

USE OF DELTA FUNCTIONS IN GENERAL RELATIVITY FOR DETERMINATION OF THE INTEGRATION CONSTANTS

By

E. SCHMUTZER

INSTITUTE FOR THEORETICAL PHYSICS OF THE UNIVERSITY OF JENA, JENA, DDR

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During the integration of the Einstein—Maxwell equations, integration constants appear and their interpretation is often very difficult. There is therefore, a requirement for a calculus of delta functions which will automatically relate integration constants to sources. In this paper a calculus of this kind is developed and applied to the spherically symmetric problem. In this way we get a method of distinction between pure mathematical and physical singularities.

I. Calculus of delta functions

In this paper a calculus of delta functions suitable for certain physical problems is presented. The work here extends the use of delta functions to be found in a paper by ROSEN and SHAMIR [1] and a book by INFELD and PLEBANSKY [2].

A general delta tensor of the rank k in space-time is defined by

$$\delta^{n_1 \dots n_k}(x^j, \xi^j) = \delta^{n_1 \dots n_k}(\xi^j, x^j) = \begin{cases} 0 & \text{for } x^j \neq \xi^j, \\ \infty & \text{for } x^j = \xi^j, \end{cases} \quad (1)$$

(k is an integer between 1 and 4). The strength of the singularity is of such a kind that

$$i \int_{V_{4-k}} \delta^{n_1 \dots n_k}(x^j, \xi^j) df_{n_1 \dots n_k}(x^j) = 1 \quad (2)$$

is valid if the singularity is situated in the $(4 - k)$ -dimensional subspace over which the integral is taken. $df_{n_1 \dots n_k}$ is the surface-element pseudo-tensor of rank k . A consequence of (1) and (2) is the equation

$$i \int_{V_{4-k}} F(x^j) \delta^{n_1 \dots n_k}(x^j, \xi^j) df_{n_1 \dots n_k}(x^j) = F(\xi^j). \quad (3)$$

(The factor i in the last two equations is a consequence of definition of $df_{n_1 \dots n_k}$) Two cases are particularly interesting: namely if the delta tensor is of the rank 0 ($k = 0$) we get

$$\int_{V_4} F(x^j) \delta(x^j, \xi^j) d^{(4)}f(x^j) = F(\xi^j) \quad (4)$$

and if the delta tensor is of the rank 1 ($k = 1$), we get

$$i \int_{V_s} F(x^j) \delta^k(x^j, \xi^j) df_k(x^j) = F(\xi^j). \quad (5)$$

In the first case the delta tensor is called Dirac's delta function. In the second case it is called a surface delta function. Our further investigations refer to Dirac's delta function.

First we study the 1-dimensional case. Heaviside's step function is defined by

$$\Theta(x) = \begin{cases} -\frac{1}{2} & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \frac{1}{2} & \text{for } x > 0 \end{cases} \quad (6)$$

so that the relations

$$\begin{aligned} a) \Theta(x)' &= \delta(x), \quad b) \Theta(x) + \Theta(-x) = 0, \quad c) \Theta(x) \Theta(-x) = \begin{cases} -\frac{1}{4} & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ -\frac{1}{4} & \text{for } x > 0 \end{cases} \\ d) \Theta(x)^2 &= -\Theta(x) \Theta(-x) \end{aligned} \quad (7)$$

hold. Dirac's delta function satisfies the differential equation

$$x\delta'(x) + \delta(x) = 0. \quad (8)$$

The 3-dimensional delta function is given by

$$\delta(\vec{r}) = \delta(x) \delta(y) \delta(z) \quad (r = \sqrt{x^2 + y^2 + z^2}). \quad (9)$$

This quantity allows the definition of the 3-dimensional radial delta function

$$\delta(r) = 4\pi r^2 \delta(\vec{r}), \quad (10)$$

for which the equations

$$a) \delta(r) = \begin{cases} \infty & \text{for } r = 0 \\ 0 & \text{for } r > 0 \end{cases}, \quad b) \int_0^\infty \delta(r) dr = 1 \quad (11)$$

are valid. The relation

$$\delta(r) = 2\bar{\Theta}(r)' \quad (12)$$

exists between the delta function and the radial step function

$$\bar{\Theta}(r) = \begin{cases} 0 & \text{for } r = 0, \\ \frac{1}{2} & \text{for } r > 0. \end{cases} \quad (13)$$

