Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 1, February 1994, pp. 49–56.


# Non-surjectivity of the Clifford invariant map

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Dedicated to the memory of Professor K G Ramanathan

**Abstract.** The question whether there exists a commutative ring $A$ for which there is an element in the 2-torsion of the Brauer group not represented by a Clifford algebra was raised by Alex Hahn. Such an example is constructed in this paper and is arrived at using certain results of Parimala–Sridharan and Parimala–Scharlau which are also reviewed here.

**Keywords.** Quadratic forms; Clifford invariant; Brauer group; canonical class.

## 1. Introduction

A celebrated theorem of Merkurjev [M] asserts that if $k$ is a field, every element in the 2-torsion of the Brauer group of $k$ is represented by the Clifford algebra of a quadratic form over $k$. The following question was raised by Alex Hahn: Does there exist a commutative ring $A$ and an element in the 2-torsion of the Brauer group of $A$ which cannot be represented by the Clifford algebra of a quadratic form over $A$? Examples of smooth projective curves $X$ over local fields for which the Clifford algebra classes do not fill up the 2-torsion in the Brauer group of $X$ were given in [P–Sr]. These were arrived at in the following manner: While comparing the graded Witt ring of a curve with graded unramified cohomology ring, necessary and sufficient conditions were given in [P–Sr], under which the 'Clifford invariant' map surjects on to the 2-torsion in the Brauer group, for a smooth projective curve over a local field. In a joint work with Scharlau [P–S], the above mentioned conditions were shown to be equivalent to the condition that the canonical class of the curve is "even" for smooth projective hyperelliptic curves over local fields. An explicit condition was also given in this case as to when the canonical class is even. This leads to the requisite examples of smooth projective curves over local fields for which the Clifford invariant map is not surjective. In this paper we review these results and use them to construct an affine algebra $A$ for which the Clifford invariant map is not surjective, thus answering the question of Alex Hahn.

Throughout this paper, $k$ denotes a field of characteristic not 2.

## 2. The Clifford invariant map

Let $X$ be a smooth integral variety over field $k$. A *quadratic space* on $X$ is a locally free sheaf $\mathscr{E}$ on $X$ together with a self-dual isomorphism $q:\mathscr{E} \to \mathscr{E}^\vee = \mathrm{Hom}(\mathscr{E}, \mathcal{O}_X)$. If the rank of $\mathscr{E}$ is even, the Clifford algebra $\mathscr{C}(q)$ of the space $(\mathscr{E}, q)$ is a sheaf of Azumaya algebras on $X$ and its class in the Brauer group of $X$ is called *the Clifford invariant* of $(\mathscr{E}, q)$, denoted by $e_2(q)$ (the *second* invariant, the first being the discriminant). Let $W(X)$ denote the Witt group of $X$, namely the quotient of the Grothendieck group of quadratic spaces over $X$ under orthogonal sum modulo the subgroup generated by metabolic spaces [K]. Let $I_2(X)$ denote the subgroup of $W(X)$ generated by spaces of even rank and trivial discriminant. The Clifford invariant is well defined on Witt equivalence classes [Kn–Oj] and defines a homomorphism $e_2: I_2(X) \to {}_2Br(X), {}_2Br(X)$ denoting the 2-torsion subgroup of the Brauer group of $X$. If $X = \operatorname{Spec} k$, the theorem of Merkurjev mentioned earlier assures that $e_2$ is surjective. The next non-trivial case is that of a smooth integral curve $X$.

We recall from [P–Sr] some results concerning this question for curves over local fields. We look at the case when $k$ is a non-archimedean local field. Let $X$ be a smooth integral curve over $k$ and $X^{(1)}$ the set of closed points of $X$. We have an exact sequence

$$0 \to W(X) \xrightarrow{i} W(k(X)) \xrightarrow{(\delta_x)} \bigoplus_{x \in X^{(1)}} W(k(x)),$$

where $i$ is the restriction to the generic point and $\delta_x$: $W(k(X)) \to W(k(x))$ is a residue homomorphism defined with respect to a choice of the parameter for the discrete valuation corresponding to $x \in X^{(1)}$. The powers of the ideal $I(k(X))$ of even rank forms in $W(k(X))$ induces a filtration

