Addendum to: « Information Theoretic Interpretation of Forte's Entropy for the Grand Canonical Ensemble » (*).

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Inequality (5) in [2] established the following upper bound for the Shannon's entropy of an infinite discrete probability distribution $\{p_n: n \geq 0\}$

$$(1) \quad - \sum_{n=0}^{\infty} p_n \log p_n \leq \sum_{n=0}^{\infty} p_n n \log D + \log \{D/(D-1)\}.$$

ELIAS had given in [1] a different upper bound which in our notation is

$$(2) \quad - \sum_{n=0}^{\infty} p_n \log p_n \leq B'\{p_n\} := \sum_{n=0}^{\infty} p_n \log (n+1) + 2 + 2 \left\{ \sum_{n=0}^{\infty} p_n \log (n+1) \right\}^{\frac{1}{2}}$$

and which, however, holds on a proper subset $\Gamma' \subset \Gamma$ rather than on the whole Γ , where $\Gamma' := \{\{p_n\} \in \Gamma: p_0 \geq p_1 \geq \dots \geq p_n \geq \dots\}$.

It is natural to compare, on Γ' , the two bounds in order to find out whether one of them is better than the other one.

It should first be noticed that (2) is independent of the code used while (1) depends on it through the number D of alphabet letters.

Since the entropy series $-\sum p_n \log p_n$ is absolutely convergent, if it converges at all, one might easily be lead to think that the restriction to Γ' does not really matter (this is the analogue of the symmetry property for the Shannon's entropy of a finite discrete probability distribution). However, the order of the p_n 's is important when one deals with the average length $\langle n \rangle := \sum p_n n$; thus the restriction to Γ' , which is necessary if one wishes to compare (1) with (2), is an actual limitation. Attention will be restricted

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to binary codes ($D = 2$) so that (1) reads

$$-\sum_{n=0}^{\infty} p_n \log p_n \leq \sum_{n=0}^{\infty} p_n n + 1 =: B''\{p_n\}.$$

Now one has $B'\{p_n\} \leq B''\{p_n\}$ according as to whether

$$\begin{aligned} \sum_{n=0}^{\infty} p_n \log(n+1) + 1 + 2 \left\{ \sum_{n=0}^{\infty} p_n \log(n+1) \right\}^{\frac{1}{2}} = \\ = \left\{ \left[\sum_{n=0}^{\infty} p_n \log(n+1) \right]^{\frac{1}{2}} + 1 \right\}^2 \leq \sum_{n=0}^{\infty} p_n n, \quad (\{p_n\} \in \Gamma'), \end{aligned}$$

respectively. The following simple examples show that there are probability distributions in Γ' for which $B' > B''$ and other ones for which $B' < B''$.

EXAMPLE 1. Consider $p_0 = p_1 = \frac{1}{2}$, $p_n = 0$ for $n \geq 2$; then

$$\left\{ \left[\sum p_n \log(n+1) \right]^{\frac{1}{2}} + 1 \right\}^2 = (1 + \sqrt{\frac{1}{2}})^2 \cong 2.91 > 0.5 = \sum p_n n,$$

so that $B' \cong 3.91 > 1.5 = B''$.

EXAMPLE 2. Let $p_0 = p_1 = \dots = p_{14} = 1/15$, $p_n = 0$ for $n \geq 15$; then

$$\left\{ \left[\sum p_n \log(n+1) \right]^{\frac{1}{2}} + 1 \right\}^2 < 6.96 < 7 = \sum p_n n,$$

so that $B' < 7.96 < 8 = \sum p_n n$.

Thus one ought to write, for lack of a uniformly better bound,

$$-\sum_{n=0}^{\infty} p_n \log p_n \leq \min \left\{ B'\{p_n\}, \sum_{n=0}^{\infty} p_n n \log D + \log(D/(D-1)) \right\}$$

for every distribution $\{p_n\} \in \Gamma'$, while inequality (1) alone stands on all of Γ .

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REFERENCES

- [1] P. ELIAS, *Universal Codeword Sets and Representations of Integers*, IEEE Trans. Inform. Theory IT-21 (1975), pp. 194-203.
- [2] C. SEMPI, *Information Theoretic Interpretation of Forte's Entropy for the Grand Canonical Ensemble*, Ann. Univ. Ferrara, sez. VII (N.S.) 29 (1983), pp. 81-86.