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Free Arrangements and Rhombic Tilings*

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Abstract. Let Z be a centrally symmetric polygon with integer side lengths. We answer the following two questions:

(1) When is the associated discriminantal hyperplane arrangement *free* in the sense of Saito and Terao?
(2) When are *all* of the tilings of Z by unit rhombi *coherent* in the sense of Billera and Sturmfels?

Surprisingly, the answers to these two questions are very similar. Furthermore, by means of an old result of MacMahon on plane partitions and some new results of Elnitsky on rhombic tilings, the answer to the first question helps to answer the second. These results then also give rise to some interesting geometric corollaries. Consideration of the discriminantal arrangements for some particular octagons leads to a previously announced counterexample to the conjecture by Saito [ER2] that the complexified complement of a real free arrangement is a $K(\pi, 1)$ space.

1. Introduction

We begin by reviewing some terminology and history of the relation between plane partitions, rhombic tilings, zonotopes, and hyperplane arrangements. A *plane partition* π is a two-dimensional array $(\pi_{i,j})_{i,j\geq 0}$ of nonnegative integers which weakly decreases

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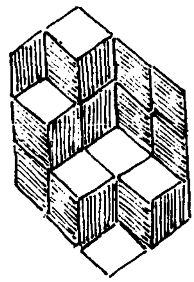


Fig. 1. Picture of a plane partition as a bunch of stacked cubes.

along rows and down columns, and has only finitely many nonzero entries, e.g.

$$\pi = \begin{matrix} 3 & 1 & 1 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

A plane partition π is said to *fit inside an* $r \times s \times t$ *box* if its first row has at most r nonzero entries, its first column has at most s nonzero entries, and all entries $\pi_{i,j}$ do not exceed t (see [Sta]). The terminology comes from picturing π as a set of unit cubes stacked $\pi_{i,j}$ high on the (i, j)-cell of the $r \times s$ floor inside an $r \times s \times t$ box. Figure 1 illustrates the partition π above inside a $3 \times 2 \times 3$ box.

The problem of counting how many plane partitions fit inside an $r \times s \times t$ box was solved by MacMahon who was further able to q-count them with $w(\pi) = \sum_{i,j} \pi_{i,j}$.

Theorem 1.1 [M].

$$\sum_{\pi} q^{w(\pi)} = \frac{H(r+s+t)H(r)H(s)H(t)}{H(r+s)H(r+t)H(s+t)},$$

where the sum runs over all plane partitions which fit inside an $r \times s \times t$ *box, and*

$$\begin{aligned} H(n) &= [n-1]!_q[n-2]!_q \cdots [2]!_q[1]!_q, \\ [n]!_q &= [n]_q[n-1]_q \cdots [2]_q[1]_q, \\ [n]_q &= 1 + q + q^2 + \cdots + q^{n-1}. \end{aligned}$$

A different point of view has been exploited more recently in [DT] and [K]. It was observed [DT] that the set of plane partitions fitting in an $r \times s \times t$ box is in bijection with the set of *rhombic tilings* of a centrally symmetric hexagon having three consecutive sides of length r, s and t. The bijection is given by viewing π as a stack of unit cubes inside the $r \times s \times t$ box from a vantage point which is far away from the origin in the direction $(1, 1, 1)$, so that the boundary of the box appears to be a hexagon, and the

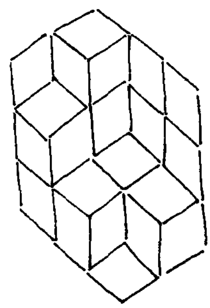


Fig. 2. Picture of the same plane partition as in Fig. 1 viewed as a rhombic tiling of a hexagon.

visible faces of the unit cubes are unit rhombi. For example, our earlier plane partition π corresponds to the rhombic tiling of a hexagon with side lengths 2, 3, and 3 in Fig. 2.

Rhombic tilings appear in a different connection in some recent work by Elnitsky on *reduced decompositions* of permutations [?]. He shows that counting rhombic tilings is equivalent to enumerating certain equivalence classes (called *C_1-equivalence classes*) of reduced decompositions of certain permutations. He also proves a formula (conjectured by Propp and Kuperberg) for the number of rhombic tilings of a centrally symmetric octagon with four consecutive sides of length r, s, 1, and 1. An example of such a tiling is shown in Fig. 3 for $r = 3$ and $s = 4$.

Furthermore, Elnitsky is able to q-count these tilings τ using a certain weight function $w(\tau)$ which is related to the rank function in the *higher Bruhat order of* Manin and Schechtmann [MS], [Zi1] which are described in Section 5 of this paper:

Theorem 1.2 [E1].

$$\sum_{\tau} q^{w(\tau)} = \frac{[2]_q[r+s+1]!_q[r+s+2]!_q}{[r]!_q[s]!_q[r+2]!_q[s+2]!_q},$$

where the sum runs over all unit rhombic tilings τ of a centrally symmetric octagon with four consecutive sides of lengths r, s, 1, and 1.

On the other hand, a certain subclass of rhombic tilings appears naturally as a very special case in the work of Billera and Sturmfels [BS] on fiber zonotopes. A polytope

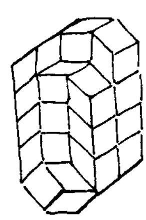


Fig. 3. Picture of a rhombic tiling of a (3, 4, 1, 1) octagon.

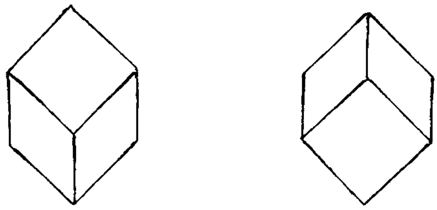


Fig. 4. The two coherent tilings of a (1, 1, 1) hexagon.

Z in $\mathbb{R}^d$ is a *zonotope* if it is the *Minkowski sum* of a generating multiset of vectors $V = \{\mathbf{v}_i\}_{i=1,\dots,n} \subset \mathbb{R}^d$, i.e.,

$$Z = \left\{ \sum_{i=1}^{n} c_i \mathbf{v}_i : 0 \le c_i \le 1 \text{ for all } 1 \le i \le n \right\}.$$

A two-dimensional zonotope is a centrally symmetric n-gon where n is the number of distinct vectors among the $\{\mathbf{v}_i\}$. A rhombic tiling of $Z \subset \mathbb{R}^2$ can be produced in the following way: Choose n generic values $\psi = (\psi_1, \psi_2, \dots, \psi_n)$ in $\mathbb{R}$, and "lift" the head of the vector $\mathbf{v}_i$ out of the plane by the height ψ_i, yielding a vector $\mathbf{v}_i$ in $\mathbb{R}^3$ which agrees with $\mathbf{v}_i$ in the first two coordinates, and has ψ_i as its third coordinate. Then form the zonotope Z' generated by $V' = \{\mathbf{v}'_i\}_{i=1,\dots,n}$. Project the "top" faces of Z' (i.e., the faces which are visible from far away on the positive x_3-axis) down onto Z in $\mathbb{R}^2$, and this gives a tiling of Z by rhombi. If a rhombic tiling of Z comes from such a choice of ψ and this projection process, it is called a *coherent* tiling of Z. Figure 4 shows the two coherent tilings of a centrally symmetric hexagon with unit sides.

Not all rhombic tilings of a two-dimensional zonotope Z are coherent, and it is an interesting problem in general to decide which tilings are coherent. The following is a special case of Corollary 4.2 of [BS].

Theorem 1.3. *Let Z be a zonotope in $\mathbb{R}^2$ with n generating vectors. There is an $(n-2)$-dimensional zonotope $F(Z)$ called its fiber zonotope, whose vertices are in bijection with the coherent rhombic tilings of Z.*

Warning. If two of the zonotope generators $\mathbf{v}_i$ and $\mathbf{v}_j$ are identical, then care must be taking in the above theorem to distinguish rhombi that come from projections of faces whose generators include $\mathbf{v}'_i$ and $\mathbf{v}'_j$, even though these faces will "look" identical. An example of the fiber zonotope $F(Z)$ for a centrally symmetric hexagon of side lengths 1, 2, and 1 is shown in Fig. 5.

Finally, we can explain how hyperplane arrangements enter the picture. The fiber zonotope $F(Z)$ is "dual" to an arrangement $\mathcal{A}_Z$ of hyperplanes in $\mathbb{R}^{n-2}$ in the sense that vertices of $F(Z)$ are in bijection to the *regions* of $\mathcal{A}_Z$ (here a region of $\mathcal{A}_Z$ means a

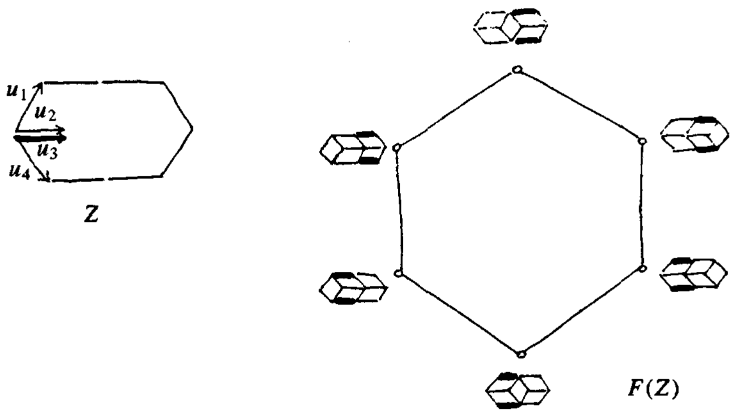


Fig. 5. The fiber zonotope $F(Z(1, 2, 1))$.

connected component of the complement of $\mathcal{A}_Z$ in $\mathbb{R}^{n-2}$). This hyperplane arrangement $\mathcal{A}_Z$ is also known as the *discriminantal arrangement* associated to the set V of vectors which generate Z (see [Ba]). The arrangement $\mathcal{A}_Z$ for the previous example is shown in Fig. 6.

The problem of counting the coherent rhombic tilings of the zonotope Z is then equivalent (bearing in mind the above warning) to counting the vertices of the fiber zonotope $F(Z)$, which is then equivalent to counting the regions of the discriminantal arrangement $\mathcal{A}_Z$. Counting regions in hyperplane arrangements is a well-studied problem (see [Za]), and has close connections to the theory of *free arrangements*. We refer the reader to the excellent book by Orlik and Terao [OT] on this subject for a definition of

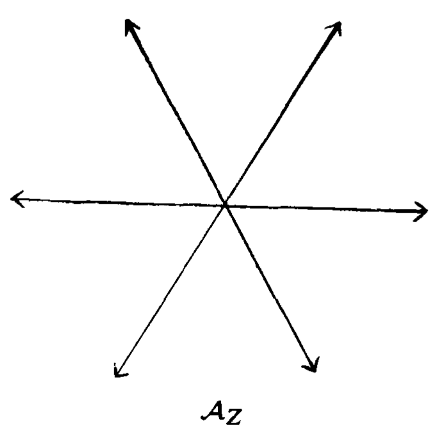


Fig. 6. The discriminantal arrangement $\mathcal{A}_{Z(1,2,1)}$.

a free arrangement and its *exponents*, but recall here the main application of freeness to counting regions:

Theorem 1.4 [OT, Theorems 2.86 and 4.137]. *Let $\mathcal{A}$ be a free arrangement of hyperplanes in $\mathbb{R}^d$ with exponents $\{e_1, e_2, \ldots, e_d\}$. Then the number of regions of $\mathcal{A}$ is*

$$\prod_{i=1}^{d}(1 + e_i).$$

With this background in mind, we now discuss the contents of this paper. Section 2 contains the main results of the paper (Theorems 2.5 and 2.6). Theorem 2.5 answers the question "When is the discriminantal arrangement $\mathcal{A}_Z$ free?", where Z is a two-dimensional zonotope whose zonotope generators are of unit length, i.e., a centrally symmetric polygon with integral side lengths. Theorem 2.6 answers the question "When are all the tilings of Z by unit rhombi coherent?"

The proof of Theorem 2.5 requires some lengthy *induction tables* to prove that certain arrangements are free, and we have included these in Section 3. Section 3 also contains a conjecture about the freeness of certain liftings of *Weyl arrangements* which is suggested by these induction tables.

Section 4 discusses two interesting one-parameter families of arrangements. One of these families is a previously announced counterexample to the 1975 conjecture of Saito that the complexified complement of a free arrangement is a $K(\pi, 1)$-space. The counterexample arises naturally in consideration of the discriminantal arrangements $\mathcal{A}_Z$ associated to octagons with four consecutive sides of lengths 2, 2, 1, and 1.