$$I_n(X) = W(X) \cap I^n(k(X))$$

on $W(X)$. The above exact sequence respects this filtration and yields the following exact sequence

$$0 \to I_n(X) \to I^n(k(X)) \to \bigoplus_{x \in X^1} I^{n-1}(k(x)).$$

Since the cohomological dimension of $k$ is 2, by a theorem of [A–E–J], there exist well defined surjective homomorphisms $e_n: I^n(k(X)) \to H^n(k(X)$, with kernel precisely $I^{n+1}(k(X))$ (we note that $e_2$ is simply the Clifford invariant map). The same is true for $I^n(k(x))$ for $x \in X^{(1)}$. We have a homomorphism $\partial = (\partial_x): H^n(k(X)) \to \bigoplus_{x \in X^1} H^{n-1}(k(x))$ whose kernel is the unramified cohomology group $H^0(X, \mathscr{H}^n)$, where $\mathscr{H}^n$ denotes the Zariski sheaf associated with the presheaf $U \mapsto H^n_{et}(U, \mu_2)$ [B–O]. The following diagram of exact rows

$$\begin{array}{ccccccc} 0 & \to & I_n(X) & \to & I^n(k(X)) & \xrightarrow{(\delta x)} & \bigoplus_{x \in X^{(1)}} I^{n-1}(k(x)) \\ & & \downarrow e_n & & \downarrow e_n & & \downarrow (e_{n-1}) \\ 0 & \to & H^0(X, \mathscr{H}^n) & \to & H^n(k(X)) & \xrightarrow{(\partial_x)} & \bigoplus_{x \in X^{(1)}} H^{n-1}(k(x)) \end{array}$$

is commutative [$P_3$]. By [A–E–J], for $n \geq 4$, $I^n(k(X)) = 0$, so that $e_3: I^3(k(X)) \to H^3(k(x))$ is an isomorphism. Further, for $x \in X^{(1)}$, $e_2: I^2(k(x)) \to H^2(k(x))$ is also an isomorphism. This implies that $e_3$: $I_3(X) \to H^0(X, \mathscr{H}^3)$ is an isomorphism. Further, the cokernel of the map $I^3(k(X)) \xrightarrow{\delta} \bigoplus_{x\in X^{(1)}} I^2(k(x))$ is isomorphic to the cokernel of $\partial: H^3(k(X)) \to \bigoplus_{x\in X^{(1)}} H^2(k(x))$, which, by Bloch–Ogus theory [B–O], is isomorphic to $H^1(X, \mathscr{H}^3)$, which is a subgroup of $H^4_{et}(X, \mu_2)$. If $X$ is *not projective*, since $cd_2 k = 2$, an analysis of the Hochschild–Serre spectral sequence yields that $H^4_{et}(X, \mu_2) = 0$, so that the map $\delta: I^3(k(X)) \to \bigoplus_{x\in X^{(1)}} I^2(k(x))$ is surjective. The following commutative diagram of exact rows and columns

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I_3(X) & \to & I^3(k(X)) & \to & \bigoplus_{x\in X^1} I^2(k(x)) \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I_2(X) & \to & I^2(k(X)) & \to & \bigoplus_{x\in X^1} I(k(x)) \\
 & & \downarrow e_2 & & \downarrow e_2 & & \downarrow (e_1) \\
0 & \to & H^0(X, \mathscr{H}^2) & \to & H^2(k(X)) & \to & \bigoplus_{x\in X^1} H^1(k(x)) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}
$$

for an affine curve $X$ over $k$, shows that $e_2: I_2(X) \to H^0(X, \mathscr{H}^2)$ is surjective. By purity theorem ([Gr], Prop. 2.1), $H^0(X, \mathscr{H}^2) \simeq {}_2Br(X)$ and $e_2$ is the Clifford invariant map. We thus have proved the following

**Theorem 1.** ([*P–Sr*], *Th.* 4.4) *If $X$ is an affine curve over a local field, the Clifford invariant map $e_2: I_2(X) \to {}_2\mathrm{Br}(X)$ is surjective.*

Suppose now that $X$ is a projective curve which has a $k$-rational point. Let $x_0 \in X(k)$. By ([A]) Satz. 4.16), we have a complex

$$H^i(k(X)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} H^{i-1}(k(x)) \xrightarrow{\text{cores}} H^{i-1}(k).$$

