Discrete Comput Geom 15:265–285 (1996)

Discrete & Computational
Geometry

Folding Rulers Inside Triangles*

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Abstract. An l-ruler is a chain of n links, each of length l. The links, which are allowed to cross, are modeled by line segments whose endpoints act as joints. A given configuration of an l-ruler is said to fold if it can be moved to a configuration in which all its links coincide. We show that l-rulers confined inside an equilateral triangle of side 1 exhibit the following surprising alternation property: there are three values $x_1 \approx 0.483$, $x_2 = 0.5$, and $x_3 \approx 0.866$ such that all configurations of n-link l-rulers fold if $l \in [0, x_1]$ or $l \in (x_2, x_3]$, but, for any $l \in (x_1, x_2]$ and any $l \in (x_3, 1]$, there are configurations of l-rulers that cannot fold. In the folding cases, linear-time algorithms are given that achieve the folding. Also, a general proof technique is given that can show that certain configurations—in the nonfolding cases—cannot fold.

1. Introduction

A linkage is a collection of rigid rods or links that are fastened together at their endpoints, about which they may rotate freely. Links may cross over one another. A ruler is a chain of links, that is, any endpoint is fastened to at most one other endpoint, and two links have an endpoint that is not fastened to any other endpoint.

Several papers have been written on reconfiguration problems for linkages or rulers from a geometric point of view, including a survey [9]. Hopcroft *et al.* [1] proved that reconfiguration of a linkage so that a designated joint reaches a given position is PSPACE-

* Most of this research was done while the first author was at McGill University supported by an NSERC international fellowship. The second author was supported by NSERC. The third author was supported by NSERC and FCAR.

hard. Joseph and Plantinga [3] proved a similar result for moving rulers amidst obstacles. Hopcroft *et al.* [2] proved that folding a ruler to a segment with at most a specified length is an NP-complete problem, but gave a polynomial-time algorithm for reconfiguring a ruler—of which one point is pinned down to the plane—inside a circle. The running time was improved to linear by Kantabutra and Kosaraju [5]. Kantabutra [4] studied rulers inside a square, with one end fixed and all links of length at most half the side length of the square. He gave a linear-time reconfiguration algorithm. Lenhart and Whitesides [6]–[8] studied the reconfiguration of simple closed chains of links in d dimensions and gave a linear-time reconfiguration algorithm.

We consider a reconfiguration problem for rulers that have all links of equal length and that are confined to an equilateral triangle with unit edge length. The objective is to fold the ruler onto a single link so that all links coincide. This problem is of interest because a confining region having acute angles presents difficulties that have not been studied previously. Also, our results give an additional example of a motion-planning problem that can be solved in linear time despite $n+2$ degrees of freedom.

We call a ruler whose links all have equal length l an l-ruler, and we scale the side of the confining triangle to have length 1. Of course there are l-rulers for l close to 1 that cannot be folded onto a single link, and it is not surprising that, for sufficiently small values of l, all l-rulers fold. However, we have discovered the following surprising phenomenon. For any n and any link length l in the range $[0, x_1]$ with $x_1 \approx 0.483$, any configuration of an n-link l-ruler folds. For $n \geq 3$ and l in the range $(x_1, x_2]$, where $x_2 = 0.5$, there are configurations of n-link l-rulers that do not fold. For any n and l in the range $(x_2, x_3]$, where $x_3 = \sqrt{3}/2 \approx 0.866$, any configuration of an n-link l-ruler folds. For $n \geq 2$ and l in the range $(x_3, 1]$, there are configurations of n-link l-rulers that do not fold. In the cases where the ruler can always be folded, we give linear-time algorithms that accomplish this. In the cases where not every ruler can be folded, we give a configuration that cannot be folded and prove this.

The values x_1, x_2, and x_3 are illustrated in Fig. 1. In the left triangle the l-ruler has one joint at v, the next joint on the side $\overline{uv}$, the third joint on the side $\overline{uw}$, and the last joint on the side $\overline{vw}$. Furthermore, the last link is normal to $\overline{vw}$. This configuration defines $x_1 = \frac{1}{4}(12 + 7\sqrt{3} - (6 + 3\sqrt{3})\sqrt{4\sqrt{3} - 3}) \approx 0.483$. In the middle triangle the first and last joints are at v and w, and the other two joints are on the sides $\overline{uv}$ and $\overline{uw}$. This configuration defines $x_2 = 0.5$. In the right triangle there is one link with one joint at u, the other joint on the side $\overline{vw}$, and this link is normal to $\overline{vw}$. This configuration defines $x_3 = \sqrt{3}/2 \approx 0.866$.

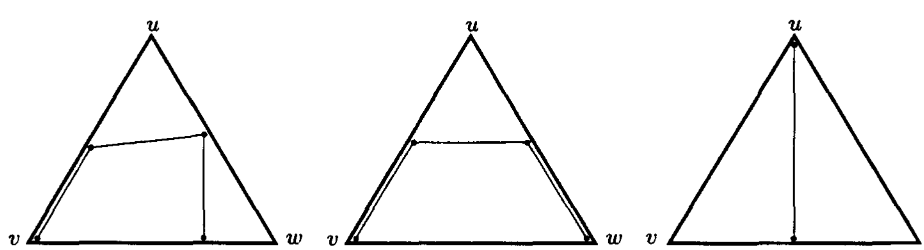


Fig. 1. Illustrations of x_1, x_2, and x_3.

The remainder of this paper is organized as follows. In Section 2 some notation is introduced, and also simple motions of the ruler. In Section 3 we give a linear-time algorithm to fold l-rulers for $l \in (x_2, x_3]$. Section 4 presents a linear-time algorithm to fold l-rulers for $l \in [0, \frac{1}{3}]$, and a sketch of the algorithm for $l \in (\frac{1}{3}, x_1]$. (The Appendix contains the long and highly technical linear-time algorithm for $l \in (\frac{1}{3}, x_1]$.) Nonfoldability of rulers is studied in Section 5. The conclusions are given in Section 6.

2. Preliminaries

We denote the links of an n-link l-ruler by $\ell_1, \ldots, \ell_n$, where link ℓ_i has endpoints j_{i-1} and j_i. The angle at j_i is the angle between links ℓ_{i-1} and ℓ_i; the angle at j_0 is the angle ℓ_1 makes with the positive x-axis. A joint j_i is open if the angle is π radians; a joint is closed if the angle is 0 radians.

We denote the unit-side triangle in which l-rulers are confined by Δ, which we visualize as having a horizontal base $\overline{vw}$ and a top vertex u. Links and joints may lie on the boundary of Δ.

For a joint j_i, we denote with C_i the circle with radius l centered at j_i. This circle may have one, two, or three connected components inside Δ, depending on the position of j_i and the value of l.

Algorithms for the reconfiguration of a ruler usually break up the motions for the whole reconfiguration into simple motions, in which only a few joints are used simultaneously [2], [7]. We allow the following type of simple motions for rulers:

- Some joint j_i of the ruler does not change its position, and at most a constant number of angles at joints between a pair of adjacent links change simultaneously.
- No angles at joints change, but the ruler may translate and rotate as a rigid object.

Note that the joints at which the angles change can be far apart in the ruler. A *dragging motion at joint* j_i is a motion in which the positions of joints j_{i+2} through j_n remain fixed, links ℓ_{i+1} and ℓ_{i+2} act as an elbow to move j_i along some specified line, and j_i drags the first i links so that they translate in the same direction as j_i, see Fig. 2.

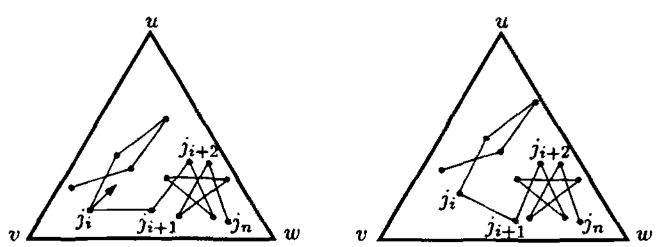


Fig. 2. Dragging motion at joint j_i.

