Correction to: Quadratic Presentations and Nilpotent Kähler Groups

By *James A. Carlson and Domingo Toledo*

First correction. It has been pointed out to us by S. Chen that the inductive step (2.3) in the construction of 1-formal, 1-minimal models is not correct. By 1-formal, 1-minimal model we mean what is called formal 1-minimal model in the first paragraph of Section 2. We proceed to describe a correct inductive step.

Let V_1 denote the vector space V of Section 2 over the field k (one of $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). Let V_2 denote the space C_0 defined by (2.1), namely the kernel of the cup-product. Assume that $i \geq 2$, that $V_1, \dots, V_i$ have been defined, let $V^i = V_1 \oplus \dots \oplus V_i$, and let $\mathcal{M}_i = \Lambda(V^i)$. Assume that the differential $d: \mathcal{M}_i \rightarrow \mathcal{M}_i$ has also been defined and that it commutes with the action of the multiplicative group k^* on $\Lambda(V^i)$ by the algebra automorphisms defined by letting $\lambda \in k^*$ act on V_j as multiplication by λ^j . If W is a vector space with a k^* action by semisimple automorphisms, let $W(j)$ denote the vector subspace where λ acts by multiplication by λ^j . Then define

$$V_{i+1} = \Lambda^2(V^i)(i+1) \cap \ker(d),$$

$\mathcal{M}_{i+1} = \Lambda(V^{i+1})$, and the differential d on $\mathcal{M}_{i+1}$ to be the Hirsch extension of the differential algebra $\mathcal{M}_i$ obtained by defining $d|_{V_{i+1}}$ to be the inclusion of V_{i+1} in $\Lambda^2(V^i)$.

This definition is clearly correct because, by the inductive assumption of a k^* action, we have a decomposition

$$H^2(\mathcal{M}_i) = \bigoplus_{j=2}^{2i} H^2(\mathcal{M}_i)(j).$$

We are inductively arranging $H^2(\mathcal{M}_i)(j) = 0$ for $2 < j \leq i$ to achieve in the limit that $H^2(\mathcal{M})(j) = 0$ for $j \neq 2$, which is equivalent to 1-formality. Now V_{i+1} is isomorphic to $H^2(\mathcal{M}_i)(i+1)$, and d is defined precisely to then make $H^2(\mathcal{M}_{i+1})(j) = 0$ for $2 < j \leq i+1$.

The reason we choose this procedure is that the k^* action is easily defined. Another correct procedure, closer to the definition of 1-formality, would have been to have a k^* action on V_i with more weights, and to define V_{i+1} to be $\bigoplus_{j>2} H^2(\mathcal{M}_i)(j)$.

With our chosen procedure, $\mathcal{M}_i$ is 1-formal if and only if $H^2(\mathcal{M}_i)(j) = 0$ for all $j > 2$. Note that this is in general a stronger condition than the vanishing of V_{i+1} .

The relation of the chosen procedure to Section 2 and the error there is now easily explained. The space C_i defined by (2.3) is the subspace $V_{i+2} \cap (V_1 \otimes V_{i+1})$ of

$$\Lambda^2(V^{i+1}) = V_1 \otimes V_{i+1} \oplus V_2 \otimes V_i \oplus \cdots,$$

which in general is a proper subspace of V_{i+2} .

Fortunately, this error leaves all but one assertion in the paper unaffected. The reason is that for small i there is little difference between the correct and incorrect procedures. Namely, we have the following assertions:

Lemma 0.1. *For any V_1, V_2 and injective $d: V_2 \rightarrow \Lambda^2 V_1$, if V_i and $\mathcal{M}_i$ are defined inductively as above, then*

1. $H^2(\mathcal{M}_2)(4) = 0$.
2. $H^2(\mathcal{M}_3)(5) = H^2(\mathcal{M}_3)(6) = 0$.
3. $H^2(\mathcal{M}_4)(7) = H^2(\mathcal{M}_4)(8) = 0$.
4. $V_3 = C_1$, where C_1 is defined in (2.2).

5. *If $\dim(V_3) = 1$ and $d(V_2)$ contains no decomposable element of $\Lambda^2 V_1$, then $V_4 \cong C_2$, where C_2 is defined as in (2.3). In fact, under these assumptions $V_4 = 0$ unless $V_3 \subset V_1 \otimes V_2$ consists of decomposable elements, in which case $\dim(V_4) = \dim(C_2) = 1$.*

Proof. We prove typical assertions 2 and 5. To prove 2, note that

$$H^2(\mathcal{M}_3)(5) = \ker(V_2 \otimes V_3 \rightarrow V_1 \otimes \Lambda^2 V_2 \oplus \Lambda^2 V_1 \otimes V_3)$$

and the projection of this map to the second summand in the target is simply

$$d \otimes 1: V_2 \otimes V_3 \rightarrow \Lambda^2 V_1 \otimes V_3,$$

which is injective since $d: V_2 \rightarrow \Lambda^2 V_1$ is injective by definition. Thus $H^2(\mathcal{M}_3)(5) = 0$. Similarly

$$H^2(\mathcal{M}_3)(6) = \ker(\Lambda^2 V_3 \rightarrow V_1 \otimes V_2 \otimes V_3 \oplus \Lambda^3 V_2),$$

which vanishes because the projection of this map to the first summand in the target is the composition of the inclusion $\Lambda^2 V_3 \rightarrow V_3 \otimes V_3$ of skew-symmetric 2-tensors, followed by the inclusion $V_3 \otimes V_3 \rightarrow (V_1 \otimes V_2) \otimes V_3$.

