Discrete Comput Geom 15:353–357 (1996)

Discrete & Computational
Geometry

Nonrevisiting Paths on Surfaces

H. Pulapaka and A. Vince

Department of Mathematics, University of Florida,
Gainesville, FL 32611, USA
hpulapak@grits.valdosta.edu

Abstract. The nonrevisiting path conjecture for polytopes, which is equivalent to the Hirsch conjecture, is open. However, for surfaces, the nonrevisiting path conjecture is known to be true for polyhedral maps on the sphere, projective plane, torus, and a Klein bottle. Barnette has provided counterexamples on the orientable surface of genus 8 and nonorientable surface of genus 16. In this note the question is settled for all the remaining surface except the connected sum of three copies of the projective plane.

1. Introduction

One of the most well-known open problems in the combinatorial theory of polytopes is the Hirsch conjecture, which gives an upper bound on the diameter of the graph of a polytope. The *graph* of a polytope P is the 1-skeleton of P. More specifically, the Hirsch conjecture states that $\Delta(d, n) \leq n - d$, where $\Delta(d, n)$ is the maximum diameter among the graphs of d-dimensional polytopes with n facets. A *facet* is a $(d - 1)$-dimensional face. The Hirsch conjecture was formulated by Hirsch in 1957 and reported by Dantzig in his book *Linear Programming and Extensions* [5]. The conjecture has implications for the complexity of linear programming algorithms like the simplex method. Since the distance between two points on the graph of a polytope P is a lower bound on the number of iterations of an edge-following algorithm for an LP problem with feasible region P, the diameter $\Delta(d, n)$ gives the worst-possible complexity for the best-possible edge-following algorithm. A nice survey on the Hirsch conjecture is the paper by Klee and Kleindschmidt [9].

Equivalent to the Hirsch conjecture is the nonrevisiting path conjecture [8] of Klee and Wolfe. If p is a path in the graph of a polytope, a *revisit* of p to a face F is a pair of vertices (x, y) such that $p[x, y] \cap F = \{x, y\}$, where $p[x, y]$ is the path along p from x to y. In other words, p visits F at x, leaves F, and subsequently, revisits F at y.

Nonrevisiting Path Conjecture. *Any two vertices of a polytope P can be joined by a path that does not revisit any facet of P.*

The Nonrevisiting Path Conjecture is known to be true for three-dimensional polytopes [1] and is open in higher dimensions. Klee and Walkup [10] showed it to be false, in general, for unbounded polyhedra. Klee [7] has asked about the validity of the Nonrevisiting Path Conjecture for more general complexes. Since the underlying topological space of the boundary complex of a polytope is a sphere, it is natural to ask whether the conjecture is true for cell complexes whose underlying space is a sphere. In this regard, the conjecture is true for 2-spheres, but there is a counterexample due to Mani and Walkup [11] for the 3-sphere.

This note concerns the Nonrevisiting Path Conjecture for polyhedral maps. By a *surface* S we mean a connected, compact 2-manifold without boundary. These comprise the orientable surfaces T_g of genus g, which are the connected sums of g tori, and the nonorientable surfaces U_h, which are the connected sums of h projective planes. Let G be a graph embedded on a surface S. The closure of a connected component of $G\backslash S$ is called a *face*. If the faces are all simply connected and the intersection of any two distinct faces is either a common edge, common vertex, or empty, then $M = (G, S)$ is called a *polyhedral map*. Two distinct faces that satisfy the condition stated above are said to meet *properly*. A surface S has the *nonrevisiting path property* if, for any polyhedral map M on S and any two vertices x and y on M, there is a path joining x to y that does not revisit any face. Recent research has been directed toward the following question.

Question. Which surfaces possess the nonrevisiting path property?

The nonrevisiting property holds for the sphere [1], [8], projective plane [2], torus [3], and Klein bottle [6], [12]. However, Barnette [4] has recently provided counterexamples for T_8 and U_{16}. In this note we settle the question for all the remaining surfaces except U_3, the connected sum of three copies of the projective plane. This may also clarify a misconception [6] that the nonrevisiting path property holds for T_2, the two-hole torus.