If $\alpha \in H^i(k(X))$ is such that $\partial(\alpha)$ has at most one non-zero component at $x_0$, then $\partial(\alpha) = 0$ since cores: $H^{i-1}(k(x_0)) \to H^{i-1}(k)$ is an isomorphism. Thus. if $Y = X \setminus \{x_0\}$, $H^0(X, \mathscr{H}^i) \simeq H^0(Y, \mathscr{H}^i)$ for all $i$. In particular, ${}_2\mathrm{Br}(X) \simeq {}_2\mathrm{Br}(Y)$. In the following commutative diagram

$$
\begin{array}{ccc}
I_3(X) & \simeq & H^0(X, \mathscr{H}^3) \\
\downarrow & & \downarrow \\
I_3(Y) & \simeq & H^0(Y, \mathscr{H}^3)
\end{array}
$$

where the vertical arrows are induced by the inclusion $Y \subset X$, all arrows, except possibly the left vertical arrow, are isomorphisms and hence the restriction map $I_3(X) \to I_3(Y)$ is an isomorphism. Finally, we look at the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & I_3(X) & \rightarrow & I_2(X) & \xrightarrow{e_2} & {}_2\mathrm{Br}(X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & I_3(Y) & \rightarrow & I_2(Y) & \xrightarrow{e_2} & {}_2\mathrm{Br}(Y) \rightarrow 0. \end{array}$$

The map $e_2 : I_2(X) \rightarrow {}_2\mathrm{Br}(X)$ is surjective if and only if the map $I_2(X) \rightarrow I_2(Y)$ is surjective. This leads to the following definition

DEFINITION

Let $X$ be a smooth projective curve over a field $k$. We say that $X$ has *extension property* (for quadratic spaces) if there exists a rational point $x_0 \in X(k)$ such that every quadratic space over $X \backslash x_0$ extends to $X$.

**Theorem 2.** ([P–Sr], Th. 4.4). *Let $X$ be a smooth projective curve over a local field with a rational point. The Clifford invariant map $e_2 : I_2(X) \rightarrow {}_2\mathrm{Br}(X)$ is surjective if and only if $X$ has extension property.*

*Proof.* Let $x_0 \in X(k)$ and $Y = X \backslash \{x_0\}$. We need only to verify that if the map $I_2(X) \rightarrow I_2(Y)$ is surjective then every quadratic space on $Y$ extends to $X$. Suppose $I_2(X) \rightarrow I_2(Y)$ is surjective. Let $q$ be any quadratic space on $Y$. We check that the second residue $\delta_{x_0}(q) = 0$. If rank $q$ is odd, we replace $q$ by $q \perp \langle 1 \rangle$ and assume that rank $q$ is even. The space disc $q \in H^1(Y, \mu_2) = H^1(X, \mu_2)$, so that disc $q$ is nonsingular on $X$. Replacing $q$ by $q \perp \langle -1, \text{disc } q \rangle$ which has the same residue as $q$ at any point, we assume that disc $q$ is trivial so that $q \in I_2(Y)$. Then by assumption, $q$ extends to $X$. □

*Remark* 1. The extension property for a smooth projective curve $X$ over any field could be defined as above with respect to a given rational point $x_0$. It is interesting to study the equivalence classes on $X(k)$ defined by $x \sim y$ if and only if $X$ has extension property 'with respect to $x$' is equivalent to $X$ has extension property 'with respect to $y$'. The theorem implies that for a smooth projective curve over a local field there is only one equivalence class: i.e., the extension property defined with respect to $x$ does not depend on the choice of $x$.

*Remark* 2. It may be shown that the map $e_2 : I_2(X) \rightarrow {}_2\mathrm{Br}(X)$ is surjective if and only if every $\xi \in {}_2\mathrm{Br}(X)$ is the class of a Clifford algebra of some even rank quadratic space over $X$. In fact if $\xi \in {}_2\mathrm{Br}(X)$ is such that $\xi = C(\mathscr{E}, q)$ with rank $q$ even, $\xi = e_2(q')$ where $q' = q \perp \langle -1, \text{disc } q \rangle \in I_2(X)$.