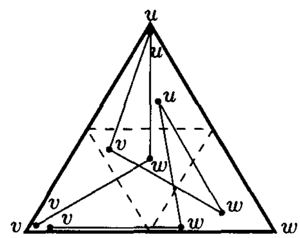


Fig. 3. Labeling the joints of moderately long links.

3. Folding Rulers With Moderately Long Links

We show that any configuration of an n-link l-ruler with $l \in (x_2, x_3]$ can be folded, where $x_2 = 0.5$ and $x_3 = \sqrt{3}/2 \approx 0.866$. The bounds are tight, that is, Section 5 shows that there are configurations of a ruler with $l = 0.5$ that cannot be folded, and the same holds for any $l > \sqrt{3}/2$.

The algorithm to fold an l-ruler with $l \in (x_2, x_3]$ has three phases. The first phase labels all joints in some appropriate way. The second phase brings an arbitrary configuration into one where the joints lie at the vertices of an equilateral triangle inside Δ. The positions correspond to the labels given to the joints. The third phase turns the triangle into a segment.

Divide Δ into four equal-sized equilateral triangles by connecting the midpoints of the sides of Δ (see Fig. 3). Let every joint in the triangle adjacent to u be labeled u, and similarly with v and w. It remains to label the joints in the middle triangle. For any such joint j_i we choose a label that is different from the labels of j_{i-1} and j_{i+1}. If j_{i-1} and j_{i+1} have the same label, say u, then we assign j_i a label depending on the direction of the link ℓ_{i+1}. If j_{i+1} lies to the left of j_i, then j_i is labeled w, otherwise j_i is labeled v.

Lemma 1. *The labeling defined above has the property that joints incident to the same link have different labels.*

Proof. Since $l > 0.5$, no two joints incident to the same link can be in the same one of the four smaller triangles. By choice, the joints in the middle triangle have a label different from the adjacent ones. □

Let Δ' be a homothetic copy of Δ with side length l and vertices u', v', and w'. Vertex u' is the top vertex of Δ', and $\overline{v'w'}$ is the horizontal bottom side. Triangle Δ' will be free to translate inside Δ. A joint j_i can *support* Δ' *at* u' if the placement of Δ' such that u' and j_i coincide is inside Δ.

Lemma 2. *For any two joints j_i, j_{i+1} labeled u, v, either j_i can support Δ' (at u'), or j_{i+1} can support Δ' (at v'), or both.*

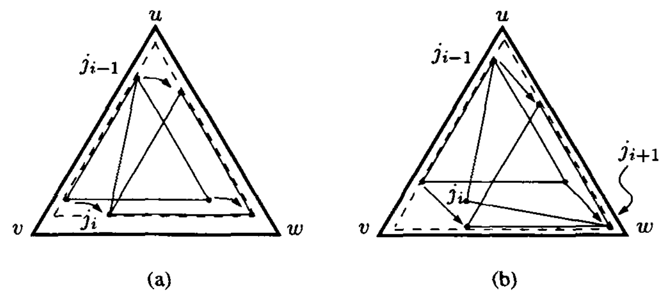


Fig. 4. The motion of Δ' stays inside the dashed triangle and thus inside Δ.

Proof. Assume first that j_i lies closer to $\overline{uv}$ than j_{i+1} does. Then j_{i+1} lies more than $l \cdot \sqrt{3}/2$ in vertical distance below j_i. The base $\overline{v'w'}$ of Δ' lies exactly $l \cdot \sqrt{3}/2$ in vertical distance below j_i. Since j_{i+1} is inside Δ, the triangle Δ' lies above $\overline{vw}$. Since $\overline{u'v'}$ and $\overline{uv}$ are parallel and j_i lies inside Δ, $\overline{u'v'}$ lies to the right of $\overline{uv}$. Similarly, $\overline{u'w'}$ lies to the left of $\overline{uw}$. It follows that Δ' is inside Δ so j_i can support Δ' at u'. The case where v' is closer to $\overline{uv}$ is similar. □

Assume without loss of generality that j_0 is labeled u and j_1 is labeled v. Rotate j_0 counterclockwise around j_1 until it hits $\overline{uv}$. From the proof of the lemma above, j_0 can now support Δ' at u'. By translating Δ' inside Δ, we will wrap the ruler onto Δ', such that any joint with label u will be at u', any joint with label v will be at v', and any joint with label w will be at w'. Assume that we have placed all joints up to j_{i-1} on the vertices of Δ'. Assume without loss of generality that j_{i-1} coincides with u' and j_i has label v. We maintain the invariant that joints $j_i, \ldots, j_n$ have not changed position yet.

First, assume that j_i can support Δ' (see Fig. 4(a)). Then, by changing the angles at joints j_{i-1} and j_i, we let j_i support Δ' at v'. Since the initial and final positions of Δ' lie inside Δ, the circular motions described by the vertices of Δ' are inside Δ. In the figure, Δ' stays inside the dashed triangle.

On the other hand, assume that j_i cannot support Δ'. Then, by Lemma 2, j_{i+1} can support Δ' (see Fig. 4(b)). If j_{i+1} has label w, then Δ' can simply be dragged to its new position where j_{i+1} and w' coincide. The motion causes j_i and v' to coincide as well. Next, assume that j_{i+1} is labeled u. Recall that since j_i is labeled v, joint j_{i+1} is straight above or to the right of j_i. Rotate j_{i-1} around j_i until j_{i-1} and j_{i+1} coincide (so Δ' translates along a circular arc). Then rotate j_i around $j_{i-1} = j_{i+1}$ until it coincides with v'.

In the third phase the ruler on Δ' is incrementally collapsed to a single segment. Consider the positions of j_0, j_1, j_2 on Δ'. If j_0 and j_2 have the same position, then the first two links are folded and the problem reduces to one for an $(n-1)$-link ruler. Otherwise, let u' be the position of j_1, the other cases being symmetrical. Then j_0 is at v' and j_2 is at w' or vice versa. Translate Δ' to make u' coincide with vertex u of Δ, and rotate j_0 around j_1 to coincide with j_2. This is possible because $l \leq \sqrt{3}/2$. Again this leaves a problem with an $(n-1)$-link ruler.

Theorem 3. *Any configuration of an n-link l-ruler with $l \in (x_2, x_3]$ can be folded in linear time, changing at most three joints simultaneously.*

4. Folding Rulers With Short Links

The folding of short n-link l-rulers is split into two algorithms—one deals with $l \in [0, \frac{1}{3}]$ and the other with $l \in (\frac{1}{3}, x_1]$. The latter algorithm is long and technical; its details can be found in the Appendix. We advise the reader not to start with the Appendix before finishing the rest of the paper. A brief sketch of the algorithm, however, is given at the end of this section.

We continue by proving that l-rulers with $l \in [0, \frac{1}{3}]$ can be folded using a linear number of simple motions. The algorithm attempts to fold the first two links, and then solve the remaining problem on an $(n-1)$-link ruler inductively. Alternatively, it can try to fold links ℓ_2, ℓ_3, and ℓ_4, which leaves a folding problem for an $(n-2)$-link ruler. We show that one of these attempts succeeds without moving $j_5, \ldots, j_n$ from their positions.

We begin with a simple observation, and then put j_2 on the boundary of Δ. Recall that C_i is the circle with radius l centered at joint j_i, and that C_i has one or more components inside Δ.

