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CORRECTION TO "PERTURBATIONS OF REGULARIZING MAXIMAL MONOTONE OPERATORS" AND A NOTE ON INJECTIVENESS

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ABSTRACT
Let A be a maximal monotone operator. Let u_f be the solution of $f(t) \in u'(t) + Au$. We investigate the injectiveness of the mapping $f \mapsto u_f$.

In this note we consider the quasiautonomous initial value problem

$$(1)\qquad \begin{aligned} f(t) \in u'(t) + Au(t) \qquad (a \leqq t < \infty),\\ x = u(a), \end{aligned}$$

where A is a maximal monotone operator in a Hilbert space $(H, |\ |)$. Here $x \in \overline{D(A)}$, $a \in \mathbf{R}$, and $f : \mathbf{R} \to H$ is assumed to be locally integrable. For any such A, x, a, f, it is well known that (1) has a unique solution $u_f : [a, \infty) \to \overline{D(A)}$. The solution depends continuously on the data, in the following sense:

$$(2)\qquad |u_f(t) - u_g(t)| \leqq |u_f(a) - u_g(a)| + \int_a^t |f(s) - g(s)|\, ds \qquad (a \leqq t < \infty)$$

(see lemma 3.1 in [1]). For later reference, we also note that

$$(3)\qquad |u_f(t) - u_g(t)|^2 \leqq |u_f(a) - u_g(a)|^2 + 2\int_a^t (f(s) - g(s), u_f(s) - u_g(s))\, ds$$

(see inequality (28) on page 65 of [1]). We assume the reader is familiar with the most basic properties of this class of operators and initial value problems. For a brief introduction to this subject see §6–10 of [4], or §II.1–II.4 and III.1–III.2 of [1].

In this paper we shall hold A fixed, and vary x, a, f, t. To display dependence on these data, we shall often denote the solution $u(t)$ of (1) by $u_f(t, a, x)$. Let $\mathcal{M}$

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be the set of mappings from $\{(t, a, x) \in \mathbf{R} \times \mathbf{R} \times \overline{D(A)} : t \geqq a\}$, into $\overline{D(A)}$. For each $f \in L^1_{\mathrm{loc}}(\mathbf{R}; H)$, the function u_f is an element of $\mathcal{M}$.

Inequality (2) shows that the mapping $f \mapsto u_f$ is continuous from $L^1_{\mathrm{loc}}(\mathbf{R}; H)$ with its usual topology, into $\mathcal{M}$ with the topology of uniform convergence on compact sets. Results in [5, 6] show that for certain classes of A's and f's, the mapping is still continuous if $L^1_{\mathrm{loc}}(\mathbf{R}; H)$ is given a weaker topology. For some classes of A's and f's, arguments in [6] and in example 2 of [5] show that the mapping $f \mapsto u_f$ is actually a homeomorphism.

Example 3 of [5] asserts that if $H = \mathbf{R}$ and

$$A(x) = \begin{cases} \operatorname{sign}(x) & \text{when } x \neq 0, \\ [-1, 1] & \text{when } x = 0, \end{cases} \tag{4}$$

then the mapping $f \mapsto u_f$ is not injective. That assertion is erroneous, as we shall see later in this note. The brief argument in example 3 of [5] actually shows only that the mapping $f \mapsto u_f(\cdot, 0, 0)$ is not injective. Clearly, this is a weaker conclusion. But there do indeed exist maximal monotone operators A for which the mapping $f \mapsto u_f$ is not injective. This fact will also follow from our main result, below.

THEOREM. *Let A be a maximal monotone operator in a real Hilbert space $(H, |\ \ |)$. Let $x_0 \in D(A)$. Let H_0 be the closed linear span of the set $D(A) - x_0$. Let P be the orthogonal projection of H onto the closed linear subspace H_0. Let $f_1, f_2 \in L^1_{\mathrm{loc}}(\mathbf{R}; H)$. Then $u_{f_1} = u_{f_2}$ if and only if $Pf_1(t) = Pf_2(t)$ for almost every $t \in \mathbf{R}$.*

REMARK. It is easy to show that H_0 depends only on the set $D(A)$, and not on the particular choice of the element x_0 in $D(A)$.