## 3. Canonical class of a curve and extension property for quadratic spaces

We now go on to analyse when the extension property holds for a smooth projective curve $X$. In view of a theorem of Geyer–Harder–Knebusch–Scharlau [G–H–K–S], a sufficient condition for the extension property to hold for $X$ is that the canonical line bundle $\Omega_X$ on $X$ even; i.e., $\Omega_X$ is a square in $\mathrm{Pic}\,X$. This is due to a certain reciprocity for quadratic spaces on $X$ if $\Omega_X$ is even. We explain this reciprocity.

Let $X$ be a smooth projective curve over a field $k$ with char $k \neq 2$ and $k$ perfect. Let $\Omega_X = \Omega$ be the canonical sheaf on $X$. We can define the Witt group $W(X,\Omega)$ of quadratic spaces on $X$ with values in the line bundle $\Omega$. There are canonical residue homomorphisms

$$\partial_x \colon W(k(X),\Omega_{k(X)}) \to W(k(x))$$

for each closed point $x$ of $X$. Any non-singular quadratic form $q$ over $k(X)$ with values in $\Omega_{k(X)}$ may be written as

$$q = q_1\, \mathrm{d}\pi \perp q_2 \left(\frac{\mathrm{d}\pi}{\pi}\right).$$

with $q_1$ and $q_2$ regular over $\mathcal{O}_{X,x}$ and $\pi$ a parameter at $x$. Then $\partial_x(q)$ is the reduction of $q_2$ modulo $\pi$, which is independent of the choice of $\pi$. It is proved in [G–H–K–S] that the sequence

$$W(k(X),\Omega_{k(X)}) \xrightarrow{(\partial x)} \bigoplus_{x \in X^{(1)}} W(k(x)) \xrightarrow{s} W(k)$$

is a complex, $s$ being the transfer induced by the trace map. If $\Omega_X$ is a square in $\mathrm{Pic}X$, we have isomorphisms of the following complexes

$$\begin{array}{ccccccccc}
0 & \to & W(X,\Omega_X) & \to & W(k(X),\Omega_{k(X)}) & \xrightarrow{(\partial_x)} & \bigoplus\limits_{x \in X^1} W(k(x)) & \xrightarrow{s} & W(k) \\
 & & z\downarrow & & z\downarrow & & z\downarrow & & \\
0 & \to & W(X) & \to & W(k(X)) & \xrightarrow{(\delta_x)} & \bigoplus\limits_{x \in X^1} W(k(x)) & &
\end{array}$$

for $\delta_x$ defined through certain choice of parameters at $x, x \in X^{(1)}$[$P_2$]. In particular, if a form $q \in W(k(X))$ has non-zero residue at possibly one rational point $x_0$, then the residue at this rational point is necessarily zero. Hence $q$ on $X \backslash x_0$ extends to $X$. Thus if $\Omega_X$ is even, $X$ has extension property.

One is led into analysing when $\Omega_X$ is even. This is purely a rationality question, since over the algebraic closure of $k$, degree $\Omega_X$ is even; $\mathrm{Pic}^0 X$ being divisible, $\Omega_{X_{\bar{k}}}$ is a square. In particular, there is extension property for curves over algebraically closed fields. In [P–S], a necessary and sufficient condition was given as to when $\Omega_X$ is even for hyperelliptic curves over any field (see also [Su]). It so happens that for smooth projective hyperelliptic curves over *local* fields, extension property is *equivalent* to $\Omega_X$ being even.

**Theorem 3.** ([*P–S*], *Th.* 2.4). *Let $k$ be a local field with char $k \neq 2$ and $X$ a smooth projective hyperelliptic curve of genus at least two with $X(k) \neq \phi$. Then the following are equivalent*:

(1) $\Omega_X$ is even.
(2) $X$ has extension property.

(3) genus $X$ is odd or genus $X$ is even and $X$ satisfies one of the following for a double covering $\pi: X \to \mathbf{P}^1$,
(a) $\pi$ has a ramification point of odd degree.
(b) All ramification points of $\pi$ have even degree and there is a quadratic extension of $k$ which is contained in the residue fields of all ramification points of $\pi$.