Theorem. *The nonrevisiting path property holds for the sphere, torus, projective plane, and Klein bottle; it does not hold for all other surfaces except possibly U_3.*

2. The Counterexamples

The proof of the theorem stated in the Introduction requires the construction of counterexamples for all surfaces except the sphere, projective plane, torus, Klein bottle, and U_3. The first counterexample is a polyhedral map M on T_2. Figure 1 shows 16 faces. The polyhedral map M is obtained by identifying like labeled edges of these faces. In order to conclude that the result is indeed a polyhedral map, it must be verified that:

(1) The neighborhood of each vertex is homeomorphic to a disk (as opposed to, say, two disks pinched together at that vertex).

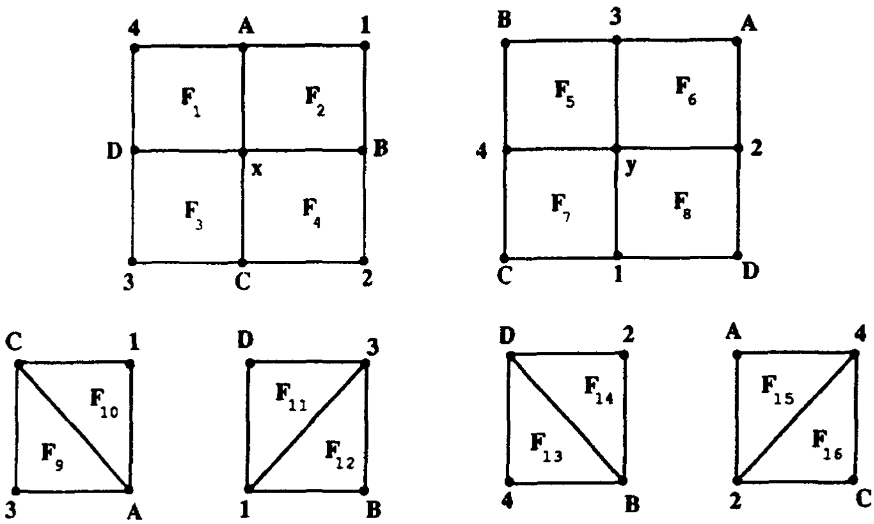

Fig. 1. Counterexample M on surface T_2.

(2) Pairs of distinct faces meet properly.
(3) The surface is T_2.

It is easy to check that the boundary cycles of the faces have a coherent orientation, i.e., an orientation such that, for each edge, the directions induced by the two incident faces are opposite. Thus the surface is orientable. Because the surface has 10 vertices, 28 edges, and 16 faces, the Euler characteristic is $\chi = 10 - 28 + 16 = -2$, which implies that the genus is $(2 - \chi)/2 = 2$. Therefore the surface is T_2. Conditions (1) and (2) above are easily checked since the example is small.

To show that M does not satisfy the nonrevisiting property, we prove that the vertices labeled x and y in Fig. 1 cannot be joined by a nonrevisiting path. Assume, by way of contradiction, that p is a nonrevisiting path joining x and y. Because of the symmetry of M there is no loss of generality in assuming that the vertex adjacent to x along p is the vertex labeled A. The path p has now left the faces labeled F_3 and F_4 and has visited the face F_6. Since p is assumed to be nonrevisiting and the vertex y lies on F_6, the remainder of p must also lie on the face F_6. There are two ways to get from A to y along F_6, via vertex 2 or via vertex 3. If p passes through vertex 2, then the face F_4 is revisited by p; if p passes through vertex 3, then the face F_3 is revisited. Either way leads to a contradiction.

In order to construct a counterexample for the surface T_g, $g \geq 3$, let $\overline{M}$ be a polyhedral map on the surface T_{g-2} such that $\overline{M}$ has a triangular face F_0. The connected sum $M\#\overline{M}$ of M and $\overline{M}$, formed by removing face F_0 from $\overline{M}$ and face F_9 of M and identifying the two triangular boundary cycles, is a map on the surface T_g. Moreover, any two faces of $M\#\overline{M}$ meet properly; otherwise if faces F in M and F' in $\overline{M}$ meet improperly, then either F and F_9 meet improperly on M or F' and F_0 meet improperly on $\overline{M}$. Both are impossible because M and $\overline{M}$ are polyhedral maps. The proof that x and y cannot be