PROOF OF THE "IF" PART. Assume $Pf_1(s) = Pf_2(s)$ for almost all $s \in \mathbf{R}$. Since u_{f_1} and u_{f_2} take values in $\overline{D(A)}$, it follows easily that $u_{f_1} - u_{f_2}$ takes values in H_0. Hence $u_{f_1}(s) - u_{f_2}(s)$ is orthogonal to $f_1(s) - f_2(s)$ for almost all s. Now apply (3); it follows that $u_{f_1} = u_{f_2}$.

PROOF OF THE "ONLY IF" PART. Assume that

$$Pf_1(a) \neq Pf_2(a), \tag{5}$$

for all a in some set of positive measure. By the vector version of Lebesgue's Theorem [3, theorem III.12.8], almost every point $a \in \mathbf{R}$ is a Lebesgue point of f_1 and f_2; that is, a number such that

(6) $\quad f_j(a)$ is defined and $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_a^{a+h}|f_j(a)-f_j(s)|\,ds=0 \quad (j=1,2).$

Fix some $a\in\mathbf{R}$ satisfying both (5) and (6); it suffices to show that $u_{f_1}(\cdot,a,\cdot)\neq u_{f_2}(\cdot,a,\cdot)$.

Let $\xi=f_1(a)-f_2(a)$. It follows from (5) that (ξ,x_1-x_0) is nonzero for some $x_1\in D(A)$. Define the operator $A+\xi$ by taking $(A+\xi)(x)=A(x)+\xi$ for all $x\in D(A)$. We claim that the operators A and $A+\xi$ are not identical.

Indeed, suppose that $A=A+\xi$. Then $A=A+k\xi$ for all integers k. Since $x_j\in D(A)$, the set Ax_j is nonempty; let $y_j\in Ax_j$ for $j=0,1$. Then $y_j\in Ax_j+\theta_jk\xi$ if $\theta_j\in\{0,1\}$. That is, $[x_j,y_j-\theta_jk\xi]\in A$. Since A is monotone,

$$\begin{aligned}0&\leqq((y_1-\theta_1k\xi)-(y_0-\theta_0k\xi),x_1-x_0)\\&=(y_1-y_0,x_1-x_0)-k(\theta_1-\theta_0)(\xi,x_1-x_0).\end{aligned}$$

Holding x_0, x_1, y_0, y_1 fixed, choose θ_0 and θ_1 so that $\theta_1-\theta_0=\operatorname{sign}(\xi,x_1-x_0)$. Then take k very large; we obtain a contradiction. Thus A and $A+\xi$ are indeed distinct. That is, $A+f_1(a)$ and $A+f_2(a)$ are distinct.

For any constant $z\in H$, the operator $A+z$ is also maximal monotone, and so it generates a nonexpansive, strongly continuous semigroup S_z on $\overline{D(A+z)}=\overline{D(A)}$. The maximal monotone operators are in one-to-one correspondence with their semigroups ([1], theorems 3.1 and 4.1). Since $A+f_1(a)\neq A+f_2(a)$, it follows that $S_{f_1(a)}\neq S_{f_2(a)}$.

For $r>0$, let

$$\delta(r)\equiv\max_{j=1,2}\sup_{0<h\leqq r}\frac{1}{h}\int_a^{a+h}|f_j(a)-f_j(s)|\,ds.$$

Then $\delta(r)\downarrow 0$ as $r\downarrow 0$, by our choice of a satisfying (6). For any $x\in\overline{D(A)}$, note that $S_{f_j(a)}(h)x=u_{f_j(a)}(a+h,a,x)$; hence from (2) we have

$$|S_{f_j(a)}(h)x-u_{f_j}(a+h,a,x)|\leqq\int_a^{a+h}|f_j(a)-f_j(s)|\,ds\leqq h\delta(h)).$$