*Remark* 3. The conditions (a) and (b) of (3) are intrinsic for $X$ since the covering $\pi: X \to \mathbf{P}^1$ is unique up to isomorphism, genus of $X$ being at least 2. Condition (3) (b) as stated in ([P–S]) includes a further condition, namely that for some choice of a rational point $\infty$ for $\mathbf{P}^1$, if $\infty$ is inert for $\pi$ and $k(\infty) = k(\sqrt{\eta})$, then $\eta$ is a norm from $\ell$. However, in our situation, $X(k) \neq \phi$ and we may choose $\infty$ to be lying below some rational point of $X$ so that $\infty$ is split and the extra condition is vacuous.

Now it is clear as to how to construct examples of curves $X$ over a local field $k$ such that $\Omega_X$ is not even. Let $k = \mathbb{Q}_3$, $p_1(t)$ an irreducible polynomial of degree 2 and $p_2(t)$ an irreducible polynomial of degree 4 over $\mathbb{Q}_3$ such that $\mathbb{Q}_3[t]/p_1(t)$ is totally ramified and $\mathbb{Q}_3[t]/(p_2(t))$ is unramified. (e.g. $p_1(t) = t^2 - 3$, $p_2(t) = t^4 + t^3 + t^2 + t + 1$). The hyperelliptic curve $y^2 = p_1(t)p_2(t)$ of genus 2 has two points of ramification. Clearly the residue fields at these points do not have a common quadratic extension since one is unramified and the other totally ramified. Hence by the above theorem, $\Omega_X$ is not even. Further there are choices for $p_1(t)$ and $p_2(t)$ (e.g., the example above) for which $X(k) \neq \phi$. One can even construct explicitly, a quadratic space over $X\setminus$ a rational point, which does not extend to $X$.

## 4. The example

In this section, we construct a commutative ring $A$ for which there is an element in ${}_2\mathrm{Br}(A)$ which is not the class of the Clifford algebra of any quadratic space over $A$.

We recall that a *locally trivial affine fibration* $\pi: Y \to X$ of schemes is one for which, locally, at each point $x \in X$, $\pi: Y \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is isomorphic to the projection $\mathbf{A}^r \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$.

**Theorem 4.** *Let $X$ be a smooth projective curve over a field $k$ and $\pi: \operatorname{Spec} A \to X$ a locally trivial affine fibration. If $\xi \in {}_2\mathrm{Br}(X)$ is not the Clifford invariant of any quadratic space on $X$, then $\pi^*\xi \in {}_2\mathrm{Br}(A)$ is not the Clifford invariant of any quadratic space over $A$.*

*Proof.* Suppose $q$ is a quadratic space of even rank over $A$ such that $\mathscr{C}(q)$ defines the class of $\pi^*\xi$. We may assume, by adding a hyperbolic plane, if necessary, that $q$ contains a hyperbolic plane. We may also assume, by adding $\langle -1, \operatorname{disc} q \rangle$, if necessary, that $[q] \in I_2(A)$ and $e_2(q) = \pi^*\xi$. Since the fibration $\pi: \operatorname{Spec} A \to X$ is locally trivial, for each closed point $x \in X^{(1)}$, $\pi: \operatorname{Spec} A \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \theta_{X,x}$ is given by $\mathcal{O}_{X,x} \subsetneq \mathcal{O}_{X,x}[T_1, \ldots T_r]$ where $T_i$ are indeterminates. Since $\mathcal{O}_{X,x}$ is a discrete valuation ring, in view of ([P$_1$], Th. 3.2), there exists a quadratic space $q_x$ over $\mathcal{O}_{X,x}$ such that $\pi^* q_x \xrightarrow{\sim} q|_{\operatorname{Spec} A \times_X \operatorname{Spec} \mathcal{O}_{X,x}}$. The spaces $q_x$ become isometric, at the generic point $x_0$ of $X$ to the space $q_0$ defined by $\pi^* q_0 \simeq q|_{\operatorname{Spec} A \times_X \operatorname{Spec} k(X)}$. (Observe that $\operatorname{Spec} A \times_X \operatorname{Spec} k(X) \simeq \operatorname{Spec} k(X)[T_1, T_2, \ldots, T_r]$ and $q$ restricted to $\operatorname{Spec} A \times_X \operatorname{Spec} k(X)$ comes from the space $q_0$ over $k(X)$ $q$ being isotropic ([Oj]). Through a typical dimension one argument, one sees that there is a quadratic space $q_1$ over $X$, which, when restricted to the generic point $x_0$ of $X$ becomes isometric to $q_0$. We note that disc $q_1 = 1$, since, locally, disc $q_1 \otimes \mathcal{O}_{X,x} = \operatorname{disc} \pi^*(q_1 \otimes \mathcal{O}_{X,x})$ is trivial, so that $q_1 \in I_2(X)$. We show that $e_2(q_1) = \xi$ which leads to a contradiction through our choice of $\xi$. Since $e_2$ is functorial and the map ${}_2\mathrm{Br}(X) \to {}_2\mathrm{Br}(k(X))$ is injective ([Gr], Prop. 2.1], it suffices to show that $e_2(q_{1k(x)}) = \xi_{k(X)}$. The map $\pi^*: \mathrm{Br}(k(X)) \to \mathrm{Br}(\operatorname{Spec} A \times_X \operatorname{Spec} k(X))$ is a (split) injection and hence it suffices to show that $\pi^*(e_2(q_{1_{k(X)}})) = \pi^*(\xi_{k(X)})$. We have