Section 5 discusses the connection of MacMahon and Elnitsky's q-counting results to the higher Bruhat orders of Manin and Schechtmann, and to the *weak order* [Ed] on the regions of the discriminantal arrangements $\mathcal{A}_Z$. From these connections we deduce an interesting consequence (Theorem 5.3) about the factorization of the rank-generating function for these weak orders in certain instances.

2. The Main Results

We begin with some definitions, and a few simple observations about two-dimensional zonotopes and their discriminantal arrangements. We frequently use the notation $[n] := \{1, 2, \ldots, n\}$ and $[a, b] := \{a, a+1, \ldots, b-1, b\}$ for integers $a < b$. For terminology and results on hyperplane arrangements the main reference is [OT].

Let $Z(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be a two-dimensional zonotope having unit length zonotope generators $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^2$, i.e., a centrally symmetric polygon in the plane with integral side lengths. If Z has its distinct zonotope generators the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_l$ with multiplicities $r_1, r_2, \ldots, r_l$, respectively, then by negating some of these vectors and renumbering, we may assume that each $\mathbf{u}_i$ points into the right half-plane, and that they are in clockwise order starting from the positive y-axis, as shown in Fig. 7.

In this case we say that Z is a $Z(r_1, r_2, \ldots, r_l)$ (ignoring the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_l$), which is equivalent to saying that one can find a clockwise consecutive sequence of half the sides of the polygon Z having lengths $r_1, r_2, \ldots, r_l$ in order. By an obvious bijection,

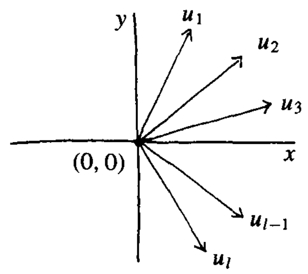


Fig. 7. The convention for ordering the zonotope generators.

any two $Z(r_1, r_2, \ldots, r_l)$'s have the same number of tilings by unit rhombi, hereafter simply called *tilings*.

If $Z = Z(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a two-dimensional zonotope, then the *discriminantal arrangement* $\mathcal{A}_Z$ lives inside a codimension 2 subspace of $\mathbb{R}^n$ and consists of the hyperplanes defined by the linear forms

$$l_S = \det(\mathbf{v}_{s_1}, \mathbf{v}_{s_2})x_{s_3} - \det(\mathbf{v}_{s_1}, \mathbf{v}_{s_3})x_{s_2} + \det(\mathbf{v}_{s_2}, \mathbf{v}_{s_3})x_{s_1}$$

as $S = \{s_1, s_2, s_3\}$ runs over all 3-subsets of $[n]$ (see Formula 5.3 of [BS]). Depending upon the number of coincidences among the three vectors $\{\mathbf{v}_{s_1}, \mathbf{v}_{s_2}, \mathbf{v}_{s_3}\}$, this formula either gives a defining form of 0 and is omitted, or a multiple of one of the forms $x_i - x_j$, or a multiple of the forms

$$ax_{s_1} + bx_{s_2} + cx_{s_3},$$

where

$$a\mathbf{v}_{s_1} + b\mathbf{v}_{s_2} + c\mathbf{v}_{s_3} = 0$$

is the unique (up to scalar multiple) linear dependence among the three distinct vectors $\mathbf{v}_{s_1}, \mathbf{v}_{s_2}, \mathbf{v}_{s_3}$. Since each of these linear forms is a dependence on the vectors $\{\mathbf{v}_i\}$, $\mathcal{A}_Z$ is actually an arrangement inside the subspace $\mathbb{R}^{n-2}$ given by the nullspace of the $2 \times n$ matrix having the $\mathbf{v}_i$'s as columns.

Given two sets of vectors $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ both in $\mathbb{R}^d$ we say that V is *projectively equivalent* to U if an invertible linear transformation $L: \mathbb{R}^d \to \mathbb{R}^d$ and a sequence of nonzero scalars $d_1, \ldots, d_n$ exist such that

$$L\mathbf{v}_i = d_i\mathbf{u}_i$$

for all $1 \leq i \leq n$. We say that two zonotopes are projectively equivalent if their generators are projectively equivalent. We also say that two hyperplane arrangements are projectively equivalent if their linear forms are projectively equivalent in the dual space.

Lemma 2.1. *If Z_1 and Z_2 are projectively equivalent two-dimensional zonotopes, then the discriminantal arrangements $\mathcal{A}_{Z_1}$ and $\mathcal{A}_{Z_2}$ are projectively equivalent as well.*

Proof. Suppose that Z_1 is generated by $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, Z_2 is generated by $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and

$$L\mathbf{v}_i = d_i\mathbf{u}_i$$

for all $1 \leq i \leq n$ where L is an invertible linear transformation and $\{d_i\}$ is a set of nonzero scalars. Then for a fixed 3-subset $S = \{s_1, s_2, s_3\} \subset [n]$ we have

$$l_S = \det(\mathbf{u}_{s_1}, \mathbf{u}_{s_2})x_{s_3} - \det(\mathbf{u}_{s_1}, \mathbf{u}_{s_3})x_{s_2} + \det(\mathbf{u}_{s_2}, \mathbf{u}_{s_3})x_{s_1}$$

is equal to

$$\begin{aligned} l_S = {} & \frac{1}{d_{s_1}d_{s_2}} \det L \det(\mathbf{v}_{s_1}, \mathbf{v}_{s_2})x_{s_3} - \frac{1}{d_{s_1}d_{s_3}} \det L \det(\mathbf{v}_{s_1}, \mathbf{v}_{s_3})x_{s_2} \\ & + \frac{1}{d_{s_2}d_{s_3}} \det L \det(\mathbf{v}_{s_2}, \mathbf{v}_{s_3})x_{s_1}. \end{aligned}$$

If we apply the linear transformation $x_i \mapsto (1/d_i)x_i$ and scale l_S by multiplying by $d_{s_1}d_{s_2}d_{s_3}(1/\det L)$ we get

$$\det(\mathbf{v}_{s_1}, \mathbf{v}_{s_2})x_{s_3} - \det(\mathbf{v}_{s_1}, \mathbf{v}_{s_3})x_{s_2} + \det(\mathbf{v}_{s_2}, \mathbf{v}_{s_3})x_{s_1}$$

and hence $\mathcal{A}_{Z_1}$ and $\mathcal{A}_{Z_2}$ are projectively equivalent. □

In fact, something stronger is true in the case where the two-dimensional zonotope is generated by only three distinct generators. The proof is left to the reader.

Lemma 2.2. *If Z_1 and Z_2 are two-dimensional zonotopes each of the form $Z(r_1, r_2, r_3)$, then $\mathcal{A}_{Z_1}$ is projectively equivalent to $\mathcal{A}_{Z_2}$ and both are projectively equivalent to the arrangement $\mathcal{A}_{r_1 \times r_2 \times r_3}$ defined by the linear forms*

$$\begin{aligned} &\{x_i - x_j \mid i < j, \{i, j\} \subseteq [1, r_1] \text{ or } [r_1 + 1, r_1 + r_2] \text{ or } [r_1 + r_2 + 1, r_1 + r_2 + r_3]\}, \\ &\{x_i + x_j + x_k \mid (i, j, k) \in [1, r_1] \times [r_1 + 1, r_1 + r_2] \times [r_1 + r_2 + 1, r_1 + r_2 + r_3]\}. \end{aligned}$$

Given a hyperplane arrangement $\mathcal{A}$ in a vector space V and a subspace X which is the intersection of some subset of its hyperplanes, the *localization* arrangement $\mathcal{A}_X$ is the arrangement in the quotient space V/X defined by

$$\{H/X\colon\ H \in \mathcal{A}, X \subseteq H\}.$$

The Localization Theorem (see Theorem 4.37 of [OT]) asserts that any localization $\mathcal{A}_X$ of a free arrangement $\mathcal{A}$ is free. Say that Z is a *subzonotope* of Z' if they have the same zonotope generators $\mathbf{u}_1, \ldots, \mathbf{u}_l$, and multiplicities $r_1, \ldots, r_l$ and $r'_1, \ldots, r'_l$, respectively, with $0 \leq r_m \leq r'_m$ for all m, i.e., some of the zonotope generators of Z' may not appear in Z. A fundamental fact in the proofs of the main results will be that subzonotopes of Z can form "obstructions" to both the freeness of $\mathcal{A}_Z$ and to the property of Z having all tilings coherent.

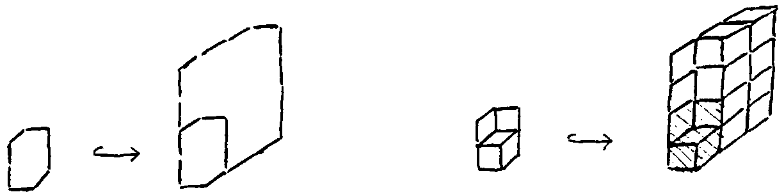


Fig. 8. Extending a tiling of Z to one of Z′.

Proposition 2.3. *If Z is a subzonotope of Z', then:*

(1) *$\mathcal{A}_Z$ is projectively equivalent to a localization of $\mathcal{A}_{Z'}$, and hence if $\mathcal{A}_Z$ is not free then $\mathcal{A}_{Z'}$ is not free.*
(2) *If Z has an incoherent tiling, then Z' has an incoherent tiling.*

Proof. (1) Assume the zonotopal generators (with multiplicities) of Z' are numbered $\mathbf{v}_1, \ldots, \mathbf{v}_{n'}$ for the purposes of computing the discriminantal arrangement, and that the n-subset N of $[n']$ gives the indices of those $\mathbf{v}_i$'s which generate Z. Then $\mathcal{A}_Z$ is projectively equivalent to the localization of $\mathcal{A}_{Z'}$ to the subspace X which is the intersection of all hyperplanes defined by l_S where S is a 3-subset of N.

(2) Given an incoherent tiling of Z, draw the outline of Z' around it, and extend this to a tiling of Z'. This can always be done and this tiling of Z' is easily seen to be incoherent. An example of this extension technique is shown in Fig. 8. □

We come now to the first main result, which characterizes when $\mathcal{A}_Z$ is a *free arrangement.* We refer the reader to Chapter 4 of [OT] for the definition of a free arrangement $\mathcal{A}$, the definition of the *exponents*, $\exp \mathcal{A}$, the *characteristic polynomial* $\chi(\mathcal{A}, t)$ of an arrangement, and for the Factorization Theorem of Terao.

Theorem 2.4 (Factorization Theorem [OT, 4.137]). *If $\mathcal{A}$ is a free arrangement in $\mathbb{R}^d$ with $\exp \mathcal{A} = (e_1, \ldots, e_d)$, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{d} (t - e_i).$$

Theorem 2.5. *Let Z be a two-dimensional zonotope as before. Then $\mathcal{A}_Z$ is free if and only if one of the following four cases holds, with the exponents $\exp(\mathcal{A}_Z)$ listed:*

(1) *Z is a $Z(r, s)$ parallelogram, and the exponents are*

$$\{1, 2, \ldots, r-1, 1, 2, \ldots, s-1\}.$$

(2) *Z is a $Z(r, s, t)$ hexagon in which either r, s, or t is at most 2 (up to projective equivalence, it may be assumed $t \leq 2$), and the exponents are*

$$\{1, 2, \ldots, r+s-1\} \qquad \textit{if} \quad t = 1,$$

$$\{1, r+1, r+2, \ldots, r+s-1, s+1, s+2, \ldots, r+s\} \qquad \textit{if} \quad t = 2.$$