By differentiation one verifies the important equation

$$\Delta \left(\frac{\bar{\Theta}(r)}{2 \pi r} \right) = - \delta(\vec{r}), \tag{14}$$

where Δ is the Laplace operator in spherical polar coordinates. In distribution theory literature, this is replaced by the notation

$$\Delta \left(\frac{1}{4 \pi r} \right) = - \delta(\vec{r}).$$

The 2-dimensional radial delta function is defined by

$$\delta(\sigma) = 2 \pi \sigma \delta(x) \delta(y) \quad (\sigma = \sqrt{x^2 + y^2}). \tag{15}$$

For this the relations

$$a) \delta(\sigma) = \begin{cases} \infty & \text{for } \sigma = 0 \\ 0 & \text{for } \sigma > 0 \end{cases}, \quad b) \int_0^\infty \delta(\sigma) d\sigma = 1 \tag{16}$$

are valid. By calculation one verifies that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(- \frac{\bar{\Theta}(\sigma) \ln \sigma}{2 \pi} \right) = - \delta(x) \delta(y). \tag{17}$$

II. Investigation of Reissner—Weyl—Schwarzschild field

The square of the line element is used in the form

$$(ds)^2 = e^{\nu(r)}(dr)^2 + r^2 [\sin^2 \Theta (d\varphi)^2 + (d\Theta)^2] - e^{\nu(r)}(dx^4)^2, \tag{18}$$

$$(x^1 = r, x^2 = \varphi, x^3 = \Theta).$$

We write the field equations in the form

$$a) R_i^j = \kappa \left(T_i^j - \frac{1}{2} g_i^j T_m^m \right), \quad (\text{Einstein equation})$$

$$b) (E^\mu \sqrt{-g})_{1,\mu} = \rho \sqrt{-g}, \tag{19}$$

$$c) (E_3 \sqrt{-g_{44}})_{,1} = (E_1 \sqrt{-g_{44}})_{,3} = 0, \quad (E_2 \sqrt{-g_{44}})_{,1} = (E_1 \sqrt{-g_{44}})_{,2},$$

$$(E_3 \sqrt{-g_{44}})_{,2} = (E_2 \sqrt{-g_{44}})_{,3}, \quad (\overset{4}{g} = -|g_{ij}|, \overset{3}{g} = -|g_{\mu\nu}|) \quad \text{equation}$$

taking into account that we have to treat a static problem. (Greek indices run from 1 to 3, latin indices from 1 to 4.) $E^\mu = g^{\mu\nu} E_\nu$ is the 3-dimensional electric field strength, and ρ is the 3-dimensional charge density.

The Einstein equations can be written in the following way:

$$\begin{aligned}
 a) \quad & \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\mu'}{r} - \frac{\nu' \mu'}{4} = \kappa e^\mu \left(T_1^1 - \frac{1}{2} T_m^m \right), \\
 b) \quad & \frac{\nu' - \mu'}{2} r + 1 - e^\mu = \kappa r^2 e^\mu \left(T_2^2 - \frac{1}{2} T_m^m \right), \\
 c) \quad & \frac{\nu' - \mu'}{2} r + 1 - e^\mu = \kappa r^2 e^\mu \left(T_3^3 - \frac{1}{2} T_m^m \right) \\
 d) \quad & \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{r} - \frac{\nu' \mu'}{4} = \kappa e^\mu \left(T_4^4 - \frac{1}{2} T_m^m \right).
 \end{aligned} \tag{20}$$

From the second and third it follows that

$$T_2^2 = T_3^3, \tag{21}$$

while the first and the fourth yield

$$\frac{\nu' + \mu'}{r} = \kappa e^\mu (T_4^4 - T_1^1). \tag{22}$$

Using the abbreviation

$$\gamma = e^{-\mu} \tag{23}$$

we get from (20) after some calculation

$$T_m^m = -\frac{\gamma}{\kappa} \left[\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{r} \right] + \frac{1}{\kappa} \left[\frac{2}{r^2} (1 - \gamma) - \frac{\nu' \gamma'}{2} - \frac{2\gamma'}{r} \right], \tag{24}$$

$$T_4^4 = \frac{1}{\kappa} \left[-\frac{\gamma'}{r} + \frac{1}{r^2} (1 - \gamma) \right], \tag{25}$$

$$T_1^1 = \frac{1}{\kappa} \left[\frac{1}{r^2} (1 - \gamma) - \frac{\nu' \gamma'}{r} \right], \tag{26}$$

$$T_2^2 = -\frac{\gamma}{2\kappa} \left[\nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} \right] - \frac{\gamma'}{2\kappa} \left[\frac{1}{r} + \frac{\nu'}{2} \right]. \tag{27}$$