$$\begin{aligned}\pi^*(e_2(q_{1_{k(x)}})) &= \pi^* e_2(q_0)\\ &= e_2(\pi^* q_0)\\ &= e_2(q_{k(X)})\end{aligned}$$

However, by choice, $e_2(q) = \pi^*\zeta$ so that $e_2(q_{k(X)}) = \pi^*\xi_{k(X)}$. Thus $\pi^* e_2(q_{1_{k(X)}}) = \pi^*(\xi_{k(X)})$, leading to a contradiction.

Let $X$ be a smooth projective curve over a field $k$. We recall the construction of a locally trivial affine fibration $T: W \to X$ with $W$ affine, due to Jouanolou [J]. Let $j: X \hookrightarrow \mathbb{P}^r_k = \mathbf{P}$ be a closed immersion. Let $W(r)$ be the Stiefel variety over $k$ given by the equation $\{E^2 = E, \operatorname{Trace} E = 1\}$ where $E$ is the $(r+1) \times (r+1)$ generic matrix $(x_{ij})$ over $k$. Clearly $W(r)$ is affine and is a principal homogeneous space for the vector bundle Hom $(\mathscr{F}, \mathscr{L})$ where $\mathscr{L}$ is the canonical line bundle $\mathcal{O}(-1)$ on $\mathbf{P}$ and $\mathscr{F}$ is defined by the exact sequence

$$0 \to \mathscr{L} \to \mathcal{O}_{\mathbf{P}}^{r+1} \to \mathscr{F} \to 0.$$

The natural map $\pi$: $W(r) \to \mathbf{P}$, given by $E \mapsto \operatorname{image}(E)$ is a locally trivial affine fibration. We define $\pi$: $W \to X$ by the Cartesian square

$$\begin{array}{ccc} W & \to & W(r)\\ \pi\downarrow & & \downarrow\pi\\ X & \overset{j}{\hookrightarrow} & \mathbf{P}\end{array}$$

The map $\pi$: $W \to X$ is a locally trivial affine fibration with each fibre an affine $r$-space and $W$ is affine.

We now give the promised example. Let $X$ be the smooth projective hyperelliptic curve over $\mathbb{Q}_3$ defined by $y^2 = (t^2 - 3)(t^4 + t^3 + t^2 + t + 1)$. Let $\pi$:Spec $A \to X$ be an $\mathbf{A}^2$-fibration described above. Then $A$ is an affine algebra over $\mathbb{Q}_3$ of dimension 3. By the above Theorem and the discussion at the end of §2, there is a Brauer class in $A$ which is not the class of the Clifford algebra of any quadratic space over $A$.

*Remark* 4. In view of Theorem 1, for any affine curve over a local field, the Clifford invariant map is surjective. The following example, pointed out to us by Kapil Paranjape, gives a locally trivial $\mathbf{A}^1$-fibration of a smooth projective curve $C$ with total space affine. Let $Y$ be the complement of a non-constant section in $C \times \mathscr{P}^1$ and $\pi$: $Y \to C$ the projection. This once again leads to examples of affine surfaces over $p$-adic fields for which the Clifford invariant map is not surjective. For an arbitrary ground field $k$, there are even perhaps examples of affine curves for which the Clifford invariant map is not surjective.

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