(3) *Z is projectively equivalent to a $Z(r, s, 1, 1)$ octagon, and the exponents are*

$$\{1, r+2, r+3, \ldots, r+s, s+2, s+3, \ldots, r+s+1\}.$$

(4) *Z is projectively equivalent to a $Z(r, 1, s, 1)$ octagon in which the four distinct zonotopal generators $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ of Z are not projectively equivalent to*

$$(0, 1), (1, 1), (1, 0), (1, -1),$$

and the exponents are

$$\{1, r+2, r+3, \ldots, r+s, s+2, s+3, \ldots, r+s+1\}.$$

Proof. We first narrow the possible arrangements $\mathcal{A}_Z$ which can be free by showing that certain "minimal obstructions" are *not free*, and then applying Proposition 2.3(1). Let Z_1 be any $Z(1, 1, 1, 1, 1)$ decagon, Z_2 any $Z(2, 2, 2, 1)$ octagon, Z_3 the $Z(2, 1, 2, 1)$ octagon which is ruled out by case (4) in the theorem (i.e., generated by $(0, 1)$, $(1, 1)$, $(1, 0)$, and $(1, -1)$), and Z_4 any $Z(3, 3, 3)$ hexagon. It is not hard to compute what the associated discriminantal arrangements for these Z_1, Z_2, Z_3, Z_4 look like up to projective equivalence (although the answer sometimes depends on the choice of their generators $\{\mathbf{u}_i\}$). It is known [Zi3, Proposition 2.3] that if an arrangement is projectively equivalent to a free arrangement, then it is free as well. Thus it is enough to check the freeness of an arrangement up to projective equivalence. For each case we have computed the characteristic polynomial $\chi(\mathcal{A}_{Z_i})$ (using the PASCAL program "Matroid," available from the first author on request), and observed that they do *not* factor over the integers. This implies by the Factorization Theorem (Theorem 2.4) that the arrangements $\mathcal{A}_{Z_i}$ are not free:

$$\begin{aligned}
\chi(\mathcal{A}_{Z_1}, t) &= (t-1)(t^2 - 9t + 21), \\
\chi(\mathcal{A}_{Z_2}, t) &= (t-1)(t-5)(t-6)(t^2 - 11t + 32) \\
&\quad \text{if } \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \text{ are projectively equivalent} \\
&\quad \text{to } (0, 1), (1, 1), (1, 0), (1, -1), \\
\chi(\mathcal{A}_{Z_2}, t) &= (t-1)(t^4 - 22t^3 + 183t^2 - 686t + 992) \text{ otherwise}, \\
\chi(\mathcal{A}_{Z_3}, t) &= (t-1)(t-3)(t^2 - 10t + 26), \\
\chi(\mathcal{A}_{Z_4}, t) &= (t-1)(t-5)(t-7)(t^4 - 23t^3 + 200t^2 - 784t - 1188).
\end{aligned}$$

By Proposition 2.3(1), if $\mathcal{A}_Z$ is free, it cannot contain any of these Z_i's as a subzonotope. We use this to show that the only possibilities for Z with $\mathcal{A}_Z$ free are the ones listed in the theorem.

Z cannot have more than four distinct zonotope generators $\mathbf{u}_i$, or else it would contain Z_1. If Z has exactly four distinct zonotope generators $\mathbf{u}_1, \ldots, \mathbf{u}_4$, then at least two of them must have multiplicity $r_i = 1$, or else it would contain a zonotope projectively equivalent to Z_2. This implies Z must either be projectively equivalent to a $Z(a, b, 1, 1)$ which is on our list, or a $Z(a, 1, b, 1)$. In the latter case this $Z(a, 1, b, 1)$ must look like case (4) of the theorem, or else it would contain a Z_3. If Z has exactly three distinct zonotope generators, then one of them must have multiplicity at most 2, or else Z would contain a Z_4. Hence Z is on the list. Lastly, if Z has exactly two distinct zonotope generators, then it is a $Z(r, s)$, which is on the list. This shows that the only possibilities for $\mathcal{A}_Z$ to be free are the ones listed in the theorem, proving the forward implication.

To prove the reverse implication, namely that the arrangements $\mathcal{A}_Z$ listed are free with the stated exponents, we proceed case by case. In each case, up to projective equivalence, $\mathcal{A}_Z$ can be written in a convenient form, which is either readily identifiable as free, or can be proven free in Section 3 using the *induction table* technique (see p. 119 of [OT]).

If Z is a $Z(r, s)$ parallelogram, then $\mathcal{A}_Z$ is the arrangement defined by linear forms

$$\{x_i - x_j \mid i < j, \{i, j\} \subseteq [1, r] \text{ or } [r+1, r+s]\},$$

which is the *product* [OT, Definition 2.13] $A_{r-1} \times A_{s-1}$, where A_{r-1} is the *Coxeter arrangement* of type A. A_{r-1} is well known to be free with exponents $\{1, 2, \ldots, r-1\}$, and the product of two free arrangements is free, with exponents equal to the multiset union of exponents for the factors [OT, Proposition 4.28]. This agrees with the asserted exponents for $\mathcal{A}_Z$.

If Z is a $Z(r, s, t)$ hexagon, then $\mathcal{A}_Z$ is easily seen (Lemma 2.2) to be projectively equivalent to the arrangement we call $\mathcal{A}_{r\times s\times t}$, defined by linear forms

$$\{x_i - x_j \mid i < j, \{i, j\} \in [1, r] \text{ or } [r+1, r+s] \text{ or } [r+s+1, r+s+t]\},$$

$$\{x_i + x_j + x_k \mid (i, j, k) \in [1, r] \times [r+1, r+s] \times [r+s+1, r+s+t]\}.$$

If $t = 1$, then $\mathcal{A}_{r\times s\times 1}$ is projectively equivalent to the Coxeter arrangement A_{r+s}, whose exponents were just discussed and agree with those asserted by the theorem in this case. If $t = 2$, then $\mathcal{A}_{r\times s\times 2}$ is proven free with the asserted exponents by an induction table in Section 3.

If Z is a $Z(r, s, 1, 1)$ or $Z(r, 1, s, 1)$ octagon then, up to a projective equivalence in the plane $\mathbb{R}^2$, it may be assumed its four distinct zonotope generators $\mathbf{u}_i$ look like

$$(0, 1), (1, 1), (1, 0), (1, a),$$

where a is some strictly negative real number, and in the special $Z(r, 1, s, 1)$ case, (4) from the theorem, a cannot equal -1. In both of these cases, it is not hard to check that $\mathcal{A}_Z$ is projectively equivalent to the arrangement we call $\mathcal{A}_{r,s}^{(b)}$ defined by the linear forms

$$\{x_i\}_{i\in[r+s]} \cup \{x_i - x_j\}_{i<j\in[r+s]} \cup \{x_i - bx_j\}_{i\in[1,r], j\in[r+1,r+s]}$$

for some value of the parameter b. In the $Z(r, s, 1, 1)$ case, b is related to a by $b = 1 - a$, so that b is a real number strictly greater than 1. In the special $Z(r, 1, s, 1)$ case, (4) from the theorem, b is equal to a, so that b is a strictly negative real number *not* equal to -1. The arrangement $\mathcal{A}_{r,s}^{(b)}$ will be proven free with the asserted exponents whenever b is a real number not equal to 0, 1, or -1 by an induction table in Section 3. This completes the proof that $\mathcal{A}_Z$ is free with the asserted exponents in all of the cases of the theorem. □

The other main result characterizes when Z has the property that all of its tilings by unit rhombi are coherent.

Theorem 2.6. *Let Z be a two-dimensional zonotope as before. Then all tilings of Z by unit rhombi are coherent if and only if one of the following four cases holds (see Theorem 2.5):*

(1) *Z is a $Z(r, s)$ parallelogram.*
(2) *Z is a $Z(r, s, t)$ hexagon in which at least one of r, s, t is at most 2.*
(3) *Z is projectively equivalent to a $Z(r, s, 1, 1)$ octagon.*
(4) *Z is a $Z(1, 1, 1, 1, 1)$ decagon.*

Proof. As in the proof of the previous theorem, we first narrow the possible zonotopes Z for which all tilings can be coherent, by noting that certain previously studied "minimal obstructions" have been found to have incoherent tilings, and then apply Proposition 2.3(2). Let Z_1 be any $Z(1, 1, 1, 1, 1, 1)$ dodecagon, Z_2 any $Z(2, 1, 1, 1, 1)$ decagon, Z_3 any $Z(2, 1, 2, 1)$ octagon, and Z_4 any $Z(3, 3, 3)$ hexagon. For Z_1, it was observed by Sturmfels [Stu], [HG] that while Z_1 has exactly 908 rhombic tilings, the number of coherent tilings depends upon the choice of zonotope generators $\mathbf{u}_1, \ldots, \mathbf{u}_6$, and is either equal to 876, 880, 884, 888, or 892 depending upon five cases. (The existence of these 908 different tilings forms the basis of the puzzle **Hexa-Grid** [HG].) This analysis can be extended to Z_2 and Z_3. For Z_2 there are either 264 or 266 coherent tilings (depending on the choice of generators) out of a total of 268 tilings. For Z_3 there are either 74 or 75 coherent tilings out of a total of 76.

For any $Z(3, 3, 3)$ hexagon Z_4, it has been shown by Richter-Gebert [Ri] that of the 980 tilings, exactly four of them are incoherent, regardless of the choice of zonotope generators $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Figure 9 shows the four tilings which are incoherent for a particular $Z(3, 3, 3)$.

Richter-Gebert actually proves a much stronger result, characterizing exactly which tilings of a $Z(r, s, t)$ are coherent.

By proposition 2.3(2), if Z has all of its tilings coherent, it cannot contain any of these Z_i's as subzonotope. We use this to show that the only possibilities for Z with all tilings coherent are the ones listed in the theorem.

Z cannot have more than five distinct zonotope generators, or else it would contain Z_1 as a subzonotope. if Z has exactly five distinct zonotope generators, then all of the multiplicities r_i must equal 1, or else it would contain Z_2. Hence Z is on the list in this

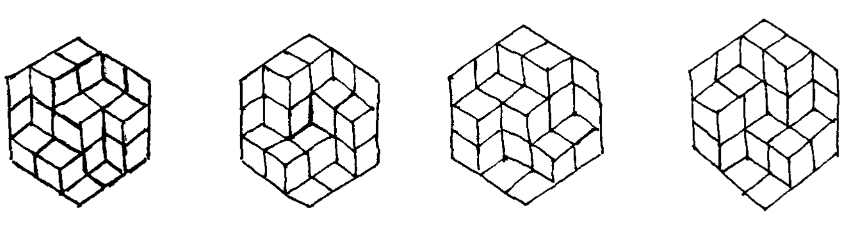


Fig. 9. The four incoherent tilings of a regular $Z(3, 3, 3)$ hexagon.

case. If Z has exactly four distinct zonotope generators, then it must be projectively equivalent to a $Z(r, s, 1, 1)$ octagon, or else it would contain Z_3. Hence Z is on the list in this case. If Z has exactly three distinct zonotope generators, then it must be a $Z(r, s, t)$ hexagon with at least one of r, s, or t at most 2, or else it would contain Z_4. Hence Z is on the list in this case. If Z has exactly two distinct zonotope generators, then it is a $Z(r, s)$ quadrilateral, and is on the list. This shows that the only possibilities for Z to have all tilings coherent are the ones listed in the theorem, proving the forward implication.

To prove the reverse implication, our main tool is the result of Zaslavsky [Za, Theorem A], [OT, Theorem 2.68] which states that the number of regions in an arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is $(-1)^d\chi(A, -1)$, where $\chi(\mathcal{A}, t)$ is the characteristic polynomial of $\mathcal{A}$. When the arrangement is free this is particularly simple to compute since the Factorization Theorem tells us that

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{d}(t - e_i)$$

so that

$$(-1)^d\chi(\mathcal{A}, -1) = \prod_{i=1}^{d}(1 + e_i).$$

By Theorem 1.3 and the Warning following it, if Z is a $Z(r_1, r_2, \dots, r_l)$, then each coherent tiling of Z gives rise to exactly $\prod_i r_i!$ vertices in the fiber zonotope $F(Z)$, which has as many vertices as there are regions of the discriminantal arrangement $\mathcal{A}_Z$. Therefore if we can calculate this characteristic polynomial, we conclude that the number of coherent tilings of Z is

$$\frac{(-1)^d\chi(\mathcal{A}_Z, -1)}{\prod_i r_i!}$$

and then compare this to the total number of tilings of Z if it is available. Luckily (or is it just luck?), for all the zonotopes Z on the list, this number of tilings *is* available.

If Z is a $Z(r, s)$ quadrilateral, then $\mathcal{A}_Z$ is free, so we can use the exponents from Theorem 2.5(1) to compute that there is exactly one coherent tiling. However, it is also clear in this case that there is exactly one tiling, so all tilings are coherent.

If Z is a $Z(r, s, t)$ hexagon with one of r, s, t at most 2, then we may assume without loss of generality that $t \leq 2$. In this case $\mathcal{A}_Z$ is free, so we can use the exponents from Theorem 2.5(2) to compute the number of coherent tilings. We can also set $q = 1$ in MacMahon's result (Theorem 1.1) to compute the number of all tilings, and it is easy to check that the answers agree. Hence all tilings are coherent.

If Z is a $Z(r, s, 1, 1)$ octagon, then $\mathcal{A}_Z$ is free, so we can use the exponents from Theorem 2.5(3) to compute the number of coherent tilings. We can also set $q = 1$ in Elnitsky's result (Theorem 1.2) to compute the number of all tilings, and it is easy to check that the answers agree, so that all tilings are coherent.