This inequality holds for $j=1,2$. Taking the difference of these two results, we obtain

$$|S_{f_1(a)}(h)x-S_{f_2(a)}(h)x|\leqq 2h\delta(h)\qquad\text{for all } x\in\overline{D(A)} \text{ and } h\geqq 0.$$

Now, let any $y\in\overline{D(A)}$ and $t\in(0,\infty)$ be given. Temporarily fix any positive integer n. For $j=0,1,2,\cdots,n$, let $y_j=S_{f_2(a)}(jt/n)y$. Then

$$\begin{aligned}
&| S_{f_1(a)}(t)y - S_{f_2(a)}(t)y | \\
&= \left| \sum_{j=1}^{n} [S_{f_1(a)}(t/n)^j S_{f_2(a)}(t/n)^{n-j} y - S_{f_1(a)}(t/n)^{j-1} S_{f_2(a)}(t/n)^{n-j+1} y] \right| \\
&\leqq \sum_{j=1}^{n} | S_{f_1(a)}(t/n) y_{n-j} - S_{f_2(a)}(t/n) y_{n-j} | \\
&\leqq \sum_{j=1}^{n} 2 \frac{t}{n} \delta\left(\frac{t}{n}\right) = 2t\delta\left(\frac{t}{n}\right)
\end{aligned}$$

which tends to 0 as $n \to \infty$. Thus $S_{f_1(a)}(t)y = S_{f_2(a)}(t)y$ for all $t > 0$ and $y \in \overline{D(A)}$, a contradiction. This completes the proof of the theorem.

COROLLARY. *Let A be a maximal monotone operator in a Hilbert space H. Let $x_0 \in D(A)$. Then the mapping $f \mapsto u_f$, from $L^1_{loc}(\mathbf{R}; H)$ into $\mathcal{M}$, is injective if and only if the closed linear span of the set $D(A) - x_0$ is all of H.*

REMARKS ON SOME CONSEQUENCES

(1) If A is defined as in (4), then $D(A) = \mathbf{R}$. Hence, by the corollary above, the mapping $f \mapsto u_f$ is injective from $L^1_{loc}(\mathbf{R})$ into $\mathcal{M}$.

(2) We easily construct examples in which the mapping $f \mapsto u_f$ is not injective. For instance, suppose A is a maximal monotone operator in a Hilbert space H, and $\tilde{H}$ is another Hilbert space. Let $B(x) = A(x) \oplus \tilde{H}$ for all $x \in D(A)$. Then B is easily seen to be maximal monotone in $H \oplus \tilde{H}$, with $D(B) = D(A) \subseteq H \subseteq H \oplus \tilde{H}$. If $\tilde{H}$ is not the trivial space $\{0\}$, then H is a proper subspace of $H \oplus \tilde{H}$; and so by the corollary above, the mapping $f \mapsto u_f$ for B is not injective.

SOME QUESTIONS FOR FURTHER STUDY

(1) Suppose $H = H_0$, so that $f \mapsto u_f$ is injective. What topologies on $\mathcal{M}$ and $L^1_{loc}(\mathbf{R}; H)$ make the inverse map $u_f \mapsto f$ continuous? What topologies make it a homeomorphism?

(2) Suppose $H \neq H_0$. Then the map $Pf \mapsto u_{Pf}$ is injective. What topologies make its inverse continuous?

(3) The proof above used the fact that strongly continuous, nonexpansive semigroups on a closed convex subset C of a Hilbert space are in one-to-one correspondence with the maximal monotone operators A satisfying $\overline{D(A)} = C$; That fact seems to be stronger than is really needed for the proof. Indeed, the proof given above does not involve *all* maximal monotone operators A satisfying $\overline{D(A)} = C$; it only involves those which differ by a constant. Can a more direct proof be given, using less powerful tools and yielding more insight into the map $f \mapsto u_f$?

(4) To what extent does the theorem above extend to m-accretive operators in an arbitrary Banach space? It is not clear what should replace the notion of orthogonal projections in that setting. Also, m-accretive operators are *not* in one-to-one correspondence with their semigroups (pages 295–297 in [2]).

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