Lemma 4. *If C_1 has j_0 and j_2 on the same component inside Δ, then ℓ_1 and ℓ_2 can be folded without changing the position of j_1.*

Proof. Simply rotate j_0 around j_1 onto j_2. □

Lemma 5. *Without changing the position of j_3, links ℓ_1 and ℓ_2 can be folded, or joints j_1 and j_2 can be put against sides of Δ.*

Proof. Translate j_0 toward j_2. If j_0 reaches j_2, then ℓ_1 and ℓ_2 are folded, otherwise, j_1 has hit a side of Δ. Assume without loss of generality that j_1 has hit $\overline{vw}$, and that j_1 is closer to v. Drag j_1 rightward along $\overline{vw}$ toward the middle, with $j_3 j_2 j_1$ acting as an elbow; note that j_0 cannot hit any side of Δ during this motion. If j_1 reaches the middle, then j_0 can be rotated onto j_2 because C_1 has only one component inside Δ. Otherwise, j_2 has hit the side of Δ, or j_2 is open and the angle $v j_1 j_3$ is at most $\pi/2$ radians. However, then j_1 is at least at distance $2l/\sqrt{3}$ from v, and C_1 has only one component inside Δ. □

Define the *u-triangle* as the equilateral triangle inside Δ with a vertex at u and with side length $l/\sqrt{3}$. Define the v-triangle and the w-triangle similarly. We continue in one of two ways, depending on whether j_2 is in a u-, v-, or w-triangle, or outside all of them.

Lemma 6. *If j_1 and j_2 are on sides of Δ, and j_2 is outside the u-, v-, and w-triangle, then ℓ_1 and ℓ_2 can be folded without changing the position of j_2.*

Proof. Assume without loss of generality that j_2 is on the side $\overline{vw}$, and closer to v than to w (see Fig. 5). If j_1 is on $\overline{vw}$, then either j_0 can be rotated onto j_2 directly, or j_0 can be rotated against $\overline{vw}$ and then dragged toward j_2.

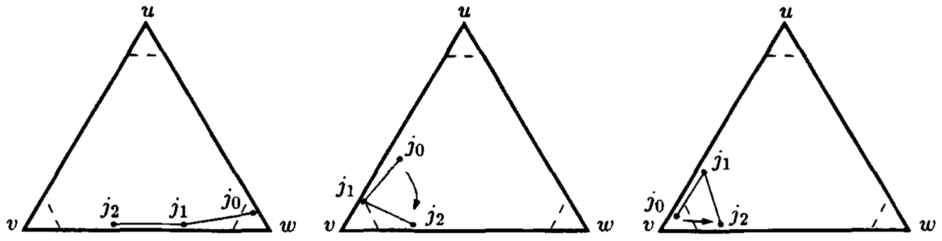


Fig. 5. Three cases of folding ℓ_1 and ℓ_2 when j_2 is on $\overline{vw}$, closer to v, and outside the v-triangle. In the leftmost case, j_0 is rotated clockwise onto $\overline{vw}$ and then dragged to j_2.

If j_1 is against $\overline{uv}$ and below the perpendicular to $\overline{uv}$ through j_2, then j_0 can be rotated around j_1 onto j_2 because C_1 has only one component inside Δ. If j_1 is against $\overline{uv}$ and above the perpendicular to $\overline{uv}$, then the link $j_1 j_2$ divides Δ into two parts. If j_0 is in the triangle $j_1 j_2 v$, then j_0 can be translated onto j_2. If j_0 is in the quadrilateral part, then j_0 can be rotated onto j_2. □

If the above method fails to fold ℓ_1 and ℓ_2, then we will drag j_2 and possibly also j_3 and j_4. First, we wish not to worry about the first two links hitting sides as long as j_2 is in the v-triangle. To this end, we make the links ℓ_1 and ℓ_2 parallel to $\overline{vw}$ with joint j_1 open, and we keep these links this way until specified otherwise. Note that j_1 and j_0 cannot hit any side (in particular, $\overline{uw}$) unless j_2 leaves the v-triangle.

Lemma 7. *If j_2 is in the v-triangle and on $\overline{vw}$, then j_2 can be put outside the v-triangle, or j_2 and j_3 can be put on the same side of Δ, without changing the position of j_4.*

Proof. Drag j_2 along $\overline{vw}$ toward w, keeping j_4's position fixed (see Fig. 6). If j_2 does not get out of the v-triangle, then j_3 has hit $\overline{uv}$ or $\overline{vw}$. If j_3 is on $\overline{uv}$, then rotate j_2 around j_3 to that side as well. □

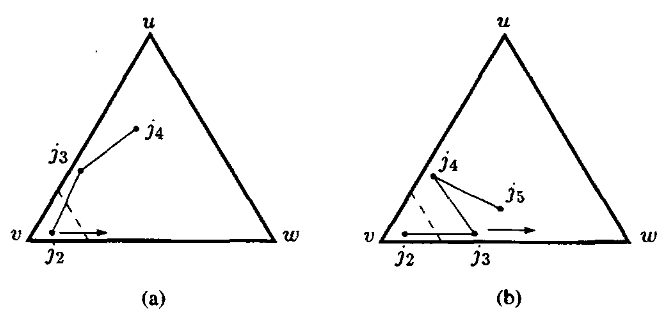


Fig. 6. (a) Putting j_3 on a side, or getting j_2 outside the v-triangle. (b) Getting j_2 outside the v-triangle by dragging j_3.

If j_2 is put outside the v-triangle, then ℓ_1 and ℓ_2 can be folded according to Lemma 6. Otherwise, assume without loss of generality that j_2 and j_3 are both on $\overline{vw}$.

Lemma 8. *If j_2 is in the v-triangle on side $\overline{vw}$ and j_3 is on side $\overline{vw}$, then j_2 can be moved outside the v-triangle, or ℓ_2, ℓ_3, and ℓ_4 can be folded, without changing the position of j_5.*

Proof. Drag j_3 along $\overline{vw}$ toward w, with $j_5 j_4 j_3$ acting as an elbow (see Fig. 6). If j_2 does not leave the v-triangle, then j_4 must have hit a side of Δ. This side cannot be $\overline{uw}$, since the distance from the v-triangle to the side $\overline{uw}$ is greater than $2l$. If the side is $\overline{vw}$ and joint j_3 is open, then j_2 can be dragged toward j_4 (and w), with j_3 leaving $\overline{vw}$. This will bring j_2 outside the v-triangle. If the side is $\overline{vw}$ and joint j_3 is closed, then ℓ_3 and ℓ_4 coincide, and we can make ℓ_2 coincide with these links as well by rotating j_3 around $j_2 = j_4$. If the side hit by j_4 is $\overline{uv}$, then drag j_2 toward w with j_3 leaving the side $\overline{vw}$, and j_2 will leave the v-triangle. This is possible since the angle $\angle j_2 j_3 j_4$ is between $\pi/6$ and $\pi/3$ radians in this case. □

Theorem 9. *Any configuration of an n-link l-ruler with $l \in [0, \frac{1}{3}]$ can be folded in linear time, changing at most three joints simultaneously.*

Proof. The lemmas above show that with only a constant number of simple motions, either ℓ_1 and ℓ_2 can be folded, or ℓ_2, ℓ_3, and ℓ_4 can be folded. Thus the problem reduces to an $(n-1)$-link or $(n-2)$-link l-ruler. The theorem follows by induction. The base cases are easy (observe for instance that imaginary links can be added to one end to reduce the number of cases). □

The remainder of this section contains a brief sketch of the algorithm of which the details are given in the Appendix. The algorithm to fold l-rulers with $l \in (\frac{1}{3}, x_1]$ has some resemblance with the algorithm for folding rulers with moderately long links. From the initial configuration of the ruler, we attempt to reach a situation where all links coincide with the edges of a *trellis* with edge length l, see Fig. 7. The trellis is translated inside Δ to reach this situation. So the trellis plays the same role as Δ' in the algorithm for folding moderately long link rulers. After all links are on the trellis, the ruler is first collapsed to a triangle and then to a single segment. These steps are relatively simple.