If Z is a $Z(1, 1, 1, 1, 1)$ decagon, then $\mathcal{A}_Z$ is not free, but $\chi(\mathcal{A}_Z, t)$ was computed as part of the proof of Theorem 2.5, so we can plug in $t = -1$ to get that there are exactly 62 coherent tilings. By brute-force enumeration, it can be shown that there are exactly 62 tilings of Z in this case (see Fig. 3 of [Zi1]), so all tilings are coherent. This completes the proof that all tilings are coherent for each of the zonotopes Z listed in the theorem. □

Remark 1. Theorems 2.5 and 2.6 show that although the notions of Z having $\mathcal{A}_Z$ free, and Z having all tilings coherent are related, neither implies the other. For example, if Z is a $Z(1, 1, 1, 1, 1)$ decagon, then it has all tilings coherent but does not have $\mathcal{A}_Z$ free, while if Z is a $Z(2, 1, 2, 1)$ octagon with zonotope generators not projectively equivalent to $(0, 1)$, $(1, 1)$, $(1, 0)$, $(1, -1)$, then $\mathcal{A}_Z$ is free, but not all tilings of Z are coherent.

Remark 2. The reader should not be surprised to learn that much of what we have discussed in this section can be generalized to higher dimensions. For an excellent introduction to this see Section 2.2 of [BLS$^+$]. Given an arbitrary zonotope $Z \subset \mathbb{R}^d$ we define a *cubical subdivision* of Z to be a polytopal subdivision of Z whose cells are affine images of cubes. Certain of these subdivisions, called the coherent ones, can be obtained by the natural extension of lifting described in Section 1. These subdivisions are in natural correspondence with the vertices of the fiber zonotope $F(Z)$ defined in general in Theorem 4.1 of [BS], and this fiber zonotope $F(Z)$ is still dual to the discriminantal arrangement $\mathcal{A}_Z$. Thus we can still use the chamber-counting machinery of Zaslavsky [Za] to count the coherent subdivisions.

The collection of *all* cubical subdivisions is in one-to-one correspondence with certain one-element liftings of the related oriented matroid (this follows from the theorem of Dress [Bo], [?], [BLS$^+$, Theorem 2.2.13]). As discussed above, in the example of the dodecagon, which subdivisions are coherent may depend on exactly which generators are chosen and not just on the oriented matroid that they generate. On the other hand, some subdivisions may not be coherent for any choice of generators, as in the case of $Z(3, 3, 3)$. So it is possible for a zonotope to have all of its liftings coordinatizable (in the oriented matroid sense) without having all of its tilings coherent, but not vice versa.

The advantages to us of working only on two-dimensional zonotopes is that there are formulas (Theorems 1.1 and 1.2) which allow us to compute the total number of rhombic tilings for the cases that turn out to be of interest. We know of no higher-dimensional analogues to those formulas.

3. Induction Tables

This section completes the unfinished business of proving that the two families of arrangements $\mathcal{A}_{r\times s\times 2}$ and $\mathcal{A}^{(b)}_{r,s}$ which appear in the proof of Theorem 2.5 are free with exponents as asserted earlier. The reader who does not wish to be bored by details may find it convenient to skip this section.

Our strategy for showing that arrangements are free is to show that they are *inductively free*, using an induction table and the Addition Theorem of Terao [OT, Theorem 4.50], which we now explain. Given an arrangement $\mathcal{A}$ in $\mathbb{R}^d$ and a hyperplane H in $\mathcal{A}$, there are two associated arrangements, the *deletion* $\mathcal{A} - H$ and the *restriction*

$$\mathcal{A}^H = \{H_i \cap H\colon\ H_i \in \mathcal{A}\},$$

where the restriction $\mathcal{A}^H$ is thought of as an arrangement of hyperplanes inside $H \cong \mathbb{R}^{d-1}$.

Theorem 3.1 (Addition Theorem [OT, Theorem 4.50]). *Assume the deletion $\mathcal{A} - H$ and restriction $\mathcal{A}^H$ are both free, and* $\exp(\mathcal{A}^H) \subseteq \exp(\mathcal{A} - H)$ *so that we can index in such a way that*

$$\begin{aligned} \exp(\mathcal{A}^H) &= \{e_1, \dots, e_{d-1}\}, \\ \exp(\mathcal{A} - H) &= \{e_1, \dots, e_{d-1}, e_d\}. \end{aligned}$$

Then $\mathcal{A}$ is also free, and

$$\exp(\mathcal{A}) = \{e_1, \dots, e_{d-1}, e_d + 1\}.$$

An *induction table* for $\mathcal{A}$ is simply a sequence of applications of the Addition Theorem that begins with the empty arrangement and shows that $\mathcal{A}$ is free. $\mathcal{A}$ is called *inductively free* if it has such an induction table.

Theorem 3.2. *The arrangement $\mathcal{A}_{r\times s\times 2}$ is free with exponents*

$$\exp(\mathcal{A}_{r\times s\times 2}) = \{1, r+1, r+2, \dots, r+s-1, s+1, s+2, \dots, r+s\}.$$

Proof. We prove the freeness of this arrangement by constructing an induction table. Our strategy for building such a table is to embed these arrangements in a larger family, in such a way that this larger family contains all the restriction arrangements needed to prove that they are *all* inductively free. To this end, we give below the defining forms for the hyperplanes of three families of arrangements which will be shown to be free, and list their purported exponents:

For $l \leq r$ and I any interval in $\mathbb{Z}$, let

$$\mathcal{A}_{r\times s\times 2,l} = \mathcal{A}_{r\times s\times 2} \cup \{x_i - x_j + x_{r+s+1} - x_{r+s+2}\}_{i\in[l], j\in[r], i\neq j}$$
$$\{1, r+1, r+2, \dots, r+s+l-1, s+l+1, s+l+2, \dots, s+r\},$$

$$\mathcal{A}_{r\times 2,l} = \{x_{r+1} - x_{r+2}\} \cup \{x_i - x_j\}_{i<j\in[r]} \cup \{x_i - x_j + x_{r+1} - x_{r+2}\}_{i\in[l], j\in[r], i\neq j}$$
$$\{1, l+1, l+2, \dots, l+r-1\},$$

$$\mathcal{A}_{r,l}^{(I)} = \{x_{r+1}\} \cup \{x_i - x_j + zx_{r+1}\}_{i<j\in[2,r], z\in[-l,l]} \cup \{x_1 - x_j + zx_{r+1}\}_{j\in[2,r], z\in I}$$

$$\begin{cases} \{0, 1, l(r-2) + \#I, l(r-2) + \#I + 1, \dots, l(r-2) + \#I + r - 2\} & \text{if } \#I \geq l, \\ \{0, 1, (r-1)\#I, l(r-1) + 1, l(r-1) + 2, \dots, l(r-1) + r - 2\} & \text{if } \#I < l. \end{cases}$$

The induction table is built up by showing these arrangements are free in the following order:

$$\mathcal{A}_{r,l}^{(I)} \Rightarrow \mathcal{A}_{r\times 2,r} \Rightarrow \mathcal{A}_{r\times 2,l} \Rightarrow \mathcal{A}_{r\times 1\times 2,l} \Rightarrow \mathcal{A}_{r\times s\times 2,l} \Rightarrow \mathcal{A}_{r\times s\times 2},$$

noting that the first and last links in this chain are trivial because

$$\mathcal{A}_{r\times 2,r} \cong \mathcal{A}^{(1)}_{r,[-1,1]},$$
$$\mathcal{A}_{r\times s\times 2} = \mathcal{A}_{r\times s\times 2,0}.$$

We begin by showing that $\mathcal{A}^{(l)}_{r,I}$ is free by induction on $r + \#I$ where $\#I$ is the cardinality of the set I. There are three cases depending on the cardinality of $I = [a, b]$. In the first case, $\#I \geq l + 1$. Beginning with $\mathcal{A}^{(l)}_{r,[a,b-1]}$, which is free with exponents

$$\{0, 1, l(r-2) + \#I - 1, l(r-2) + \#I, \ldots, l(r-2) + \#I + r - 3\}$$

by induction on $r + \#I$, we add in the hyperplanes

$$x_1 - x_m + bx_{r+1}$$

for $m = 2, \ldots, r$ in any order. Each time, the restriction arrangement is projectively equivalent to $\mathcal{A}^{(l)}_{r-1,[-l,b-a]}$ which has exponents

$$\{0, 1, l(r-2) + \#I, l(r-2) + \#I + 1, \ldots, l(r-2) + \#I + r - 3\}$$

by induction on $r + \#I$, and hence the effect is to raise the exponent $l(r-2) + \#I - 1$ up to $l(r-2) + \#I + r - 2$, giving us

$$\{0, 1, l(r-2) + \#I - 1, l(r-2) + \#I, \ldots, l(r-2) + \#I + r - 2\},$$

which are the exponents claimed for $\mathcal{A}^{(l)}_{r,I}$.

In the second case, $1 \leq \#I \leq l$. Beginning with $\mathcal{A}^{(l)}_{r,[a,b-1]}$, which is free with exponents

$$\{0, 1, (r-1)(\#I - 1), l(r-1) + 1, l(r-1) + 2, \ldots, l(r-1) + r - 2\}$$

by induction on $r + \#I$, we add in the hyperplanes

$$x_1 - x_m + bx_{r+1}$$

for $m = 2, \ldots, r$ in any order. Each time, the restriction arrangement is projectively equivalent to $\mathcal{A}^{(l)}_{r-1,[-l,l]}$ which has exponents

$$\{0, 1, l(r-1) + 1, l(r-1) + 2, \ldots, l(r-1) + r - 2\}$$

by induction on $r + \#I$, and hence the effect is to raise the exponent $(r-1)(\#I - 1)$ up to $(r-1)\#I$, giving us

$$\{0, 1, (r-1)\#I, l(r-1) + 1, l(r-1) + 2, \ldots, l(r-1) + r - 2\},$$

which are the exponents claimed for $\mathcal{A}^{(l)}_{r,I}$.

In the third case, I is the empty interval $\varnothing$, and we have

$$\mathcal{A}^{(l)}_{r,\varnothing} \cong (0) \times \mathcal{A}^{(l)}_{r-1,[-l,l]},$$

where (0) denotes the empty arrangement in $\mathbb{R}^1$ and $\times$ denotes the product of arrangements [OT, Definition 2.13]. By induction on $r + \#I$ we know that $\mathcal{A}^{(l)}_{r-1,[-l,l]}$ is free with exponents

$$\{0, 1, l(r-1)+1, l(r-1)+2, \ldots, l(r-1)+r-2\}$$

and taking its product with (0) preserves freeness and adds on the exponent $\{0\}$, which gives the asserted exponents for $\mathcal{A}^{(l)}_{r,\varnothing}$. This completes the three cases, and hence the proof that $\mathcal{A}^{(l)}_{r,l}$ is free with the asserted exponents is complete.

