{\varrho(z,r)}\,c^2\,dt^2 \tag{1}$$

(abbreviations:

$$z = x^1,\ r = x^2,\ \vartheta = x^3,\ ct = x^4,\ \sqrt{\overset{3}{g}} = e^{\lambda}\,r\,;$$

Greek indices run from 1 to 3, Latin indices from 1 to 4) with the infinity conditions:

$$\lambda \to 0,\ \varrho \to 0,\ \text{if}\ (z^2 + r^2) \to \infty\,.$$

The field equations are

$$R^n_m = \varkappa\left(T^n_m - \frac{1}{2}\,g^n_m\,T^r_r\right), \tag{2a}$$

$$H^{mn}{}_{;\,n} = \frac{1}{c}\,j^m\,, \tag{2b}$$

$$B_{<mn;\,l>} = 0\,. \tag{2c}$$

We split the energy-momentum tensor into two parts

$$T_{mn} = E_{mn} + \Theta_{mn}\,. \tag{3}$$

E_{mn} corresponds to the pure magnetic field without polarization, Θ_{mn} to the magnetization and the additional matter in the point singularity.

Therefore E_{mn} can be written as

$$E_{mn} = B_{mi} B^i_n + \frac{1}{4} g_{mn} B_{ij} B^{ij} . \tag{4}$$

The tensor Θ_{mn} consists of two parts:

$$\Theta_{mn} = S_{mn} - {}^{(\Theta)} \mu u_m u_n . \tag{5}$$

S_{mn} describes the magnetizational and mechanical stress part in the singularity. ${}^{\Theta}\mu$ represents the rest-mass density. In a rest-system ($u^1 = u^2 = u^3 = 0$, $u_4 u^4 = -c^2$) Θ^n_m is expressed in the form

$$(\Theta^n_m) = \begin{pmatrix} \alpha & \varepsilon & 0 & 0 \\ \varepsilon & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \mu c^2 \end{pmatrix} , \tag{6}$$

if we use the notation

$$\frac{1}{c^2} S^4_4 + {}^{(\Theta)} \mu = \mu .$$

The magnetic dipole may be orientated in z-direction. Now we introduce the four-potential A_m and use $B_{12} \sim B_\vartheta = 0$. Then we need only $A_3 = \Phi(z, r) \neq 0$. As a (non essential) consequence of the line element chosen, we obtain $\alpha = \bar{\alpha}$. Using the transformation

$$\begin{aligned} \Phi_{,1} &= r e^{-\varrho} \chi_{,2} \\ \Phi_{,2} &= - r e^{-\varrho} \chi_{,1} \end{aligned} \tag{7}$$

after some calculation we get the following system of Einstein's equation

$$\begin{aligned} &(a)\ \Delta\varrho = \varkappa e^{-\varrho} \left[(\chi_{,1})^2 + (\chi_{,2})^2\right] + \varkappa e^{\lambda} (\mu c^2 - \beta) , \\ &(b)\ \gamma_{,1,1} + \gamma_{,2,2} + \frac{1}{2} \left[(\varrho_{,1})^2 + (\varrho_{,2})^2\right] = \varkappa e^{-\varrho} \left[(\chi_{,1})^2 + (\chi_{,2})^2\right] - 2 \varkappa e^{\lambda} \beta , \\ &(c)\ (\varrho_{,1})^2 - (\varrho_{,2})^2 + 2 r^{-1} \gamma_{,2} = 2 \varkappa e^{-\varrho} \left[(\chi_{,1})^2 - (\chi_{,2})^2\right] + 4 \varkappa e^{\lambda} \alpha , \\ &(d)\ \varrho_{,1} \varrho_{,2} - r^{-1} \gamma_{,1} = 2 \varkappa e^{-\varrho} \chi_{,1} \chi_{,2} + 2 \varkappa e^{\lambda} \varepsilon \end{aligned} \tag{8}$$

(abbreviations:

$$\Delta\varrho = \varrho_{,1,1} + \varrho_{,2,2} + r^{-1} \varrho_{,2}, \ \gamma = \lambda + \varrho .$$