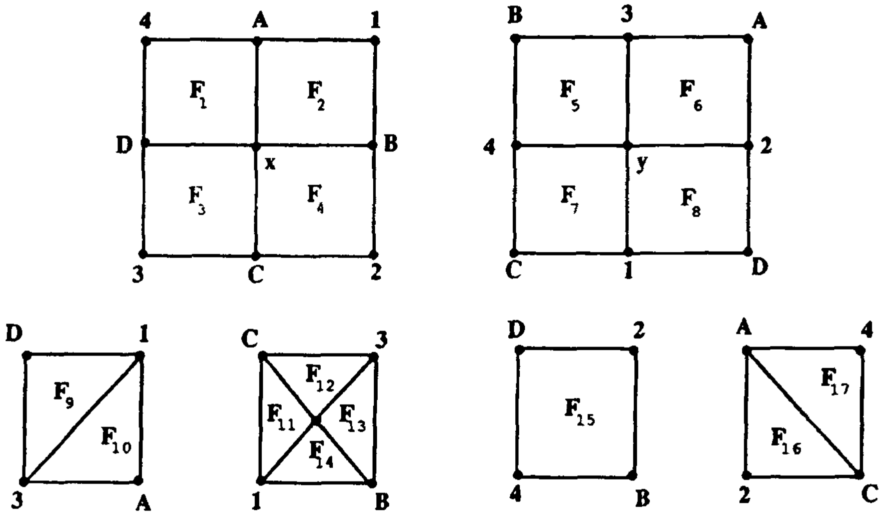

Fig. 2. Counterexample N on surface U_4.

joined by a nonrevisiting path in $M\#\overline{M}$ is identical to the proof for M alone. Thus the theorem is proved for the orientable surfaces.

For the nonorientable case consider the 17 faces in Fig. 2. As in the orientable case, identify edges with the same labels to obtain a polyhedral map N. It is easy to verify that the surface is nonorientable because there is no coherent orientation of the boundary cycles of the faces. The Euler characteristic is $11 - 30 + 17 = -2$, so the underlying surface is U_4, the connected sum of four projective planes (or, equivalently, two Klein bottles or a torus and Klein bottle). The proof that N does not satisfy the nonrevisiting path property is identical to the one given for M. Counterexamples for the surfaces U_h, $h \geq 5$, are also obtained in a manner similar to the orientable case, by taking the connected sum of N and a polyhedral map $\overline{N}$ on U_{h-4}.

References

1. D. W. Barnette, W_v paths on 3-polytopes, *J. Combin. Theory* **7** (1969), 62–70.
2. D. W. Barnette, W_v paths in the projective plane, *Discrete Math.* **62** (1986), 127–131.
3. D. W. Barnette, W_v Paths on the torus, *Discrete Comput. Geom.* **5** (1990), 603–608.
4. D. W. Barnette, A 2-manifold of genus eight without the W_v-property, *Geom. Dedicata* **46** (1993), 211–214.
5. G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, 1963.
6. E. Engelhardt, Some Problems on Paths in Graphs, Ph.D. Thesis, University of Washington, Seattle, WA, 1988.
7. V. Klee, Problem 19, *Proc. Colloquium on Convexity*, Copenhagen, 1965.
8. V. Klee, Paths on polyhedra I, *SIAM J.* **13** (1965), 946–956; Paths on polyhedra II, *Pacific J. Math.* **17** (1966), 249–262.
9. V. Klee and P. Kleindschmidt, The d-step conjecture and its relatives, *Math. Oper. Res.* **12**(4) (1987), 718–755.

10. V. Klee and D. Walkup, The d-step conjecture for polyhedra of dimension $d < 6$, *Acta Math.* **133** (1967), 595–598.
11. P. Mani and D. Walkup, A 3-sphere counterexample to the W_v-path conjecture, *Math. Oper. Res.* **5**(4) (1980), 595–598.
12. H. Pulapaka, Nonrevisiting Paths and Cycles on Polyhedral Maps, Ph.D. Thesis, University of Florida, Gainesville, FL, 1995.

Received February 2, 1995.