We next proceed to prove that $\mathcal{A}_{r\times 2,l}$ is free by induction on r. We recall our earlier observation that $\mathcal{A}_{r\times 2,r} \cong \mathcal{A}^{(1)}_{r,[-1,1]}$ so the case $l = r$ is already proven and we may assume $l < r$. We begin with the arrangement

$$(0) \times \mathcal{A}_{(r-1)\times 2,l},$$

which is free by induction on r with exponents

$$\{0, 1, l+1, l+2, \ldots, l+r-2\}.$$

Adding in the hyperplanes $\{x_j - x_r\}_{j\in[r-1]}$ one at a time in any order, and then adding in the hyperplanes $\{x_r - x_j + x_{r+1} - x_{r+2}\}_{j\in[l]}$ in any order, the restriction arrangements are all projectively equivalent to $\mathcal{A}_{(r-1)\times 2,l}$, which by induction on r is free with exponents

$$\{1, l+1, l+2, \ldots, l+r-2\}.$$

After adding all these hyperplanes in we have the arrangement $\mathcal{A}_{r\times 2,l}$. This raises the exponent 0 up to $l+r-1$, so that $\mathcal{A}_{r\times 2,l}$ is free with the desired exponents

$$\{1, r, l+1, l+2, \ldots, l+r-2, l+r-1\}.$$

We next proceed to prove that $\mathcal{A}_{r\times 2\times 2,l}$ is free by induction on r, by working our way up from $(0)\times(0)\times\mathcal{A}_{r\times 2,l}$ through several intermediate arrangements defined below:

$$\begin{aligned}
\mathcal{A}_0 &= (0)\times(0)\times\mathcal{A}_{r\times r,l},\\
\mathcal{A}_1 &= A_0 \cup \{x_j + x_{r+1} + x_{r+3}\}_{j\in[r]},\\
\mathcal{A}_2 &= A_1 \cup \{x_j + x_{r+1} + x_{r+4}\}_{j\in[l]},\\
(0)\times A^{(l)}_3 &= A_2 \cup \{x_j + x_{r+1} + x_{r+4}\}_{j\in[l+1,r]},\\
\mathcal{A}_4 &= A_3 \cup \{x_{r+1} - x_{r+2}\},\\
\mathcal{A}_5 &= A_4 \cup \{x_j + x_{r+2} + x_{r+3}\}_{j\in[r]},\\
\mathcal{A}_6 &= A_5 \cup \{x_j + x_{r+2} + x_{r+4}\}_{j\in[l]},\\
\mathcal{A}_7 &= A_6 \cup \{x_j + x_{r+2} + x_{r+4}\}_{j\in[l+1,r]}.
\end{aligned}$$

To abbreviate the proof, we only say for each i what the restrictions look like (up to projective equivalence) as one adds in each of the hyperplanes H in $\mathcal{A}_i - \mathcal{A}_{i-1}$ *in any order*:

i	Restriction	Exponents of restriction
1	$(0) \times \mathcal{A}_{r\times 2,l}$	$\{0, 1, l+1, l+2, \ldots, l+r-1\}$
2	$(0) \times \mathcal{A}_{r\times 2,l}$	$\{0, 1, l+1, l+2, \ldots, l+r-1\}$
3	$(0) \times \mathcal{A}_{r\times 2,l+1}$	$\{0, 1, l+2, l+3, \ldots, l+r\}$
4	$\mathcal{A}_3^{(l)}$	$\{1, l+2, l+3, \ldots, l+r, r+1\}$
5	$\mathcal{A}_3^{(l)}$	$\{1, l+2, l+3, \ldots, l+r, r+1\}$
6	$\mathcal{A}_3^{(l)}$	$\{1, l+2, l+3, \ldots, l+r, r+1\}$
7	$\mathcal{A}_3^{(l+1)}$	$\{1, l+3, l+4, \ldots, l+r+1, r+1\}$

To summarize the effect this has on the exponents, we start with $\mathcal{A}_0 = (0) \times (0) \times \mathcal{A}_{r\times 2,l}$ having exponents

$$\{0, 0, 1, l+1, l+2, \ldots, l+r-1\}.$$

Passing from $\mathcal{A}_0$ to $\mathcal{A}_2$ raises one of the 0 exponents to $l+r$, yielding

$$\{0, 1, l+1, l+2, \ldots, l+r\}.$$

Passing from $\mathcal{A}_2$ to $\mathcal{A}_3$ raises the $l+1$ exponent to $r+1$, yielding

$$\{0, 1, l+2, l+3, \ldots, l+r, r+1\}.$$

Passing from $\mathcal{A}_3$ to $\mathcal{A}_6$ raises the remaining 0 exponent to $l+r+1$, yielding

$$\{1, l+2, l+3, \ldots, l+r+1, r+1\}.$$

Passing from $\mathcal{A}_6$ to $\mathcal{A}_7$ raises the $l+2$ exponent to $r+2$, yielding

$$\{1, l+3, l+4, \ldots, l+r+1, r+1, r+2\},$$

which one can check agrees with the asserted exponents for $\mathcal{A}_{r\times 2\times 2,l} = \mathcal{A}_7$.

Finally we prove that $\mathcal{A}_{r\times s\times 2,l}$ is free by induction on s, using the just-proven base case $s=2$. Define the intermediate arrangements

$$\begin{aligned}
\mathcal{A}_0 &= (0) \times \mathcal{A}_{r\times s-1\times 2,l},\\
\mathcal{A}_1 &= \mathcal{A}_0 \cup \{x_{r+m} - x_{r+s}\}_{m\in[1,s-1]},\\
\mathcal{A}_2 &= \mathcal{A}_1 \cup \{x_j + x_{r+s} + x_{r+s+1}\}_{j\in[1,r]},\\
\mathcal{A}_3 &= \mathcal{A}_2 \cup \{x_j + x_{r+s} + x_{r+s+2}\}_{j\in[1,l]},\\
\mathcal{A}_4 &= \mathcal{A}_3 \cup \{x_j + x_{r+s} + x_{r+s+2}\}_{j\in[l+1,r]}.
\end{aligned}$$

As before we only say for each i what the restrictions look like (up to projective equivalence) as one adds in each of the hyperplanes H in $\mathcal{A}_i - \mathcal{A}_{i-1}$ *in any order*:

i	Restriction	Exponents of restriction
1	$\mathcal{A}_{r\times s-1\times 2,l}$	$\{1, r+1, r+2, \ldots, r+s+l-2, l+s, l+s+1, \ldots, r+s-1\}$
2	$\mathcal{A}_{r\times s-1\times 2,l}$	$\{1, r+1, r+2, \ldots, r+s+l-2, l+s, l+s+1, \ldots, r+s-1\}$
3	$\mathcal{A}_{r\times s-1\times 2,l}$	$\{1, r+1, r+2, \ldots, r+s+l-2, l+s, l+s+1, \ldots, r+s-1\}$
4	$\mathcal{A}_{r\times s-1\times 2,l+1}$	$\{1, r+1, r+2, \ldots, r+s+l-1, l+s+1, l+s+2, \ldots, r+s-1\}$

To summarize the effect this has on the exponents, we start with $\mathcal{A}_0 = (0) \times \mathcal{A}_{r\times s-1\times 2,l}$, which is free by induction on s with exponents

$$\{0, 1, r+1, r+2, \ldots, r+s+l-2, s+l, s+l+1, \ldots, s+r-1\}.$$

Passing from $\mathcal{A}_0$ and $\mathcal{A}_3$ raises the 0 exponent to $r+s+l-1$, yielding

$$\{1, r+1, r+2, \ldots, r+s+l-2, r+s+l-1, s+l, s+l+1, \ldots, s+r-1\}.$$

Passing from $\mathcal{A}_3$ to $\mathcal{A}_4$ raises the $s+l$ exponent to $s+r$, yielding

$$\{1, r+1, r+2, \ldots, r+s+l-2, r+s+l-1, s+l+1, \ldots, s+r-1, s+r\}.$$

which agrees with the asserted exponents for $\mathcal{A}_{r\times s\times 2,l} = \mathcal{A}_4$.

This completes the proof that $\mathcal{A}_{r\times s\times 2}$ is inductively free, and has the exponents asserted earlier. □

Before proceeding to prove that $\mathcal{A}_{r,s}^{(b)}$ is free, we stop to take a closer look at the class of arrangements $\mathcal{A}_{r,I}^{(l)}$ which were proven free along the way, in the special case when $I = [-l, l]$. These arrangements may be viewed as a special case of a certain "symmetrical lifting" construction which we now describe. Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{R}^d$ with defining linear forms $\{l_H\}$ for the hyperplanes $\{H\}$. Let $\mathcal{L}^{(l)}(\mathcal{A})$ (resp. $\mathcal{L}_+^{(l)}(\mathcal{A})$) be the arrangement of $(2l+1)n+1$ hyperplanes in $\mathbb{R}^{d+1}$ with hyperplanes defined by the forms

$$\{l_H + zx_{d+1}\}_{H\in\mathcal{A}, z\in[-l,l]} \cup \{x_{d+1}\}$$

(resp.

$$\{l_H + zx_{d+1}\}_{H\in\mathcal{A}, z\in[-l,l+1]} \cup \{x_{d+1}\}).$$

It can be checked that the arrangement $\mathcal{A}_{r,[-l,l]}^{(l)}$ is the same as $\mathcal{L}^{(l)}(A_{r-1})$ where A_{r-1} is the Coxeter arrangement $\{x_i - x_j\}_{i<j\in[r]}$. The exponents which were computed for this case are consistent with the following conjecture:

Conjecture 3.3. *Let $\mathcal{A}_W$ be the hyperplane arrangement associated to an irreducible Weyl group W, with exponents $\{e_1, \ldots, e_d\}$ and Coxeter number h (see Definition* 6.99 *of* [OT]). *Then $\mathcal{L}^{(l)}(\mathcal{A}_W)$ is free, with exponents*

$$\{1, e_1 + l\cdot h, e_2 + l\cdot h, \ldots, e_d + l\cdot h\},$$

and $\mathcal{L}_+^{(l)}(\mathcal{A}_W)$ is free, with exponents

$$\{1, (l+1)h, (l+1)h, \ldots, (l+1)h\}.$$

As pointed out above, the conjecture for $\mathcal{L}^{(l)}(\mathcal{A}_W)$ is true for all l when $W = A_n$, and both assertions of the conjecture have been checked for many small values of l when $W = B_n, C_n, D_n, E_6, F_4$, and G_2. Also, the characteristic polynomial of $\mathcal{L}_+^{(1)}(\mathcal{A}_W)$ has been shown to factor over the integers with the conjectured exponents when $W = A_n, B_n, C_n$, and D_n by Headley [H]. Here Headley considers these arrangements as subarrangements of the *cone* [OT, p. 14] over the affine Weyl arrangements $\mathcal{A}_{\tilde{W}}$.

The connection to affine Weyl arrangements is also suggested by the fact that the conjecture does *not* hold in general when W is an irreducible *Coxeter group*. For example, if W is the dihedral group of order 10 acting in its defining representation as the symmetries of a regular pentagon, it can be checked that the characteristic polynomial $\chi(\mathcal{L}^{(1)}(\mathcal{A}_W, t)$ does not factor completely over $\mathbb{Z}$.

We now return to the discussion of the arrangements $\mathcal{A}_{r,s}^{(b)}$.

Theorem 3.4. *The arrangement $\mathcal{A}_{r,s}^{(b)}$, for $b \neq 0, 1, -1$, is free with exponents*

$$\exp(\mathcal{A}_{r,s}^{(b)}) = \{1, r+2, r+3, \ldots, r+s, s+2, s+2, \ldots, r+s+1\}.$$

Proof. Similarly to the proof of the previous theorem, our strategy in building an induction table for $\mathcal{A}_{r,s}^{(b)}$, where $b \neq 0, 1, -1$, is to embed these arrangements in a larger family, in such a way that this larger family contains all the restriction arrangements needed to prove that they are *all* inductively free. To this end, we give below the defining forms for the hyperplanes of two families of arrangements which will be shown to be free, and list their purported exponents:

For r, s, k nonnegative integers, let

$$\begin{aligned}\mathcal{A}_{=,r,s,k}^{(b)} = \{x_i\}_{i\in[r+s+1]} &\cup \{x_i - x_j\}_{i<j\in[r+s+1]} \cup \{x_i - bx_j\}_{i\in[1,r],j\in[r+1,r+s]}\\ &\cup \{x_i - b^m x_{r+s+1}\}_{i\in[1,r+s],m\in[0,k]}\end{aligned}$$

$$\{1, r+2+k, r+3+k, \ldots, r+s+k+1, s+2+k, s+3+k, \ldots, r+s+k+1\},$$

$$\begin{aligned}\mathcal{A}_{+,r,s,k}^{(b)} = \{x_i\}_{i\in[r+s+1]} &\cup \{x_i - x_j\}_{i<j\in[r+s+1]} \cup \{x_i - bx_j\}_{i\in[1,r],j\in[r+1,r+s]}\\ &\cup \{x_i - b^m x_{r+s+1}\}_{i\in[1,r+s],m\in[0,k]}\\ &\cup \{x_i - b^m x_{r+s+1}\}_{i\in[r+1,r+s],m\in[0,k-1]}\end{aligned}$$

$$\{1, r+2+k, r+3+k, \ldots, r+s+k+1, s+3+k, s+4+k, \ldots, r+s+k+2\}.$$

The induction table is built up in the following order: we show first that $\mathcal{A}^{(b)}_{=,r,s,k}$, $\mathcal{A}^{(b)}_{+,r,s,k}$ are free by an intertwining induction on the minimum of $\{r, s, k\}$, and then show how the freeness of $\mathcal{A}^{(b)}_{=,r,s,0}$, $\mathcal{A}^{(b)}_{=,r,s,1}$ imply the freeness of $\mathcal{A}^{(b)}_{r,s}$ by induction on s.

To begin the intertwining induction for $\mathcal{A}^{(b)}_{=,r,s,k}$, $\mathcal{A}^{(b)}_{+,r,s,k}$, we note that if either $r = 0$ or $s = 0$, then both of these families of arrangements degenerate into an arrangement of the form

$$\{x_i\}_{i\in[r+1]} \cup \{x_i - b^m x_{r+1}\}_{i\in[1,r],m\in[0,k]}.$$

This arrangement is easily seen to be *supersolvable* with M-chain given by

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots,$$

where $\mathcal{A}_k$ is the subarrangement consisting of all hyperplanes whose linear forms only involve the coordinates x_{r+s+1} and x_i where $i \in [1, k]$. We refer the reader to Sections 2.1 and 4.3 of [OT] for a definition of supersolvability, M-chains, and why this implies the arrangement is inductively free.