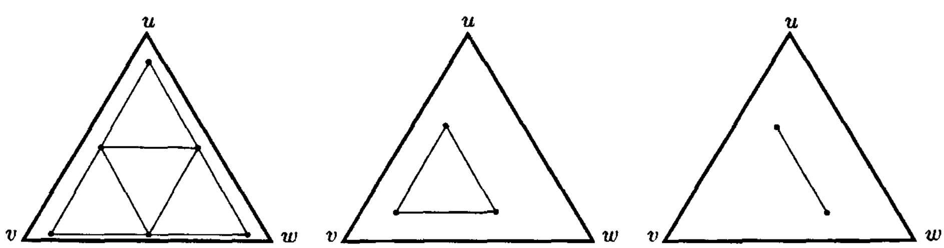


Fig. 7. A trellis with side length l onto which the ruler is put. Then the ruler is collapsed to a triangle and finally to a segment.

To get all joints on the vertices of the trellis, an extensive analysis of 2-link rulers $j_0 j_1 j_2$ is made where joint j_0 is kept fixed and joint j_2 drags along a side of Δ. The analysis contains a study of the cases where this dragging motion is stopped. For rulers with more links, the first joint j_0 is put on a vertex of the trellis and we try to continue to put the next joints onto the vertices. The analysis of two 2-link rulers is used on $j_0 j_1 j_2$ and $j_2 j_3 j_4$, so we know what can happen when j_2 is dragged along a side. This leads to the result that either the next joint j_1 can be put on a vertex of the trellis, or three consecutive links can be folded onto one.

5. Nonfoldable Rulers

It will be shown that not every configuration of an l-ruler is foldable if $l \in (x_3, 1]$ where $x_3 = \sqrt{3}/2 \approx 0.866$, or if $l \in (x_1, x_2]$ where $x_1 \approx 0.483$ and $x_2 = 0.5$. A distinction can be made between two types of nonfoldability. It may be that the ruler is rigidly stuck, or it may be that small motions are possible, but not enough to fold it. Besides giving examples of stuck rulers, we also provide a proof technique to show that a ruler is stuck.

The first example of a rigidly stuck ruler (see Fig. 8) consists of two links of length 1, one coinciding with the side $\overline{uv}$ of Δ, and the other coinciding with $\overline{vw}$. It is easy to see that this configuration cannot be folded, and that it is rigidly stuck. Next, assume that the link length is less than 1, joint j_1 coincides with v, link ℓ_1 lies on the side $\overline{uv}$ and link ℓ_2 lies on the side $\overline{vw}$. This configuration is not rigidly stuck. However, if $l > x_3 = \sqrt{3}/2$, then joint j_0 cannot rotate past the bisector of v to reach joint j_2. Nor can j_2 reach j_0. So the given configuration is nonfoldable.

The second example of a rigidly stuck ruler consists of three links of length 0.5. Joint j_0 coincides with v, joint j_1 coincides with the midpoint of $\overline{uv}$, joint j_2 coincides with the midpoint of $\overline{uw}$, and j_3 coincides with w. As in the previous example, one can decrease the link length slightly and start with roughly the same configuration, and obtain a nonfoldable ruler that is not rigidly stuck. We prove that this example provides a nonfoldable ruler when $l \in (x_1, x_2]$ where $x_1 \approx 0.483$ and $x_2 = 0.5$, by using a proof technique which we explain after the third example.

The third example of a rigidly stuck ruler has nine links of length ≈ 0.483576. This value is slightly larger than $x_1 \approx 0.483481$. Joint j_0 coincides with w, joint j_1 lies on

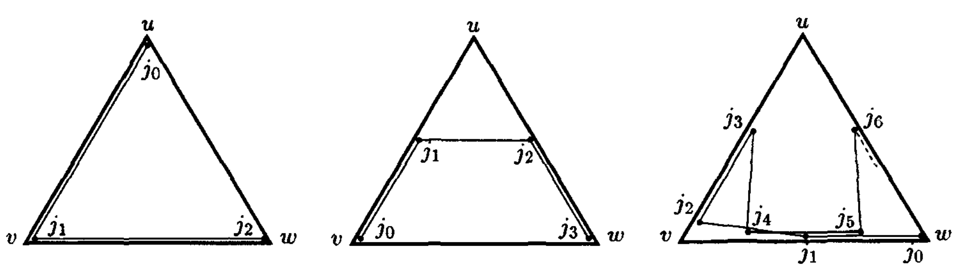


Fig. 8. Three rulers that are rigidly stuck.

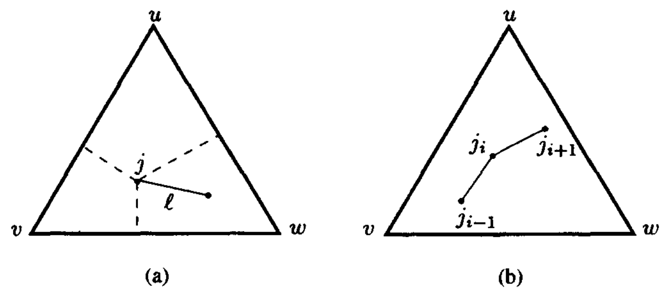


Fig. 9. (a) ℓ is in the w-sector of j. (b) $j_{i-1} j_i j_{i+1}$ make a right turn.

the side $\overline{vw}$, joint j_2 lies on $\overline{uv}$, joint j_3 also lies on $\overline{uv}$, joint j_4 lies on $\overline{vw}$ and, of the two possibilities, closest to v. Joints $j_9, \ldots, j_5$ are the mirror images of $j_0, \ldots, j_4$ when reflected in the bisector at u.

To prove that a configuration of a ruler is stuck, we define the *state* of a configuration, which is a discretization of it. We use the states to show that a given configuration cannot change to a different state. We study the possible *state transitions* for any configuration, and show that none can take place first. A state of a configuration consists of the following items (see Fig. 9):

1. For any joint j and incident link ℓ, draw from the joint j the perpendiculars to the three edges of the triangle Δ. The link ℓ can be in any of the three sectors centered at j, which define one item of the state of the ruler. We denote the sectors as the u-sector, v-sector, and w-sector. The boundaries of the sectors are assigned arbitrarily to one of the incident sectors.
2. For three consecutive joints j_{i-1}, j_i, and j_{i+1}, the sidedness of the triangle $j_{i-1} j_i j_{i+1}$ (a left turn or a right turn) is an item of the state. If joint j_i is open or closed, then one of the possible item instances is assigned arbitrarily.

It follows that any configuration of an n-link ruler with at least two links has $3n - 1$ items in its state. There are two possible state transitions for a configuration of a ruler, for which the following states are critical (in other words, when an item is about to change):

1. A link ℓ makes an angle of $\pi/2$ radians with one of the edges of Δ.
2. Three consecutive joints are collinear (the middle joint is open or closed).

If two consecutive links, both incident to some joint j, are in the same sector, then one need not test whether the three joints incident to these links are collinear with j open. For this to happen, one of the links must leave the sector first. Similarly, if two consecutive links, both incident to some joint j, are in different sectors, then one need not test whether the three joints incident to these links are collinear with j closed.

For a proof that a configuration of a ruler is nonfoldable, the following ideas can be used. It is necessary that the initial and final configurations be in separate connected components. It is sufficient that the initial configuration be in an isolated vertex of the state graph that is different from the final configuration. Following this approach, we show that the configurations of the rulers of the first and second examples are nonfoldable for the appropriate link lengths.