The tensors H^{mn} and B^{mn} are connected with the magnetization tensor M^{mn}:

$$H^{mn} = B^{mp} + M^{mn} . \tag{9}$$

The tensor M^{mn} has the structure

$$M_{\mu\nu} = i\varepsilon_{\mu\nu\lambda} M^{\lambda}, \; M_{\mu 4} = 0 \; (\varepsilon_{\mu\nu\lambda} : \text{3-dimensional Levi-Civita pseudotensor}). \qquad (10)$$

The non vanishing Maxwell equation (the Lorentz condition is fulfilled identically) is

$$\Delta\chi - \varrho_{,1}\chi_{,1} - \varrho_{,2}\chi_{,2} = e^{\varrho}(e^{-\varrho/2} M_1)_{,1}\,. \qquad (11)$$

Outside of the singularity the system for equations (8a,b,c,d; 11) is already solved (cf. 1, 2).

Using the abbreviations

$$p^2 = z^2 + r^2, \quad D = \sqrt{\frac{2}{\varkappa} - \frac{A^2}{4}},$$

the general solution for a magnetic dipole can be written as

$$(a) \qquad \varrho = \ln\left\{\frac{\varkappa D^2}{2\cos^2\left[\varkappa\dfrac{D\psi}{2} + \operatorname{arctg}\dfrac{A}{2D}\right]}\right\},$$

$$(b) \qquad \chi = D \operatorname{tg}\left\{\frac{\varkappa D\psi}{2} + \operatorname{arctg}\frac{A}{2D}\right\} - \frac{A}{2}, \qquad (12)$$

$$(c) \qquad \lambda = -\frac{\varkappa^2 D^2 \bar{\mu}^2 r^2}{128\pi^2 p^3}\left[2 - \frac{9r^2}{4p^2}\right] - \varrho,$$

where ψ is the magnetic potential for the field of a magnetic dipole in Minkowskian space

$$\psi = \frac{\bar{\mu} z}{4\pi p^3} \qquad (13)$$

($\bar{\mu}$ is the z-component of m^a, m^a being the vector of the magnetic moment.) A (or D) is a free constant of integration.

§ 2. The method of extending the solution

The 4-dimensional complex of the magnetic moment we define by

$$m^k = \int \check{M}^{kr}\, df_r\,, \qquad (14)$$

where

$$\check{M}^{kr} = -\frac{i}{2}\,\varepsilon^{kr}_{ij}\, ijM^{ij}$$

is the dual tensor of electromagnetic polarization and df_r the surface element tensor. If we define the 3-dimensional space by $x^4 = \text{const}$, we get

$$df_1 = df_2 = df_3 = 0\,,\; df_4 = i\sqrt{-g_{44}}\,d^{(3)}V$$

($d^{(3)}\,V$ is the 3-dimensional volume element).

This means

$$m^k = i\int \check{M}^{k4}\sqrt{-g_{44}}\,d^{(3)}V. \tag{15}$$

Observing the relation between $\check{M}^{k4}$ and the 3-dimensional magnetization M_α, namely

$$\check{M}_{\alpha 4} = i\sqrt{-g_{44}}\,M_\alpha\,, \tag{16}$$

we find

$$m^\alpha = i\int g^{\alpha\beta}\,g^{44}\,\check{M}_{\beta 4}\sqrt{-g_{44}}\,d^{(3)}V = \int g^{\alpha\beta}\,M_\beta\,d^{(3)}V. \tag{17}$$

In our metric, for a magnetic dipole in z-direction this equation becomes

$$m^1 = \bar{\mu} = \int e^{-\varrho/2}M_1\,rdzdrd\vartheta = 2\pi\int e^{-\varrho/2}M_1\,rdzdr. \tag{18}$$

This equation for a point singularity is satisfied by

$$M_1 = \frac{\bar{\mu}e^{\varrho/2}\,\delta(z)\,\delta(r)}{2\pi r} = \frac{\bar{\mu}e^{\varrho/2}\,\delta(p)}{4\pi p^2}\,. \tag{19}$$