In the first of the two intertwining inductive steps, we assume that $\mathcal{A}^{(b)}_{+,r,s-1,k}$, $\mathcal{A}^{(b)}_{+,r,s,k-1}$ are free with the correct exponents, and show that $\mathcal{A}^{(b)}_{=,r,s,k}$ is free. Beginning with $\mathcal{A}^{(b)}_{+,r,s,k-1}$, which is free with exponents

$$\{1, r+1+k, r+2+k, \ldots, r+s+k, s+2+k, s+3+k, \ldots, r+s+k+1\}$$

by induction, we add in the hyperplanes

$$x_m - b^k x_{r+s+1}$$

for $m = r+1, \ldots, r+s$ in any order. Each time, the restriction arrangement is projectively equivalent to $\mathcal{A}^{(b)}_{+,r,s-1,k}$, which has exponents

$$\{1, r+2+k, r+3+k, \ldots, r+s+k, s+2+k, s+3+k, \ldots, r+s+k+1\},$$

and hence the effect is to raise the exponent $r+1+k$ up to $r+s+1+k$, giving us

$$\{1, r+2+k, \ldots, r+s+k, r+s+1+k, s+2+k, s+3+k, \ldots, r+s+k+1\},$$

which are the exponents claimed for $\mathcal{A}^{(b)}_{=,r,s,k}$. In the second of the intertwining inductive steps, we assume that $\mathcal{A}^{(b)}_{=,r-1,s,k+1}$, $\mathcal{A}^{(b)}_{=,r,s,k}$ are free with the correct exponents, and show that $\mathcal{A}^{(b)}_{+,r,s,k}$ is free. Beginning with $\mathcal{A}^{(b)}_{=,r,s,k}$, which is free with exponents

$$\{1, r+2+k, r+3+k, \ldots, r+s+k+1, s+2+k, s+3+k, \ldots, r+s+k+1\}$$

by induction, we add in the hyperplanes

$$x_m - b^{k+1} x_{r+s+1}$$

for $m = 1, \ldots, r$ in any order. Each time, the restriction arrangement is projectively equivalent to $\mathcal{A}^{(b)}_{=,r-1,s,k+1}$, which has exponents

$$\{1, r+2+k, r+3+k, \ldots, r+s+k+1, s+3+k, s+4+k, \ldots,$$
$$r+s+k+1\},$$

and hence the effect is to raise the exponent $s+2+k$ up to $r+s+k+2$, giving us

$$\{1, r+2+k, r+3+k, \ldots, r+s+k+1, s+3+k, \ldots,$$
$$r+s+k+1, r+s+k+2\},$$

which are the exponents claimed for $\mathcal{A}^{(b)}_{+,r,s,k}$.

This proves that $\mathcal{A}^{(b)}_{=,r,s,k}$, $\mathcal{A}^{(b)}_{+,r,s,k}$ are both free with the correct exponents. We now use $\mathcal{A}^{(b)}_{=,r,s,0}$, $\mathcal{A}^{(b)}_{=,r,s,1}$, to prove that $\mathcal{A}^{(b)}_{r,s}$ is free by induction on s. Beginning with $\mathcal{A}^{(b)}_{=,r,s-1,0}$, which is free with exponents

$$\{1, r+2, r+3, \ldots, r+s, s+1, s+2, \ldots, r+s\}$$

by induction, we add in the hyperplanes

$$x_m - bx_{r+s+1}$$

for $m = 1, \ldots, r$ in any order. Each time, the restriction arrangement is projectively equivalent to $\mathcal{A}^{(b)}_{=,r-1,s-1,1}$, which has exponents

$$\{1, r+2, r+3, \ldots, r+s, s+2, s+3, \ldots, r+s\},$$

and hence the effect is to raise the exponent $s+1$ up to $r+s+1$, giving us

$$\{1, r+2, r+3, \ldots, r+s, s+2, \ldots, r+s, r+s+1\},$$

which are the exponents claimed for $\mathcal{A}^{(b)}_{r,s}$. This completes the proof. □

Remark. The induction table used to prove $\mathcal{A}^{(b)}_{r,s}$ is free suggests that the following might be an interesting problem: characterize all free arrangements of hyperplanes in which the linear forms defining the hyperplanes are all of the form

$$x_i - b^m x_j$$

for some fixed real number b and m is allowed to vary through the integers. Undoubtedly this is an interesting family, since it contains all of the classical Coxeter arrangements (when $b = -1$), the minimal-dimensional counterexample to Orlik's conjecture found in [ER1], and the counterexample to Saito's conjecture discussed in the next section.

4. Saito's Conjecture

In this section we discuss two very interesting, closely related one-parameter families of arrangements arising from the induction tables of the previous section. The first is a one-parameter family of free arrangements for which most parameter values yield a free arrangement, but one value yields a nonfree arrangement. The second is a previously announced [ER2] one-parameter family of counterexamples to Saito's conjecture that the *complexified complement* of a free arrangement is a $K(\pi, 1)$ space (see [OT, Conjecture 5.18] and [Sa]).

The first family is $\mathcal{A}_{2,2}^{(b)}$ where b is any real number, i.e., the discriminantal arrangement for a $Z(2, 2, 1, 1)$ or $Z(2, 1, 2, 1)$ octagon, depending upon the value of b. It has hyperplanes defined by the linear forms

$$\{x_i\}_{i\in[4]} \cup \{x_i - x_j\}_{1\leq i<j\leq 4} \cup \{x_i - bx_j\}_{i\in[1,2], j\in[3,4]}.$$

As b varies over all real numbers, we have that $\mathcal{A}_{2,2}^{(b)}$ is

$$\begin{aligned} \text{free with exponents } \{1, 4, 4, 5\} \text{ if } b &\neq 0, 1, -1, \\ \text{free with exponents } \{1, 2, 3, 4\} \text{ if } b &= 0, 1, \\ \text{not free if } b &= -1. \end{aligned}$$

These assertions follow from the induction tables in the previous section for $b \neq 0, 1, -1$, from the fact that the arrangement degenerates to something projectively equivalent to the Coxeter arrangement A_4 if $b = 0, 1$, and by computing that the characteristic polynomial is

$$(t-1)(t-3)(t^2 - 10t + 26)$$

if $b = -1$.

Theorem 4.1. *The set of free arrangements is a constructible subset which is neither Zariski-closed nor Zariski-open.*

Proof. The example $\mathcal{A}_{2,2}^{(b)}$ just discussed shows that the set of free arrangements is not Zariski-closed in the set of all arrangements. It follows from Corollary 7.6 of [Zi2] that there is a Zariski-open set of arrangements which are not free (the *general position* arrangements), and hence the set of free arrangements is not Zariski-open in all arrangements. On the other hand, a result of Yuzvinsky [Y] shows that among all arrangements with a fixed intersection lattice, the subset of free arrangements is Zariski-open. Since the set of arrangements with a fixed intersection lattice is easily seen to be a constructible set, we conclude that the set of all free arrangement is constructible. □

Restricting this arrangement $\mathcal{A}_{2,2}^{(b)}$ to any one of the hyperplanes

$$\{x_i - x_j, x_i - bx_j\}_{i\in[1,2], j\in[3,4]}$$

yields an arrangement projectively equivalent to $\mathcal{A}^{(b)}_{=,1,1,1}$, whose hyperplanes are defined by the forms

$$\{x_i\}_{i\in[3]} \cup \{x_i - x_j\}_{i<j\in[3]} \cup \{x_1 - bx_2, x_1 - bx_3, x_2 - bx_3\}.$$

As b varies over all real numbers, we have that $\mathcal{A}^{(b)}_{=,1,1,1}$ is

$$\begin{aligned}
&\text{free with exponents } \{1, 4, 4\} \text{ if } b \neq 0, 1, -1,\\
&\text{free with exponents } \{1, 2, 3\} \text{ if } b = 0, 1,\\
&\text{free with exponents } \{1, 3, 5\} \text{ if } b = -1.
\end{aligned}$$

These assertions follow from the induction tables in the previous section for $b \neq 0, 1, -1$, and for the cases $b = 0, 1$ or $b = -1$ from the fact that the arrangement degenerates to something projectively equivalent to the Coxeter arrangements A_3 or B_3, respectively. Figure 10 depicts these arrangements $\mathcal{A}^{(b)}_{=,1,1,1}$ in the real projective plane $\mathbf{RP}^2$ for various values of b. Here $\mathbf{RP}^2$ has been identified with the top hemisphere of the unit circle

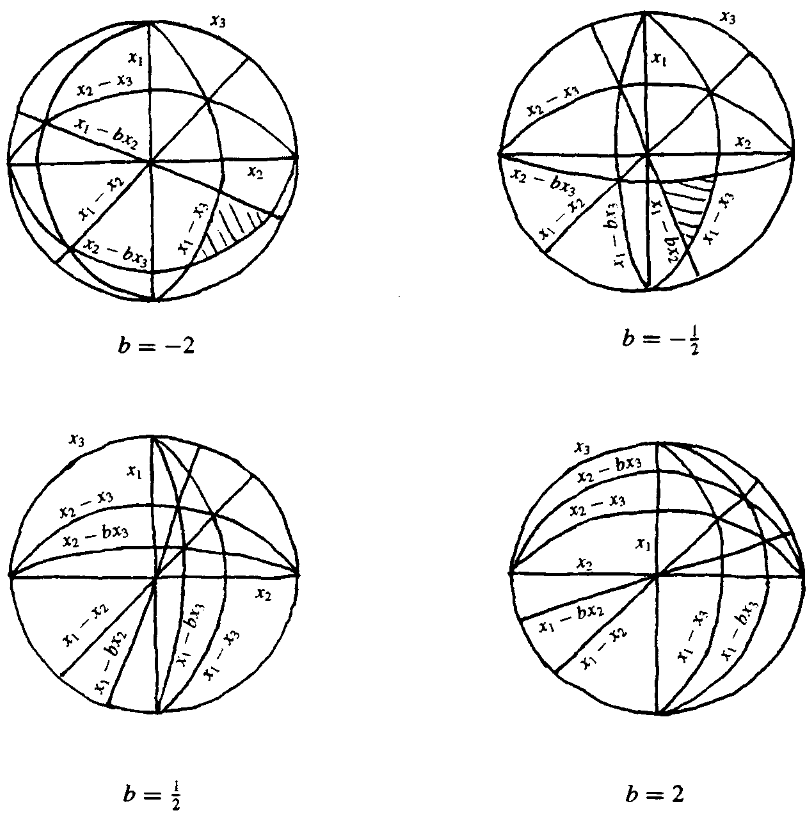


Fig. 10. $\mathcal{A}^{(b)}_{=,1,1,1}$ for $b = -2, -\frac{1}{2}, \frac{1}{2}$, and 2.

$x_1^2 + x_2^2 + x_3^2 = 1$, in which we have identified opposite points on the equator (the line at infinity) $x_3 = 0$.

Theorem 4.2. *The arrangement $\mathcal{A}^{(b)}_{=,1,1,1}$ is free for all values of b but its complexified complement is not a $K(\pi, 1)$ for $b \neq 0, 1, -1$.*

Proof. We have already seen that $\mathcal{A}^{(b)}_{=,1,1,1}$ is free for all values of b. To show that, for $b \neq 0, 1, -1$, the complexified complement is not a $K(\pi, 1)$ space we begin with the case that $b < 0$. When $b < 0$, the triangular region bounded by the hyperplanes

$$\{x_1 - bx_2, x_1 - x_3, x_2 - bx_3\}$$

touches no other hyperplanes in the arrangement, even at its vertices. We should note that the existence of such a region is dependent on the *oriented matroid* structure of the arrangement *not* just on the *matroid* structure specified by the intersection lattice. Thus two arrangements can have the same intersection lattice but one may have such a triangular region and the other may not. It is known that the existence of such a region implies that the complexified complement of the arrangement is not a $K(\pi, 1)$ space [FR, Corollary 3.3], and we thank R. Randell for the following argument: Let ι be the inclusion map of the complexified complement of this arrangement into the complement of the arrangement which only has the hyperplane $x_3 = 0$ at infinity and the three which bound this triangular region. Then it can easily be shown that ι induces a surjection on all homotopy groups, and, in particular, on the second homotopy group π_2. However, the complexified complement of the arrangement with four hyperplanes is well known (see Corollary 5.23 of [OT]) to be homotopy equivalent to the Cartesian product of a circle and the 2-skeleton of a 3-torus. This space has a nontrivial second homotopy group, and therefore the complexified complement of $\mathcal{A}^{(b)}_{=,1,1,1}$ also has a nontrivial second homotopy group, so it is not a $K(\pi, 1)$ space.