Lemma 10. *For each $l \in (x_3, 1]$, there is a configuration of an l-ruler that cannot be folded (where $x_3 = \sqrt{3}/2$).*

Proof. Consider the configuration of example 1. In a folded configuration of this ruler, links ℓ_1 and ℓ_2 are in the same state with respect to joint j_1. For the initial configuration of example 1, this is not the case. We consider which critical state can occur as the first one (possibly, simultaneously with others). Consider the state of joint j_1 and link ℓ_1. The link ℓ_1 is in the u-sector with respect to j_1. If ℓ_1 were to change its state to be in the w-sector, then ℓ_1 must make an angle of $\pi/2$ radians with the side uw, but this is impossible, because Δ cannot contain a link with the given link length perpendicular to any of its sides. The other transitions of the first type cannot occur for the same reasons. A transition of the second type can occur in one of two forms. Joint j_1 can be open, i.e., j_0 and j_2 are distance $2l$ apart, or joint j_2 can be closed, i.e., j_0 and j_2 coincide. Clearly, Δ cannot contain a configuration of this ruler with j_1 open. Also, j_1 cannot close unless another state transition occurs before or simultaneously, because when j_1 closes the links ℓ_1 and ℓ_2 are in the same state with respect to j_1. □

Lemma 11. *For each $l \in (x_1, x_2]$, there is a configuration of an l-ruler that cannot be folded (where $x_1 = \frac{1}{4}(12 + 7\sqrt{3} - (6 + 3\sqrt{3})\sqrt{4\sqrt{3} - 3}) \approx 0.483$ and $x_2 = 0.5$).*

Proof. Consider the configuration of example 2 in the middle of Fig. 8. In a folded configuration of this ruler, links ℓ_1 and ℓ_2 are in the same state with respect to joint j_1. For the initial configuration this is not the case. We consider which critical state can occur as the first one (possibly, simultaneously with others), and for what values of l. Consider link ℓ_3, which is in the w-sector with respect to joint j_2. Assume that the first state transition brings ℓ_3 in the v-sector. Then j_2 must lie at least a distance l above the side $\overline{vw}$ in the critical state. Since link ℓ_2 is in the v-sector with respect to j_2, link ℓ_1 is in the v-sector with respect to j_1, and j_0, j_1, j_2 make a right turn, the ruler in this critical configuration only fits inside Δ if $l \leq x_1$ (from straightforward calculations using the left configuration in Fig. 1 the value x_1 is obtained).

Assume that the first state transition brings ℓ_2 in the u-sector with respect to j_1. It can be calculated that in this case $l \leq 2\sqrt{3} - 3 \approx 0.464$.

Next, assume that the first state transition brings ℓ_3 in the u-sector with respect to j_2. This state transition can never occur as the first, since the state of ℓ_3 with respect to j_3 will always change before. The other possible state transitions of this type can be handled similarly.

Consider joints j_0, j_1, j_2, which make a right turn, and assume that the first state transition brings this into a left turn. Since ℓ_1 and ℓ_2 are in different sectors with respect to j_1, joint j_1 cannot close without having another state transition before or simultaneously. Furthermore, ℓ_1 is in the u-sector of j_0 and ℓ_1 is in the w-sector of j_1. If joint j_1 is open, these sectors must be the same. Therefore, another state transition must occur before or simultaneously. Hence, we need not consider state changes for three consecutive joints as the first state change. □

6. Conclusions

We have studied folding an n-link ruler with equal length links inside an equilateral triangle. This paper gives one of the first results on the reconfiguration of rulers when there are acute angles that constrain the motion of the ruler. Even in the simple setting of this paper, a surprising result shows up: rulers with short links can always be folded, rulers with midsize links cannot always be folded, rulers with fairly long links can always be folded, and rulers with long links cannot always be folded. We showed these results using techniques that can be used in other ruler-folding situations as well.

We have not considered the question whether a given configuration can be folded in the ranges of the link length where folding is not necessarily possible. In the case of long links the question is easy to answer, but for midsize links the problem is open. We also do not know whether there are more rigidly stuck configurations than the three we found.

When considering other confining regions than equilateral triangles, the situation may be quite different. We do not know whether the alternation property on link lengths with respect to foldability also shows up for regular k-gons with $k \geq 4$. When an arbitrary triangle is the confining region, or when different link lengths are allowed, a brief study showed that the situation is exceedingly difficult.

Acknowledgments

The authors thank Bill Lenhart for interesting discussions on the reconfiguration of rulers, and the referees for their excellent comments and careful refereeing.

Appendix. Folding l-Rulers for $\frac{1}{3} < l \leq x_1 \approx 0.483$

In Section 3 we folded moderately long l-rulers onto an equilateral triangle with side length l and then folded this triangle. In this appendix we fold l-rulers for $0 < l \leq x_1 \approx 0.483$ onto a *trellis*. Then we fold the trellis to a triangle and fold the triangle. We prove in the reverse order that these three foldings are possible.

A *trellis* is composed of four equilateral triangles of side length l—three *corner* triangles homothetic to Δ and one upside-down *center* triangle, as in Fig. 10. If we translate a trellis in Δ, keeping sides parallel, then the six vertices of the trellis sweep out six equilateral *frame triangles*, also shown in Fig. 10. These are called the u, v, and w frame triangles for the corners, and the uv, uw, and vw frame triangles for the others.

Recall that C_i denotes the circle with radius l that is centered at joint j_i, and A_i denotes the set of circular arcs that are the connected components of $C_i \cap \Delta$. The *vw-fence* is the line segment that is the intersection of Δ with a line parallel to $\overline{vw}$ at distance l from

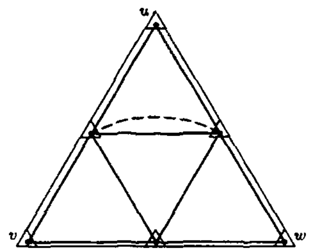


Fig. 10. The trellis and frame.

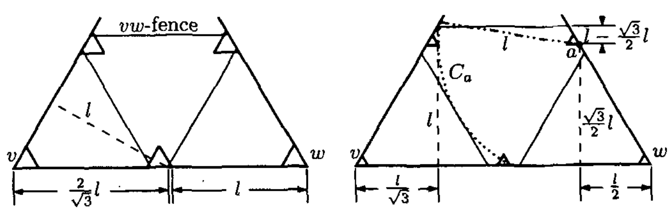
$\overline{vw}$. We say that a joint j_i is *above the* vw*-fence* if the disk bounded by circle C_i does not intersect the line $\overline{vw}$. Define the uv-fence and uw-fence similarly.

There are critical values for l that determine the relationship between middle frame triangles and fences. We assume throughout this appendix that $\frac{1}{3} < l \leq x_1$, which is the larger critical value.

Lemma 12. *If* $l \leq 2\sqrt{3} - 3 \approx 0.464$, *then any point of* $\overline{vw}$ *is above the* uv*- or* uw*-fence or is in the* vw *frame triangle. Let* a *be the corner of the* uw *frame triangle nearest* w. *If* $l \leq x_1 \approx 0.483$, *then the circle* C_a *of radius* l *intersects* $\overline{uv}$ *above the* vw*-fence.*

Proof. Figure 11(a) illustrates the first part of the lemma: the fences touch the middle frame triangles when $l \leq 1 - 2l/\sqrt{3}$. Figure 11(b) illustrates the second part: the lemma is satisfied if l is at most the distance between the lower right corner of the uw frame triangle and the left end of the vw-fence. That is, if

$$l^2 \leq \left(l - \left(\frac{\sqrt{3}}{2}\right)l\right)^2 + \left(1 - \frac{l}{\sqrt{3}} - \frac{l}{2}\right)^2. \qquad \square$$



(a) (b)

Fig. 11. The relationship between fences and frame depends on l.

A.1. *Folding a Triangle onto a Link, Folding a Trellis onto a Triangle*

To begin, we prove that any prefix of links that lie on edges of the center triangle in a trellis can be folded without moving the trellis to a single link.

Lemma 13. *Let $\ell_1, \ell_2, \ldots, \ell_k$ be a ruler on the center triangle τ of a trellis. Then the ruler can be folded onto ℓ_k inside the trellis.*

Proof. The circular sector formed by pivoting link ℓ_1 about joint j_1 onto link ℓ_2 is entirely within the trellis. By induction, we can therefore fold all links onto ℓ_k. □

Lemma 14. *A ruler on a trellis can be folded to a single segment if $l \leq x_1 \approx 0.483$.*

Proof. As an induction hypothesis, suppose that all links from ℓ_1 to ℓ_i, for some $i \geq 1$, lie on τ, which is a corner triangle of the trellis. This is easy to obtain in the base case: link ℓ_1, being on the trellis, is an edge of a unique corner triangle that can be chosen as τ.