Inserting this result into (11) we obtain

$$\Delta\chi - \varrho_{,1}\,\chi_{,1} - \varrho_{,2}\,\chi_{,2} = e^\varrho\left(\frac{\bar{\mu}\,\delta(p)}{4\pi p^2}\right)_{,1}\,. \tag{20}$$

Eliminating χ and ϱ by (12a,b) we get

$$\begin{aligned}\Delta\psi &= \frac{2}{\varkappa D^2}\cos^2\left(\frac{\varkappa D\psi}{2} + \operatorname{arctg}\frac{A}{2D}\right)e^\varrho\left(\frac{\bar{\mu}\,\delta(p)}{4\pi p^2}\right)_{,1} = \\ &= \left(\frac{\bar{\mu}\delta(p)}{4\pi p^2}\right)_{,1} = -\frac{3\bar{\mu}}{4\pi}\,\frac{\delta(p)\,z}{p^4}\,.\end{aligned} \tag{21}$$

This equation is solved by

$$\psi = \frac{\bar{\mu}}{2\pi}\,\frac{\bar{\Theta}(p)\,z}{p^3}\,. \tag{22}$$

Our method to extend the solution (12a,b,c) into the singularity is the following: We use the form (22) for the potential ψ in all three equations (12a,b,c) and find with the help of the abbreviation

$$s = \frac{\varkappa \bar{\mu} z D \bar{\Theta}(p)}{4\pi p^3} + \text{arc tg} \frac{A}{2D} \tag{23}$$

from (12a,b)

$$\begin{aligned} &(a) \quad \varrho = -\ln\left(\frac{2\cos^2 s}{\varkappa D^2}\right), \\ &(b) \quad \chi = D \,\text{tg}\, s - \frac{A}{2}. \end{aligned} \tag{24}$$

A similar, non artificial generalization of Eq. (12c) is not known. Therefore, we now investigate the case $D = 0$.

After some calculation we obtain from (12) for $D \to 0$

$$\begin{aligned} &(a) \quad \varrho = -2\ln\left(1 \mp \sqrt{\frac{\varkappa}{2}}\,\frac{\bar{\mu} z \bar{\Theta}(p)}{2\pi p^3}\right), \\ &(b) \quad \chi = \pm\sqrt{\frac{2}{\varkappa}}\left[\left(1 \mp \sqrt{\frac{\varkappa}{2}}\,\frac{\bar{\mu} z \bar{\Theta}(p)}{2\pi p^3}\right)^{-1} - 1\right], \\ &(c) \quad \lambda + \varrho = 0. \end{aligned} \tag{25}$$

Inserting these results into (8b,c,d) we obtain

$$\varepsilon = \alpha = \beta = 0. \tag{26}$$

From Eq. (8a) we get

$$\mu c^2 = \pm\sqrt{\frac{2}{\varkappa}}\left(1 \pm \sqrt{\frac{\varkappa}{2}}\,\chi\right) \Delta\chi - 2\,[(\chi_{,1})^2\,(\chi_{,2})^2]. \tag{27}$$

By elimination from (25a,b) results

$$\varrho = 2\ln\left(1 \pm \sqrt{\frac{\varkappa}{2}}\,\chi\right). \tag{28}$$

Inserting into Eq. (20) yields an expression which we use for elimination in Eq. (27). The result finally is

$$\mu c^2 = \mp \frac{3\bar{\mu}\sqrt{\frac{2}{\varkappa}}}{4\pi\left(1 \mp \frac{\sqrt{\frac{\varkappa}{2}}\,\bar{\mu}\bar{\Theta}(p)\,z}{2\pi p^3}\right)}\,\frac{\delta(p)\cdot z}{p^4}. \tag{29}$$

An additional calculation shows that the solution (25a,b,c) fulfils the condition

$$T^n_{m;n} = 0. \tag{30}$$