Furthermore, by results of Randell [Ra], this also implies that, for any $b \neq 0, 1, -1$, the complexified complement of $\mathcal{A}^{(b)}_{=,1,1,1}$ is not $K(\pi, 1)$. The reason is that if we allow b to be a *complex* parameter, then it is easy to see that the *intersection lattice* of the arrangement is isomorphic for all values $b \neq 0, 1, -1$. Since the complex plane with 0, 1, -1 deleted is path-connected, a *lattice isotopy* can be constructed between any two such arrangements, and it can be concluded that their complements are all *diffeomorphic* (see [Ra] or [OT, Theorem 5.28]). □

We remark on one last interesting feature of the arrangement $\mathcal{A}^{(b)}_{=,1,1,1}$: it is the discriminantal arrangement for the three-dimensional zonotope $Z_{(b)}$ with six zonotopal generators

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (b, 1, 1), (b, b, 1)\}.$$

The *matroid dual* (see Section 3.4 of [BLS$^+$]) of this zonotope is another three-dimensional zonotope Z with six generators, which, somewhat surprisingly, turns out to be projectively equivalent to $Z_{(b)}$ itself!

5. Higher Bruhat Order and q-Counting

In this section we connect the theorems of MacMahon and Elnitsky which q-count tilings to the higher Bruhat orders of Manin and Schechtmann, and to the *weak order* on the regions of the discriminantal arrangements A_Z. A very pleasant consequence of this connection is a factorization for the rank-generating function of these weak orders in some of the cases where $\mathcal{A}_Z$ is free, which relates to a question of Björner, Terao, and Wagreich [W] (see the Remark after Theorem 5.4). In this section we also resolve a question posed by Ziegler concerning higher Bruhat orders (see Digression below).

In order to define the higher Bruhat order, it is convenient to replace a tiling of a $Z(r_1, \ldots, r_l)$ polygon with an equivalent object which we call the *braid picture* of the tiling. Informally, a braid picture consists of $n = \sum_i r_i$ *strands* which begin on the left side of the page and move across to the right side of the page, and each pair of strands cross each other zero or one times as they move across, with at most two strands crossing at a single point. Our notion of a braid picture is closely related to Goodman and Pollack's *wiring diagram*, see p. 260 of [BLS$^+$]. If we order the strands $s_1, \ldots, s_n$ from bottom to top on the left side of the page, then two strands s_i, s_j will cross if and only if the indices i, j do not lie in a common interval of the form $[r_1, r_2+\cdots+r_m+1, r_1+r_2+\cdots+r_m+r_{m+1}]$. The correspondence between such braid pictures and tilings of a $Z(r_1, \ldots, r_l)$ polygon Z with unit rhombi is hard to explain in words, but easy in pictures; see Fig. 11 and [HG, Instructions, Figures 3 and 4]. Roughly speaking, if the (distinct) zonotope generators for Z are numbered $\{\mathbf{u}_1, \ldots, \mathbf{u}_l\}$ in order of weakly decreasing slopes, then, after rotating Z counterclockwise 90°, the strands

$$\{s_i\}_{i\in[r_1+r_2+\cdots+r_m+1, r_1+r_2+\cdots+r_m+r_{m+1}]}$$

correspond to those rhombus edges in the tiling which have slopes the same as $\mathbf{u}_m$. The tiling may be considered the *dual planar graph* of the planar graph which has vertices at all the crossing points of the strands, and edges along strand segments between the crossing points.

Given such a braid picture for a tiling T of Z, we define the *inversion set* $\mathrm{Inv}(T)$ to be the set of triples (i, j, k) with $1 \le i < j < k \le n$ for which the three strands s_i, s_j, s_k all do cross each other, and for which the order of crossing is first s_j, s_k, then s_i, s_k, and then s_j, s_i. Figure 12 shows the two different ways the three strands s_i, s_j, s_k can cross, one of which is an inversion, the other not.

We then define the *higher Bruhat order* $HB(r_1, \ldots, r_l)$ on the set of all tilings T of a $Z(r_1, \ldots, r_l)$ polygon to be the order induced by inclusion of the inversion sets $\mathrm{Inv}(T)$.

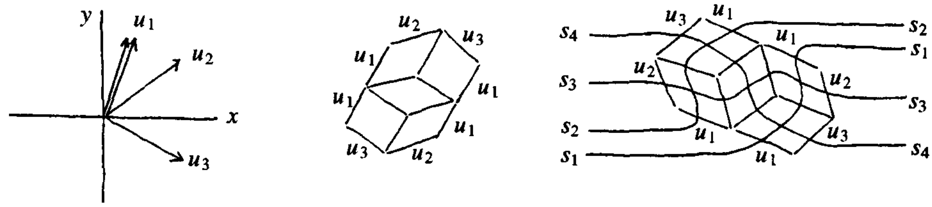


Fig. 11. A tiling and its braid picture.

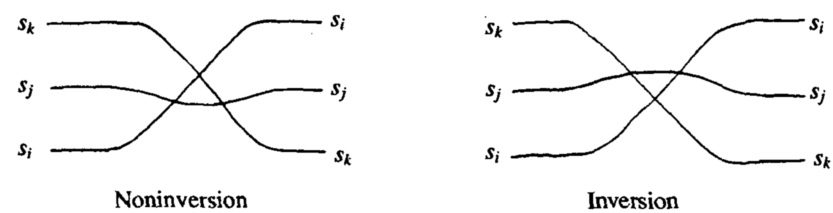


Fig. 12. The two orders in which three strands can cross.

In the special case where $r_1 = \cdots = r_l = 1$, so $l = n$ and Z is a $(2n)$-gon, this order is the same as the order $B(n, 2)$ in the family $B(n, k)$ of *higher Bruhat orders* defined by Manin and Schechtmann [MS].

Digression. In fact, what we have just said is not quite true, since, in [MS], the orders $B(n, k)$ were defined by *single-step inclusion* of inversion sets, not inclusion. More precisely, they defined a *covering relation* in $B(n, k)$ to be a pair of inversion sets (see [MS] for the definition of inversion sets in $B(n, k)$), $I' \subseteq I$ with $\#I' = \#I + 1$, and then $B(n, k)$ is the transitive closure of this relation. It was pointed out by Ziegler [Zi1] that for $k = 1$ single-step inclusion is the same as inclusion on inversion sets, but for $k = 3$ these two concepts give rise to distinct partial orders. Ziegler also asked whether for $k = 2$ single-step inclusion was the same as inclusion. We sketch here a proof that this is true, so that our definition of higher Bruhat order will indeed coincide with that of [MS].

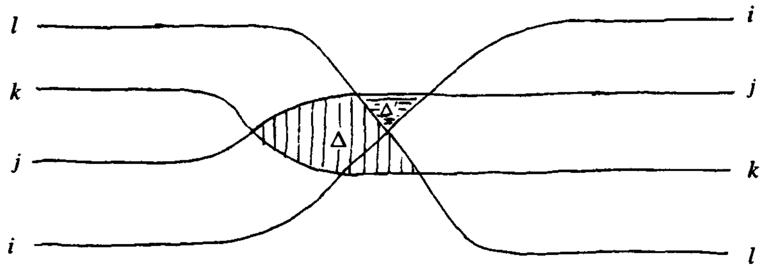
Theorem 5.1. *The partial order of $B(n, 2)$ by single-step inclusion is the same as the partial order of $B(n, 2)$ by inclusion.*

Proof. In order to show that single-step inclusion is the same as inclusion, we assume that we have two tilings T and T' with $\mathrm{Inv}(T') \subseteq \mathrm{Inv}(T)$. If we can show that there is always a triple (i, j, k) in the set difference $\mathrm{Inv}(T) - \mathrm{Inv}(T')$ for which $\mathrm{Inv}(T) - (i, j, k)$ is still an inversion set, then we will be done by induction on $\#(\mathrm{Inv}(T) - \mathrm{Inv}(T'))$. Notice that the triples (i, j, k) which can be removed from $\mathrm{Inv}(T)$ and still leave an inversion set are exactly the triples corresponding to three strands s_i, s_j, s_k which bound a *connected, triangular* region of the plane in the braid picture of T, and such triples are usually called *mutations* [BLS+, p. 267]). To rephrase then, we need to show that there is always a mutation t of the braid picture of T which lies in $\mathrm{Inv}(T) - \mathrm{Inv}(T')$.

We call a set of triples I *biconvex* if, for all quadruples $1 \le i < j < k < l \le n$, the intersection of I with the following four triples

$$(i, j, k) < (i, j, l) < (i, k, l) < (j, k, l)$$

is either an initial or final segment in the linear ordering indicated. A set of triples I is $\mathrm{Inv}(T)$ for some tiling T if and only if it is biconvex [Zi1, Lemma 2.4]. We would therefore be done if we could prove the following lemma:



$t = (j, k, l) > t' = (i, j, l)$

Fig. 13. The covering relation that generates the partial order on triples.

Lemma 5.2. *If I is a biconvex inversion set of a tiling T, then no other biconvex set contained in I which contains all of the mutation triples t of T can be* **properly** *contained in I.*

Sketch of Proof. Having fixed the tiling T and the biconvex set I of its inversions, partially order all triples $t = (t, j, k)$ in I by taking the transitive closure of the following relation: $t > t'$ if the triangles Δ and Δ' bounded by t and t' in the braid pictures share exactly two bounding strands, and the third bounding strand of Δ' cuts through the interior of Δ. An example is shown in Fig. 13.

It is not obvious that the transitive closure of the above relation is a partial order, i.e., that taking the transitive closure creates no cycles. However, it can be checked that this follows from the fact that the southernmost corner of Δ (i.e., the intersection point of strands s_i, s_k) is always weakly south of the southernmost corner of Δ', in the sense that Δ' lies in the most northern quadrant above both strands s_i, s_k.

Given this partial order, it is then shown that any biconvex subset $I' \subseteq I$ which contains all mutations of I has to contain each triple $t \in I$, using induction on this partial order on the triples t. Since the mutations of I are exactly the minimal elements in this partial order, the base of the induction is clear. For the inductive step, it is shown that given any nonminimal triple t in a biconvex set I', knowing which triples below it in the partial order are in I' and which ones are not will exactly determine whether or not $t \in I'$ by biconvexity. Hence if I' is contained in I and shares all of I's mutations, it must actually contain every triple of I. $\square$

Now that we have removed any ambiguity about how the higher Bruhat order is defined we can continue to explore the enumerative consequences of our results. Before going any further, we should point out that the higher Bruhat order on tilings of $Z(r_1, \dots, r_l)$ *does* depend on the ordering of $(r_1, \dots, r_l)$ more closely than the number of tilings does. It is clear that the number of tilings only depends on $(r_1, \dots, r_l)$ up to cyclic permutations, since the whole polygon can be rotated to get a bijection between tilings. It is even true that the graph underlying the Hasse diagrams for the corresponding higher Bruhat orders will be isomorphic. However, the notion of which triples (i, j, k) are inversions is *not* invariant under such a cyclic permutation of $(r_1, \ldots, r_l)$ and the choice of the *bottom* tiling T_0 in the higher Bruhat order will be different for different cyclic permutations of $(r_1, \ldots, r_l)$.

We now explain why MacMahon and Elnitsky's q-counting results are special cases of the *rank-generating function*

$$R(r_1, \ldots, r_l) = \sum_{T \in HB(r_1, \ldots, r_l)} q^{\mathrm{rank}(T)},$$

where $\mathrm{rank}(T) = \#\,\mathrm{Inv}(T)$ is the *rank function* for the higher Bruhat order on tilings of a $Z(r_1, \ldots, r_l)$.

MacMahon q-counts plane partitions π sitting inside an $r \times s \times t$ box with weight $q^{\sum_{i,j} \pi_{i,j}}$. When viewing the plane partition as a set of unit cubes stacked into the corner of an $r \times s \times t$ box, $\sum_{i,j} \pi_{i,j}$ is the same as the number of cubes. When viewing this set of cubes as a rhombic tiling T of a $Z(r, s, t)$ hexagon, the set of triples (i, j, k) giving the coordinates of each cube is exactly the inversion set $\mathrm{Inv}(T)$ of the tiling. Hence the weight $q^{\sum_{i,j} \pi_{i,j}}$ is the same as $q^{\mathrm{rank}(T)}$ in the order $HB(r, s, t)$. Note that in this case the higher Bruhat order $HB(r, s, t)$ has a simpler description as the *distributive lattice of order ideals* in the product of three chains $r \times s \times t$ (see [Sta]), and it is well known that MacMahon's result gives the rank-generating function for this lattice.