To reduce the number of cases in the induction step, we always fold the ruler onto a triangle τ that has one vertex in the corner of the trellis—if we ever put τ in the center of the trellis, then Lemma 13 says that we can fold the links on τ to a single link and take a new triangle τ that is incident to this link and a corner of the trellis.

If the next link ℓ_{i+1} is already on τ, then nothing needs to be done. Otherwise, we have three cases depicted in Fig. 12 for folding ℓ_{i+1} onto τ, which depend on the locations of joints j_{i+1} and j_{i-1}.

Case 1. Joints j_{i+1} and j_{i-1} are in the corners of the trellis. Then j_i is at the side between j_{i-1} and j_{i+1}. Translate τ, moving j_{i-1} toward j_{i+1} and j_i away from the side of the trellis until τ is again a corner triangle of the trellis and has vertices j_{i-1}, j_i, and j_{i+1}.

Case 2. Joints j_i and j_{i-1} lie at the sides of the trellis; joint j_{i+1} lies at the side or corner. Rotate j_i to bring j_{i-1} to the side near j_{i+1} while rotating j_{i-1} to keep τ homothetic to the corner triangles. This also makes τ a corner triangle of the trellis having vertices j_{i-1}, j_i, and j_{i+1}.

Case 3. Joint j_{i+1} is on a side. The triangle τ must touch the opposite corner of the trellis or else joint j_{i+1} and link ℓ_{i+1} are already on τ. This is the most complicated case—it

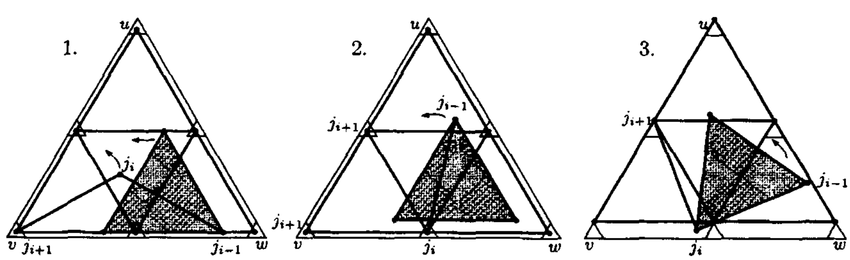


Fig. 12. Cases for folding the trellis.

cannot be folded inside the trellis, but can be folded inside a unit equilateral triangle if $l \leq x_1$. To prove this, let us be more specific about the locations of the trellis and the joints.

Let joint j_{i+1} be at the side $\overline{uv}$ of the triangle Δ, let j_i be at the side $\overline{vw}$, and let τ be near w. We translate the trellis so that one of its vertices coincides with u. Next, we pivot j_i about j_{i+1}, keeping τ homothetic to Δ, until j_i moves above the uw-fence. By Lemma 12 the triangle τ can then swing freely on C_i to hit $\overline{uv}$ at j_{i+1}. Next, rotate about j_{i+1} to bring j_i back onto the trellis, making τ the center triangle. Finally, fold τ to a segment according to Lemma 13 and choose a new τ incident to this segment and a corner of the trellis. This completes Case 3.

At the completion of these cases, we have all the links folded onto a corner triangle. We can move this triangle to the center and fold it according to Lemma 13. □

A.2. An Analysis of Two-Link Rulers

In this section we study the motion of a two link ruler when one end is dragged along the side of the triangle Δ. This dragging motion is the primary tool in the next and final section, which folds a ruler onto the trellis. We look at configurations where joints are on the sides of Δ. With a two link ruler *abc*, for example, we place *c* on a side and drag it, pivotting on *a*, until *b* hits a side (or joints go onto a trellis). This reduces the problem to three and then to two degrees of freedom—the placement of *a* (which we draw in Fig. 14). Thus, by proving lemmas about these contact configurations, we avoid having to look at the entire configuration space.

Consider a ruler consisting of two segments $\overline{ab}$ and $\overline{bc}$, where *c* is along the $\overline{vw}$ side of Δ. Let $\overline{vw}$ be horizontal with *w* on the right. We say that a *wall* is any portion of an edge of Δ that is not contained in a frame triangle. In the next lemmas we investigate how *b* can hit a wall when we drag *c* along $\overline{vw}$. Figure 13 illustrates these different cases.

Lemma 15. *Given a ruler abc with c on $\overline{vw}$. If we fix the location of a and drag c toward w, then one of the following occurs:*

1. *Joint c or b reaches a frame triangle.*

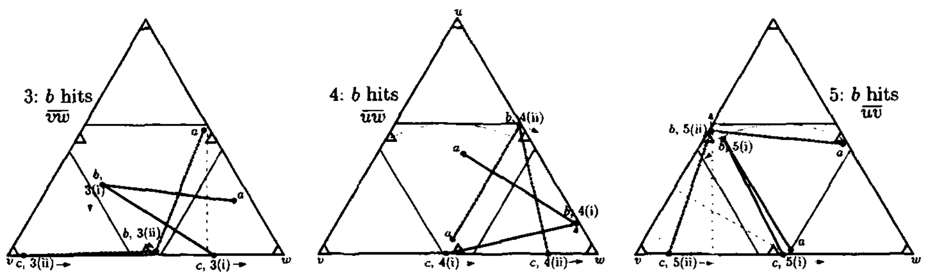


Fig. 13. Illustrations of the cases of Lemma 15, in which *b* hits a wall as *c* is dragged along $\overline{vw}$ toward *w*.

2. *Joints a, b, and c become collinear.*
3. *Joint b hits a wall on $\overline{vw}$*
   (i) *between the v and vw frame triangles, or*
   (ii) *between the vw frame triangle and the vertical line through the right endpoint of the vw-fence.*
4. *With $l > 2\sqrt{3} - 3 \approx 0.464$, joint b hits a wall on $\overline{uw}$*
   (i) *inside the circle C centered at the left corner of the vw frame triangle, or*
   (ii) *between the vw-fence and the uw frame triangle.*
5. *With $l > 2\sqrt{3} - 3 \approx 0.464$, joint b hits a wall on $\overline{uv}$*
   (i) *inside the circle C centered at the right corner of the vw frame triangle, or*
   (ii) *between the vw-fence and the uv frame triangle.*

Proof. The only events that can prevent c from reaching the frame triangle at w are joint b hitting a side of Δ or the ruler abc straightening. We can look at the cases in which b hits sides of Δ without b or c being in a frame triangle. Note that b is below the vw-fence since c is on $\overline{vw}$.

In case 3(ii) joint a must be right of the vertical line through b, or else dragging c right would move b away from the wall. However, b must then be to the left of the vertical line through the right endpoint of the vw-fence.

In 4(ii) there is room for b between the vw-fence and the uw frame triangle only if link length $l > 2\sqrt{3} - 3 \approx 0.464$, by Lemma 12. In 4(i), c must be between the uw-fence and the vw frame triangle for b to hit the wall between the uw and w frame triangles. The fact that c is to the left of the vw frame triangle means that b hits inside the circle centered at the left corner of the vw frame triangle. This case occurs only if $l > 2\sqrt{3} - 3 \approx 0.464$. Furthermore, b is above the uv-fence if $l \leq x_1 \approx 0.483$ by Lemma 12.

The cases for 5(i) and (ii) are similar to those for 4(i) and (ii). □

We can characterize the locations for a (and c) in terms of the location that b hits the wall. For example, a lies on C_b when b is on the wall—additional conditions may restrict which portion of C_b. One can determine all locations for a that cause b to hit a certain wall segment by taking the union of the appropriate portions of C_b for all positions where b hits that segment. Figure 14 illustrates the regions for a that are described in the next lemma.