To prove 5, observe that

$$V_4 = \ker(V_1 \otimes V_3 \oplus \Lambda^2 V_2 \rightarrow \Lambda^2 V_1 \otimes V_2).$$

If $\dim(V_3) = 1$, then $V_3 \subset V_1 \otimes V_2$ has a basis element b of the form

$$b = \sum x_i \otimes c_i,$$

where the c_i form a basis for V_2 and the $x_i \in V_1$. The map in question is then

$$z = (v \otimes b, a) \rightarrow \sum (v \wedge x_i) \otimes c_i + a,$$

where $a \in \Lambda^2 V_2 \subset V_2 \otimes V_2$ is a skew-symmetric tensor. If z is in the kernel of this map, then

$$\sum (v \wedge x_i) \otimes c_i = -a \in V_2 \otimes V_2$$

is a skew-symmetric tensor. But this is only possible if

$$\sum (v \wedge x_i) \otimes c_i = 0 = a$$

because for each i , $v \wedge x_i$ is decomposable, and c_i is by assumption not decomposable. Thus $v \wedge x_i = 0$ for all i . There are two possibilities: either $v = 0$, in which case $C_2 = V_4 = 0$, or $v \neq 0$, in which case there is only one nonzero x_i , call it x , and v is a non-zero multiple of x , thus C_2 and V_4 are both one-dimensional with basis $x \otimes x \otimes c$, $(x \otimes x \otimes c, 0)$ respectively. Thus we see that $C_2 \cong V_4$ as asserted.

Corollary 0.2.

1. $\mathcal{M}_2$ is 1-formal if and only if $V_3 = 0$.
2. $\mathcal{M}_3$ is 1-formal if and only if $V_4 = 0$.
3. If $\dim(V_3) = 1$, $\mathcal{M}_3$ is 1-formal if and only if $C_2 = 0$.

From these assertions we see that the proofs of Example 2.6, Proposition 2.7 remain unchanged. In fact, the error affects only one assertion in the paper. Namely, the Lie algebra of Example 3.5 is not 3-step nilpotent, as asserted there. The reason is that the verification of assertion (3.2) of Lemma 3.4 is not sufficient for 3-step nilpotency (in this case $\dim(V_3) = 5$, so part 5 of the lemma does not apply). One needs to compute the kernel of the map

$$V_1 \otimes V_3 \oplus \Lambda^2(V_2) \rightarrow \Lambda^2(V_1) \otimes V_2,$$

which turns out to be one-dimensional. Using the notation $x_1, \dots, x_5, c_1, c_2, c_3$ of Lemmas 3.3 and 3.4, a basis for the one-dimensional space V_4 is the following. Let $\alpha \in V_1 \otimes V_3 \subset V_1 \otimes V_1 \otimes V_2$

be

$$2x_4 \otimes (x_1 \otimes c_1 + x_2 \otimes c_2) + 3x_2 \otimes (x_4 \otimes c_2 - x_2 \otimes c_3) \\ + x_5 \otimes (x_2 \otimes c_1 + x_5 \otimes c_2 + x_1 \otimes c_3) + x_1 \otimes (x_4 \otimes c_1 + x_3 \otimes c_2 + x_5 \otimes c_3),$$

and let $\beta = c_2 \wedge c_1 = c_2 \otimes c_1 - c_1 \otimes c_2 \in \Lambda^2 V_2 \subset V_2 \otimes V_2$. Then

$$(\alpha, \beta) \in V_1 \otimes V_3 \oplus \Lambda^2 V_2$$

is a basis for V_4 . We believe that further computation shows that the corresponding $\mathcal{M}_4$ is 1-formal, but this would take us too far afield.

Second correction. The statement near the end of Section 5, “This is the subgroup of smallest index in the integral Heisenberg group which we know to be a Kähler group,” is incorrect and misleading. The integral Heisenberg group itself is a Kähler group. This can be seen, for instance, by taking the first concrete example at the beginning of that paragraph, using a product of elliptic curves, with fundamental group the subgroup of the integral Heisenberg group (2.6) with $x_1, \dots, x_4$ even. Its subgroup of index two where z is also even is isomorphic to the integral Heisenberg group, and is of course a Kähler group since subgroups of finite index of Kähler groups are Kähler groups. This is a simple special case of the fact, pointed out to us by P. Deligne, that any two $\mathbb{Z}$ -forms of the same nilpotent graded $\mathbb{Q}$ -group can be embedded in each other as subgroups of finite index, therefore one is a Kähler group if and only if the other one is.

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