Elnitsky q-counts tilings T of a $Z(r, s, 1, 1)$ octagon by $q^{\mathrm{dist}(T, T_0)}$, where $\mathrm{dist}(T, T_0)$ is the distance of the tiling T from some canonical tiling T_0 in a certain connected graph of tilings. If we reinterpret this as a $Z(s, 1, 1, r)$ octagon (by cyclically permuting $(r, s, 1, 1)$ and rotating the octagon and its tilings), then it can easily be checked that his graph coincides with the *Hasse diagram* of the higher Bruhat order $HB(s, 1, 1, r)$, and that T_0 is the bottom element in this order, i.e., it is the tiling with $\mathrm{Inv}(T) = \varnothing$. Hence $\mathrm{dist}(T, T_0) = \mathrm{rank}(T)$.

We now explore the relation between this higher Bruhat order and the *weak order* on the regions of a hyperplane arrangement. A *region* of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is a connected component of the complement $\mathbb{R}^d - \mathcal{A}$. Pick a *base region* C_0, and define a function S from regions C of $\mathcal{A}$ to subsets of $\mathcal{A}$ by

$$S(C_0, C) = \{H \in \mathcal{A}\colon\ H \text{ separates } C_0 \text{ from } C\}.$$

Partially ordering the set of regions C of $\mathcal{A}$ by inclusion of $S(C_0, C)$ gives a poset $P(\mathcal{A})$ defined in [Ed] and usually called the *weak order* on regions of $\mathcal{A}$ with respect to the base chamber C_0. Clearly, the rank function on $P(\mathcal{A})$ corresponds to the cardinality $\#S(C_0, C)$. The weak order derives its name from the special case where $\mathcal{A}$ is the Coxeter arrangement A_{n-1} with defining hyperplanes $\{x_i - x_j\}_{i<j \in [n]}$. In this case, regions correspond to permutations π in the symmetric group S_n, the chamber C_0 is chosen to correspond to the identity permutation, and the poset $P(\mathcal{A}_n)$ is usually referred to as the *weak order* (see [Ed]).

In the case $\mathcal{A} = \mathcal{A}_Z$ where Z is some $Z(r_1, \ldots, r_l)$ polygon, there are two kinds of hyperplanes: those of the form $x_i - x_j = 0$, where $i < j$ lie in some interval

$$[r_1 + r_2 + \cdots + r_m + 1, r_1 + r_2 + \cdots + r_m + r_{m+1}],$$

and those of the form $ax_i + bx_j + cx_k = 0$, where no two of $1 \leq i < j < k \leq n$ lie in any such interval and $a\mathbf{u}_{l_i} + b\mathbf{u}_{l_j} + c\mathbf{u}_{l_k} = 0$ is the unique linear dependence (up to scaling) among their corresponding zonotope generators. Recalling our convention that the $\mathbf{u}_i$'s all point into the right half-plane, and are numbered going clockwise from the positive y-axis, we can normalize the scaling of a, b, c so that b is always negative, and hence a, c are always positive. We choose the base chamber C_0 in $\mathcal{A}_Z$ to be defined by $x_j - x_i > 0$ for hyperplanes of the first type (and $i < j$), and $ax_i + bx_i + cx_k > 0$ for hyperplanes of the second type. Given a region C in $\mathcal{A}_Z$, its location relative to the hyperplanes of the first type uniquely defines l permutations $(\sigma^{(1)}, \ldots, \sigma^{(l)})$ where $\sigma^{(i)}$ is a permutation of the set $[r_1 + \cdots + r_{i-1} + 1, r_1 + \cdots + r_i]$ and

$$x_{\sigma^{(m)}_{r_1+\cdots+r_{m-1}+1}} < x_{\sigma^{(m)}_{r_1+\cdots+r_{m-1}+2}} < \cdots < x_{\sigma^{(m)}_{r_1+\cdots+r_{m-1}+r_m}}$$

for points x in C and each m and where $\sigma^{(m)}_t$ means the image of t under $\sigma^{(m)}$. Likewise, the location of the chamber C relative to hyperplanes of the second type uniquely defines a subset $I(C)$ of the triples $1 \leq i < j < k \leq n$ corresponding to braid strands s_i, s_j, s_k which all pairwise intersect, according to the following rule: let

$$\sigma^{(m_1)}_i = n_1, \qquad \sigma^{(m_2)}_j = n_2, \qquad \sigma^{(m_3)}_k = n_3$$

and then say that (i, j, k) is in I if and only if

$$ax_{n_1} + bx_{n_2} + cx_{n_3} < 0$$

is satisfied for points x in the chamber C. Roughly speaking, what the previous convention is saying is that if the chamber C corresponds to the permutations $(\sigma^{(1)}, \ldots, \sigma^{(l)})$, then to make it correspond to a tiling we must reorder the strands within each interval

$$[r_1 + r_2 + \cdots + r_m + 1, r_1 + r_2 + \cdots + r_m + r_{m+1}]$$

in the braid picture. This allows us to define a set map

$$\varphi\colon P(\mathcal{A}_Z) \to HB(r_1, \ldots, r_l) \times S_{r_1} \times \cdots \times S_{r_l}$$

by

$$\varphi(C) = (I(C), \sigma^{(1)}, \ldots, \sigma^{(l)}),$$

where we have identified S_{r_m} with the set of permutations of the set $[r_1 + \cdots + r_{m-1} + 1, r_1 + \cdots + r_m]$. It only needs to be checked that the set of triples $\{(i, j, k)\}$ on the right-hand side really do form the inversion set $\mathrm{Inv}(T)$ for some tiling, but this is simply a tedious unravelling of two bijections: the one between regions C in $\mathcal{A}_Z$ and coherent tilings of Z, and the one between tilings of Z and inversion sets. We omit this verification.

Proposition 5.3. *Let Z be a $Z(r_1, \ldots, r_l)$ polygon. Then the map φ defined above is a rank- and order-preserving injection*

$$\varphi\colon P(\mathcal{A}_Z) \hookrightarrow HB(r_1, \ldots, r_l) \times S_{r_1} \times \cdots \times S_{r_l}.$$

Here the right-hand side is ordered componentwise, using the weak order in each of the factors S_{r_m}.

Proof. The statement is nearly a tautology once the definitions are unravelled. The relevant point is that any chamber C is completely specified by $S(C_0, C)$, and if we have $\varphi(C) = (I(C), (\sigma^{(1)}, \dots, \sigma^{(l)}))$, then $I(C)$ specifies the intersection

$$S(C_0, C) \cap \{ax_i + bx_j + cx_k\},$$

while $\sigma^{(1)}, \dots, \sigma^{(l)}$ specify the intersections

$$S(C_0, C) \cap \{x_i - x_j\},$$

respectively. All other assertions follow trivially from this observation. □

Remark. It is not true that the map φ above embeds $P(\mathcal{A}_Z)$ as an *induced* subposet of $HB(r_1, \dots, r_l) \times S_{r_1} \times \cdots \times S_{r_l}$. This can be seen in the case of $Z(1, 2, 1)$. See Fig. 5.

Theorem 5.4. *In the following instances:*

(1) *Z is a $Z(r, s)$ parallelogram,*
(2) *Z is a $Z(r, s, t)$ hexagon in which one of r, s, t is at most 2,*
(3) *Z is a $Z(r, s, 1, 1)$, $Z(s, 1, 1, r)$, $Z(1, 1, r, s)$, or $Z(1, r, s, 1)$ octagon, and*
(4) *Z is a $Z(1, 1, 1, 1, 1)$ decagon,*

the map φ induces an equality of the rank-generating functions of $P(\mathcal{A}_Z)$ and $HB(r_1, \dots, r_l) \times S_{r_1} \times \cdots \times S_{r_l}$. Furthermore, in cases (1), (2) and the $Z(s, 1, 1, r)$ case of (3) we have that

$$\sum_{C \in P(\mathcal{A}_Z)} q^{\mathrm{rank}(C)} = \prod_i [e_i + 1]_q,$$

where $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $\exp(\mathcal{A}_Z) = \{e_i\}$ are the exponents given in Theorem 2.5.

Proof. Since $S_{r_1} \times \cdots \times S_{r_l}$ has cardinality $\prod_m r_m!$, we have already seen in the proof of Theorem 2.6 that in all the cases listed above, the domain and range of φ have the same cardinality. Hence φ induces the equality of generating functions, since it preserves rank. Furthermore, we have that

$$\begin{aligned}\sum_{C \in P(\mathcal{A}_Z)} q^{\mathrm{rank}(C)} &= \left(\sum_{\text{tilings } T \in HB(r_1, \dots, r_l)} q^{\mathrm{rank}(T)} \right) \cdot \prod_m \sum_{\sigma \in S_{r_m}} q^{\mathrm{rank}(\sigma)} \\ &= \left(\sum_{\text{tilings } T \in HB(r_1, \dots, r_l)} q^{\mathrm{rank}(T)} \right) \cdot \prod_m [r_m]!_q,\end{aligned}$$

where we have used the well-known fact (see, e.g., Theorem 2.4(3) of [BEZ]) that the weak Bruhat order on S_n has rank-generating function $[n]!_q$.

To prove the second assertion, in each case we can plug in a known expression for

$$\sum_{\text{tilings } T \in HB(r_1, \dots, r_l)} q^{\mathrm{rank}(T)}.$$

For case (1), this known expression is trivially 1, since there is only one tiling of a $Z(r, s)$. For case (2) this expression is MacMahon's theorem (Theorem 1.1), and for case (3) with $Z = Z(s, 1, 1, r)$ this expression is Elnitsky's theorem (Theorem 1.2).

It is now a simple matter left to the reader to show that, in each case, multiplication of this expression by $\prod_m [r_m]!_q$ agrees with

$$\prod_i [e_i + 1]_q,$$

where $\exp(\mathcal{A}_{r\times s\times t}) = \{e_i\}$ are the exponents given in Theorem 2.5. □

Remarks. (1) The last theorem is related to a question of Björner, and of Terao and Wagreich [W]: Assuming $\mathcal{A}$ is free with exponents $\{e_i\}$, is it always possible to choose a base chamber C_0 so that

$$\sum_{C\in P(\mathcal{A})} q^{\mathrm{rank}(C)} = \prod_i [e_i + 1]_q?$$

The answer was long known to be "yes" for *Coxeter arrangements* and *supersolvable* arrangements [BEZ, Theorem 4.4], and more recently for the *inductively factored arrangements* introduced by Jambu and Paris [JP], but an example for which the answer is "no" was found by Terao [BEZ, p. 277].

The previous theorem shows that the answer is "yes" for the arrangements listed, which would suggest that perhaps these arrangements are inductively factored. This is true and well known for case (1) and the $t = 1$ subcase of (2), but it can be checked by brute force that $\mathcal{A}_{Z(2,2,2)}$ and $\mathcal{A}_{Z(2,2,1,1)}$ do not even satisfy the weaker hypothesis of being *factored* (see Definition 2.66 of [OT]). Therefore the arrangements $\mathcal{A}_{Z(r,s,2)}$ for $t \leq 2$ and $\mathcal{A}_{Z(r,s,1,1)}$ form a new class of examples where this question can be affirmatively answered.

However, even in the cases where $\mathcal{A}_{Z(2,1,2,1)}$ is free it will not have such a base chamber. This was checked using a program written in MATHEMATICA available from the second author. What this shows is that the existence of such a base chamber for a free arrangement $\mathcal{A}$ is not solely dependent on the intersection lattice $L(\mathcal{A})$ (see p. 4 of [OT]) since $L(\mathcal{A}_{Z(2,1,2,1)})$ (when $\mathcal{A}_{Z(2,1,2,1)}$ is free) is isomorphic to $L(\mathcal{A}_{Z(2,2,1,1)})$.

(2) It is worth noting that if we multiply the polynomial in MacMahon's q-count of tilings of a $Z(r, s, t)$ by $[r]!_q \cdot [s]!_q \cdot [t]!_q$, the result

$$\frac{H(r+s+t)H(r)H(s)H(t)}{H(r+2)H(r+t)H(s+t)} \cdot [r]!_q \cdot [s]!_q \cdot [t]!_q$$

does not even factor into a product of terms of the form $[n]_q$ if $r, s, t \geq 3$. For example, if $r = s = t = 3$ one gets

$$\frac{[2]_q[5]_q[6]_q[6]_q[6]_q[7]_q[7]_q[8]_q}{[4]_q},$$

which one can check has no such factorization.

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