Lemma 16. *When b is on a wall, we have additional restrictions in the following cases of Lemma* 15:

3. *Joint a is below the vw-fence and to the right of the vertical line through b.*
4. (i) *Joint a is above the* 30° *line through b.*
   (ii) *Joint c is either* (A) *left or* (B) *right of the vertical line through b. Joint a is either* (A) *right of the vertical or else* (B) *left of the vertical through b and below the* 30° *line through b.* (*Actually, a can be coincident with c in* (B), *but then the three joints are collinear.*)
5. (i) *Joint a is below the* $-30°$ *line through b or coincident with c.*
   (ii) *Joint c is either* (A) *left or* (B) *right of the vertical line through b. Joint a is*

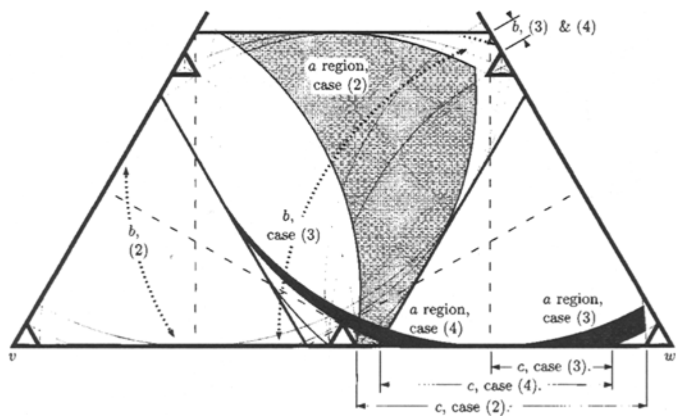


Fig. 14. Locations for a that make b hit a wall in the cases of Lemma 15 as c is dragged along $\overline{vw}$ toward w. (Small circles mark the centers of relevant arcs.)

either (A) *left of the vertical or above the* $-30°$ *line or else* (B) *right of the vertical and below the* $-30°$ *line through* b.

Proof. Since a is fixed, joint b moves along C_a in a direction determined by the motion of c. The conditions on a (and c) ensure that this motion is into the wall. □

Next we look at what can happen when we try to drag c either right or left. The cases are illustrated in Fig. 15.

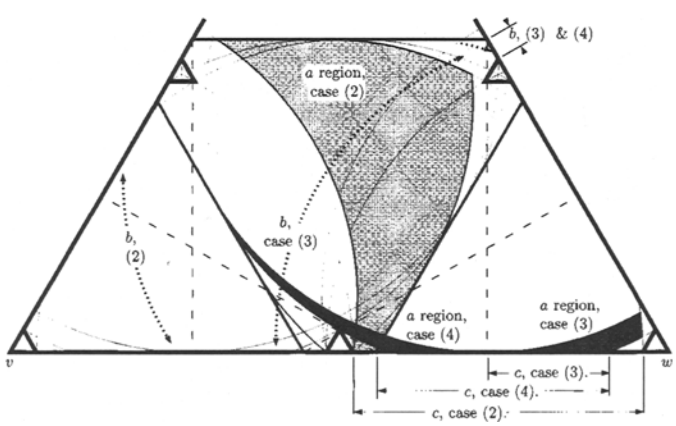


Fig. 15. Locations for a and b that prevent motion of c both left and right.

Corollary 17. *Given a ruler abc with c on $\overline{vw}$, by dragging c toward v and w we get b or c into a frame triangle unless*

(1) *joints a, b, and c become collinear,*
(2) *case* 3(i) *of Lemma* 15 *applies in one direction and* 4(i) *in the other,*
(3) *cases* 3(ii) *and* 5(iiA) *of Lemma* 15 *apply, or*
(4) *cases* 4(iiB) *and* 5(iiB) *of Lemma* 15 *apply.*

Proof. If we take the union of the a regions described in Lemma 16 and intersect them with the reflection about a vertical line, then we find the positions in which a can prevent motion in both directions. Figure 15 illustrates the combinations for which the regions for a intersect and the resulting sliding ranges for c on $\overline{vw}$ remain between the vw and w frame triangles. The motions of b are also shown. (Other potential combinations in which regions for a intersect are 3(i) and *refl*(3(ii)), 3(i) and *refl*(5(iiB)), and 4(iiA) and *refl*(5(iiA)). These do not appear because the conditions regarding vertical lines cannot be met by sliding c.) □

By way of a remark, if $l \leq 2\sqrt{3} - 3$, then cases 2–4 of Corollary 17 cannot apply.

A.3. *Folding a Ruler onto the Trellis*

We are finally ready to fold a ruler with length $l \leq x_1 \approx 0.483$ onto the trellis. We first put joint j_0 into a frame triangle (and thus onto the trellis), then we look at how the two-link rulers $j_0 j_1 j_2$ and $j_4 j_3 j_2$ work together and show that by dragging j_2 either j_1 can be moved onto the trellis or three links can be folded to one. Once we have the first joint on the trellis, frame triangles can be a big help.

Lemma 18. *Suppose that j_0 is on the trellis. If one of the joints j_1, j_2, j_3, or j_4 ever gets into a frame triangle, then we can put j_1 onto the trellis.*

Proof. If j_1 is in a frame triangle, then we can drag the trellis by moving j_0 on C_1 until j_1 is on the trellis. If joint j_2, j_3, or j_4 is in a frame triangle, then we can drag the trellis toward that joint until a lower-numbered joint enters a frame triangle. □

Now, consider the ruler $j_0, j_1, \ldots, j_n$.

Lemma 19. *Given a ruler $j_0, j_1, \ldots,$ one can move j_0 into a frame triangle or fold the first two links.*

Proof. Consider the ruler $j_2 j_1 j_0$ with the position of j_2 fixed. If j_1 or j_2 are in frame triangles, then we can put j_0 into a frame triangle. Otherwise, rotate j_0 to a wall and apply Corollary 17. The only way for $j_2 j_1 j_0$ to be collinear in Δ minus the frame triangles is to fold j_0 onto j_2. If, on the other hand, one of cases (2)–(4) holds, then dragging j_0 along the wall moves j_1 above some fence so that j_0 can rotate on C_1 to j_2. □

We make one more useful observation. If we can put two joints together above a fence, then we can fold three links to one.

Lemma 20. *Given a ruler with joints abcd, if a and c are positioned at a common point above some fence, then we can fold all three links onto $\overline{cd}$ without moving d.*

Proof. Joints b and d lie on the single arc $A_c = A_a$. □

Lemma 21. *If j_0 is on the trellis, we can put j_1 onto the trellis or fold three links to one by rotating at most seven joints.*

Proof. We apply our analysis of two-link rulers to $j_0 j_1 j_2$ and $j_4 j_3 j_2$. First, we make sure that collinearity can never prevent joint j_2 from reaching a frame triangle. Then we rotate j_2 to a wall and drag it until $j_0 j_1 j_2$ or $j_4 j_3 j_2$ stop the motion according to Corollary 17. We handle mixed cases—where $j_4 j_3 j_2$ prevents motion of j_2 in one direction and $j_0 j_1 j_2$ prevents motion in the other—by reducing them to cases where the ruler $j_0 j_1 j_2$ does not restrict the motion of j_2. Finally, we show how to solve these cases by folding three links to one or moving a joint into a frame triangle and applying Lemma 18.

If j_0 is in a corner frame triangle, then we move the trellis away from this corner, pivoting on j_2, until j_0 is at the edge of the frame triangle strictly inside Δ. (Notice that if joint j_1 hits an edge of Δ during this process, then j_1 is in a frame triangle.) Now, since Δ minus the corner frame triangles has diameter at most $2l$ and j_0 is in the frame inside this region, any future collinearity of $j_0 j_1 j_2$ will imply that j_2 has entered a frame triangle.

Since j_0 is in a frame triangle, j_1 is on an arc of A_1 that intersects a frame triangle. We can move j_1 into that frame triangle, pivoting on j_3, unless j_2 hits a wall. By rotating and reflecting Δ, we can assume that this wall is $\overline{vw}$.

Suppose, without loss of generality, that the ruler $j_0 j_1 j_2$ does not allow j_2 to slide freely to the right. We will show either how to satisfy the theorem or else arrange that one joint (j_2 or j_3) can slide without restriction from preceding links. Since j_0 is in a frame triangle, j_2 can be restricted only by cases 3(i), 3(ii), or 5(iiA) of Lemma 15—only these cases have a region for a that intersects a frame triangle. (See Figs. 14 and 16.)