The energy tensor is now split into an electromagnetic part E_i^j and a remainder part Θ_i^j :

$$T_i^j = E_i^j + \Theta_i^j, \tag{28}$$

where

$$(E_i^j) = \left(\begin{array}{c|c} E_\mu E^\nu - \frac{1}{2} g_\mu^\nu E_\lambda E^\lambda & 0 \\ \hline 0 & -\frac{1}{2} E_\lambda E^\lambda \end{array} \right), \quad (\Theta_i^j) = \left(\begin{array}{cccc} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \mu c^2 \end{array} \right). \tag{29}$$

An electrical point singularity with the charge e must be described by

$$\varrho(r) = \frac{e}{4 \pi r^2 e^{\mu/2}} \delta(r) \tag{30}$$

because the equation

$$\int \varrho d^{(3)}f = \int \varrho \sqrt{-g} d^{(3)}x = e \int_0^\infty \delta(r) dr = e \tag{31}$$

must be valid.

The integration of the Maxwell equations yields

$$E_2 = E_3 = 0, E_1 = \frac{e \bar{\theta}(r) e^{\mu/2}}{2 \pi r^2} \tag{32}$$

so that

$$\begin{aligned} E_1^1 = E_4^4 &= \frac{1}{2} E_1 E^1 = \frac{e^2 \bar{\theta}(r)^2}{8 \pi^2 r^4}, \\ E_2^2 = E_3^3 &= -\frac{1}{2} E_1 E^1 = -\frac{e^2 \bar{\theta}(r)^2}{8 \pi^2 r^4} \end{aligned} \tag{33}$$

follows.

For integration of (25) we choose for a point singularity with the mass M the rest mass density

$$\mu = \frac{M \delta(r)}{4 \pi r^2}. \tag{34}$$

Using (33) we find

$$(\gamma r)' = 1 - \frac{\kappa M c^2 \delta(r)}{4 \pi} - \frac{\kappa e^2 \bar{\theta}(r)^2}{8 \pi^2 r^2} \tag{35}$$

and further by integration

$$\gamma = 1 - \frac{\kappa M c^2 \bar{\theta}(r)}{2 \pi r} + \frac{\kappa e^2 \bar{\theta}(r)^2}{8 \pi^2 r^2} + \frac{\kappa e^2}{8 \pi^2 r} \int_0^r \bar{\theta}(r) \delta'(r) dr. \tag{36}$$

Outside the singularity this is the well known result

$$\gamma = 1 - \frac{\kappa M c^2}{4 \pi r} + \frac{\kappa e^2}{32 \pi^2 r^2}. \tag{37}$$

Up to this point the stress distribution in the singularity is not fixed. Further assumptions about $\nu(r)$ would be necessary. The formula $\nu + \mu = 0$, which holds outside the singularity, leads to non-physical stress inside the singularity.

REFERENCES

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2. L. INFELD and J. PLEBANSKI, *Motion and Relativity*, Warsaw, 1960.

ПРИМЕНЕНИЕ δ -ФУНКЦИЙ В ТЕОРИИ ОБЩЕЙ ОТНОСИТЕЛЬНОСТИ
ДЛЯ ОПРЕДЕЛЕНИЯ ПОСТОЯННЫХ ИНТЕГРИРОВАНИЯ

Е. ШМУЦЕР

Резюме

При интегрировании уравнений Эйнштейна—Маквелла появляются постоянные интегрирования, интерпретация которых часто представляется очень трудной. Отсюда возникает потребность применения в вычислениях δ -функций, которые автоматически указывают на происхождение этих постоянных. В данной работе развивается метод такого характера, дается его применение в случае проблемы, обладающей сферической симметрией. Таким путем нами дается метод для различия между чисто математической и физической сингулярностями.