Case 5(iiA). This case is the easiest—we move the trellis to have a vertex at w and j_0 moves out of the critical region and no longer restricts the motion of j_2. (This is because the arc A_0 goes above the vw-fence after the move.)

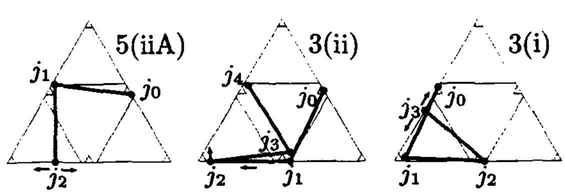


Fig. 16. Dealing with cases in which $j_0 j_1 j_2$ restricts j_2.

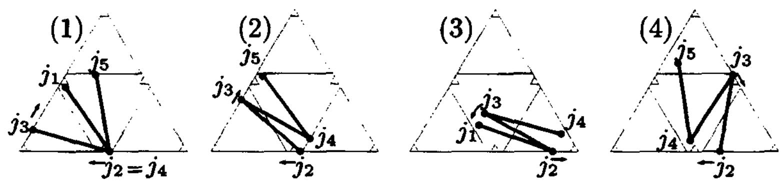


Fig. 17. Only the ruler $j_4 j_3 j_2$ restricts j_2.

Case 3(ii). Joint j_1 is between the vw frame triangle and the uw-fence, which means that j_2 is near v. If we drag j_2 toward v, Lemma 15 implies that only the ruler $j_4 j_3 j_2$ can prevent j_2's entry into the v frame triangle.

If joints j_4, j_3, and j_2 become collinear by folding, then $j_2 = j_4$ and Lemma 20 implies that we can fold three links. With any other collinearity, j_4 is in the frame. The only case of Lemma 15 that applies to the ruler $j_4 j_3 j_2$ is 3(i). (Joint j_2 is too close to v for 4(iiB).) In that case, drag j_2 and j_1 along $\overline{wv}$, pivoting on j_4 and moving the trellis as necessary. Joint j_3 hits the wall at j_1. Next, move j_2 on $C_3 = C_1$ to $\overline{uv}$ and move the trellis to u. Then the ruler $j_0 j_1 j_2$ does not restrict the motion of j_2 on $\overline{uv}$.

Case 3(i). Joint j_2 is below the uv-fence and can move to the vw frame triangle unless j_3 hits $\overline{uv}$ according to case 4(i). However, then the trellis can be moved to u so that j_2 can slide freely between the uv-fence and the vw frame triangle. Thus, j_3 can slide on $\overline{uv}$ without constraint from $j_0 j_1 j_2 j_3$.

We can now slide a joint freely along a wall, with respect to preceding links. We call the joint j_2 and assume that the wall is between the vw and v frame triangles on $\overline{vw}$. According to Corollary 17, we can put j_2 or j_3 onto the frame unless $j_4 j_3 j_2$ become collinear or $l > 2\sqrt{3} - 3 \approx 0.464$ and one of cases (2)–(4) depicted in Fig. 15 (and Fig. 17) occurs.

Case 3. This is the easiest case. Joint j_2 (as c) is always above the uw-fence, so A_2 has one connected component. Joint j_3 sweeps this component, so must hit j_1. Then the positions of j_2 and j_4 place them on the same connected component of $A_3 = A_1$; we can move j_2 to fold $j_1 j_2 j_3 j_4$ to a single link.

Case (4). In this case, joint j_3 stops j_2 from reaching the vw frame triangle by hitting $\overline{uw}$ according to 5(iiB). Move j_2 as close to the vw frame triangle as possible. Apply Lemma 15 to ruler $j_5 j_4 j_3$ in an attempt to drag j_3 into the uw frame triangle. (Notice that we can slide j_2 toward the vw frame triangle so that j_2 never prevents this motion of j_3.) One of four outcomes occurs. First, if j_3 reaches the frame, then we are done by Lemma 18. Second, if j_4 exits the case (4) region of Fig. 15, then we are done because j_2 is no longer restricted in both directions by $j_4 j_3 j_2$. Third, if j_4 hits a wall in the case (4) region, then it does so at j_2 and above the uv-fence; Lemma 20 says we can fold three links to one. Finally, if j_5, j_4, and j_3 become collinear, then $j_5 = j_3$. Joints j_2 and j_4 are on the same connected component of $A_5 = A_3$, so moving j_3 folds $j_2 j_3 j_4 j_5$ to a single link.

Case (2). In this case, joint j_3 stops j_2 from reaching the vw frame triangle by hitting

$\overline{uv}$ above the uw-fence. Attempt to drag j_3 on $\overline{uv}$; notice that we can slide j_2 so that it never prevents the motion of j_3.

Either j_3 reaches the frame triangle at v, and we are done by Lemma 18, or j_3 goes below the uw-fence and j_2 enters the vw frame triangle, or one of the cases of Corollary 17 occur for $j_5 j_4 j_3$. In case (1), joint j_3 becomes coincident with j_5 above the uw-fence and Lemma 20 says that we can fold $j_2 j_3 j_4 j_5$ to a single link. We need not consider case (2), because there j_3 goes below the uw-fence. In cases (3) and (4) we slide j_3 as far toward the uv frame triangle as possible and j_4 hits $\overline{vw}$ at j_2. Now, j_3 and j_5 are on the same connected component of $A_3 = A_5$ and we can again fold $j_2 j_3 j_4 j_5$.

Case (1). In the last case, j_4, j_3, and j_2 become collinear. If one of these joints is in a frame triangle, then Lemma 18 applies—this must occur if the ruler $j_4 j_3 j_2$ straightens. Otherwise, $j_4 j_3 j_2$ folds so that $j_2 = j_4$.

If $j_2 = j_4$ is above a fence, then Lemma 20 applies. Otherwise, we have two components of $A_2 = A_4$. If joint j_3 is on a component that intersects a frame triangle or one of joints j_1 or j_5, then we are done by Lemma 18 or by folding three links to one. In the remaining case, which is illustrated in Fig. 17(1), joint j_3 can be moved to $\overline{uv}$ and dragged into the uv frame triangle without interference from the rulers $j_1 j_2 j_3$ or $j_5 j_4 j_3$.

This completes the proof that joints can be moved onto the trellis or links folded. Since our motions affect at most three links before and three links after the freely sliding vertex, we move at most seven joints. □

References

1. Hopcroft, J., D. Joseph, and S. Whitesides, Movement problems for two-dimensional linkages, *SIAM J. Comput.* **13** (1984), 610–629.
2. Hopcroft, J., D. Joseph, and S. Whitesides, On the movement of robot arms in two-dimensional bounded regions, *SIAM J. Comput.* **14** (1985), 315–333.
3. Joseph, D., and W. H. Plantinga, On the complexity of reachability and motion planning questions, *Proc. 1st ACM Symp. on Computational Geometry*, 1985, pp. 62–66.
4. Kantabutra, V., Motions of a short-linked robot arm in a square, *Discrete Comput. Geom.* **7** (1992), 69–76.
5. Kantabutra, V., and S. R. Kosaraju, New algorithms for multilink robot arms, *J. Comput. System Sci.* **32** (1986), 136–153.
6. Lenhart, W., and S. Whitesides, Turning a polygon inside out, *Proc. 3rd Canadian Conf. on Computational Geometry*, 1991, pp. 66–69.
7. Lenhart, W., and S. Whitesides, Reconfiguration using line tracking motions, *Proc. 4th Canadian Conf. on Computational Geometry*, 1992, pp. 198–203.
8. Lenhart, W., and S. Whitesides, Reconfiguring closed polygonal chains in euclidean d-space, *Discrete Comput. Geom.*, **13** (1995), 123–140.
9. Whitesides, S., Algorithmic issues in the geometry of planar linkage movement, *Austral. Comput. J.* **24** (1992), 42–50.

Received October 2, 1994.