

The Maximum Number of Odd Integral Distances Between Points in the Plane

L. Piepmeyer*

Technische Universität Braunschweig, Diskrete Mathematik,
Pockelsstrasse 14, D-38106 Braunschweig, Germany

Dedicated to Professor Dr. H.-J. Kanold on the occasion of his eightieth birthday

Abstract. Four points in the plane with pairwise odd integral distances do not exist. The maximum number of odd distances between n points in the plane is proved to be $n^2/3 + r(r-3)/6$ for all n , where $r = 1, 2, 3$ and $n \equiv r \pmod{3}$. This solves a recently stated problem of Erdős.

For every dimension d and for every natural number n there exist n points in the Euclidean space E^d with pairwise integral distances, since n equidistributed points on a line determine by an appropriate scaling only integral distances. However, the maximum number of points with pairwise odd integral distances in E^d is $d + 2$ if $d + 2 \equiv 0 \pmod{16}$ and $d + 1$ in the remaining cases (see [2]; an independent proof was given in [4]). Thus four points in the plane with mutual odd distances do not exist.

Erdős [1] asked for the maximum number $f(n)$ of odd integral distances occurring between n points in the plane. This maximum $f(n)$ is determined in Theorem 1. The proof is based on a construction given in [3]. Moreover, we make use of Turán's theorem from graph theory (see p. 30 of [5]).

Turán's Theorem. *The maximum number of edges in a graph with n vertices that does not contain a complete subgraph with four vertices is*

$$\frac{n^2}{3} + \frac{r(r-3)}{6} \quad \text{with } r = 1, 2, \text{ or } 3, \text{ and } n \equiv r \pmod{3}.$$

The upper bound is attained if and only if the graph is isomorphic to the complete 3-partite graph T_n with three vertex classes that have almost the same size.

* Current address: Morassistrasse 24, 80469 Munich, Germany. lotharp@informix.com.

Theorem 1. *The maximum number $f(n)$ of odd integral distances between n points in the plane is*

$$f(n) = \frac{n^2}{3} + \frac{r(r-3)}{6} \quad \text{with } r = 1, 2, \text{ or } 3, \text{ and } n \equiv r \pmod{3}.$$

Proof. The upper bound $f(n) \leq n^2/3 + r(r-3)/6$ follows immediately from Turán’s theorem if we consider a set \mathcal{S} of n points in the plane with the maximum number of odd integral distances together with the graph $G(\mathcal{S})$ induced by the points and the odd distances of \mathcal{S} . Since $G(\mathcal{S})$ does not contain a complete graph with four vertices the number of edges in $G(\mathcal{S})$ and hence the upper bound for the number of odd integral distances in \mathcal{S} follows by Turán’s theorem.

The lower bound $f(n) \geq n^2/3 + r(r-3)/6$ follows by construction of a set $\mathcal{P}_m = \{p_0, \dots, p_{3m-1}\}$ of $n = 3m$ points in the complex plane with $3m^2$ odd integral distances for each natural number m . Since $G(\mathcal{P}_m)$ is isomorphic to T_{3m} by Turán’s theorem, the deletion of one or two appropriate vertices leaves a T_{3m-1} or a T_{3m-2} so that the corresponding point sets prove the lower bound for $n = 3m - 1$ or $n = 3m - 2$.

For the third root of unity $\rho = (-1 + \sqrt{3}i)/2$ with $i^2 = -1$, and for $\omega = 3 + \rho$ let $\eta_k = 7^{m-\lfloor k/3 \rfloor} \rho^k \omega^{2\lfloor k/3 \rfloor}$, and $p_k = \sqrt{3}\eta_k^2 / (3 \cdot 7^m)$ with $0 \leq k < 3m$. This is the special case $R = 7^m = (\omega\bar{\omega})^m$ of a construction given in [3] (see Fig. 1 for the case $m = 2$). The points of the construction consist of the vertices of a regular triangle of side lengths 7^m that is rotated $m - 1$ times about its center through an angle $\theta = \arccos 71/98$.

In [3] it is shown that $\bar{\eta}_s \eta_t = x + \rho 7^m |p_s - p_t|$ for $0 \leq s < t < 3m$ and with a real number x . Comparing the imaginary parts implies

$$|p_s - p_t| = \frac{\text{Im}(\bar{\eta}_s \eta_t)}{7^m \text{Im}(\rho)} = \frac{(2\sqrt{3}/3) \text{Im}(\bar{\eta}_s \eta_t)}{7^m}.$$

With $\bar{\rho} = \rho^2$ and $\omega\bar{\omega} = 7$ it follows that

$$\bar{\eta}_s \eta_t = 7^{2m+\lfloor s/3 \rfloor - \lfloor t/3 \rfloor} \rho^{t+2s} \omega^{2\lfloor t/3 \rfloor - 2\lfloor s/3 \rfloor}.$$

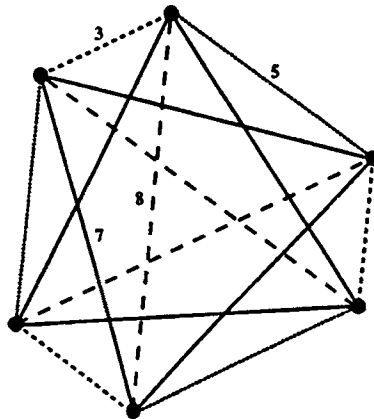


Fig. 1. Six points with 12 odd distances.

Since all occurring distances $|p_s - p_t|$ are integers as shown in [3] the number of odd distances equals the number of odd numbers $(2\sqrt{3}/3) \operatorname{Im}(\rho^{t+2s} \omega^{2\lfloor t/3 \rfloor - 2\lfloor s/3 \rfloor})$, with $0 \leq s < t < 3m$.

With $\omega^2 = 8 + 5\rho$, $\rho^2 = -1 - \rho$, and $\rho^3 = 1$, and with

$$(8 + 5\rho)^k = \sum_{v \equiv 0 \pmod{3}} 8^{k-v} 5^v \binom{k}{v} + \rho \sum_{v \equiv 1 \pmod{3}} 8^{k-v} 5^v \binom{k}{v} - (1 + \rho) \sum_{v \equiv 2 \pmod{3}} 8^{k-v} 5^v \binom{k}{v}$$

it follows that ω^{2k} and ρ^k are both of the form $a + b\rho$ with integers a, b satisfying $a \equiv 1, 0, 1 \pmod{2}$ and $b \equiv 0, 1, 1 \pmod{2}$ for $k \equiv 0, 1, 2 \pmod{3}$, respectively. Thus $(2\sqrt{3}/3) \operatorname{Im}(\omega^{2k} \rho^t)$ for fixed k is an odd number in exactly two of the three cases $1 = 0, 1, 2$. Now it follows for all pairs s, t with $0 \leq \lfloor s/3 \rfloor < \lfloor t/3 \rfloor < m$ that exactly six of all nine distances $|p_s - p_t|$ are odd. In the remaining m cases where $\lfloor s/3 \rfloor = \lfloor t/3 \rfloor$ all three distances $|p_s - p_t|$ are odd. Altogether there are $6 \binom{m}{2} + 3m = 3m^2$ odd integral distances in \mathcal{P}_m and the proof is complete. □

References

1. P. Erdős, Oral communication, Conference on Graph Theory, Combinatorics, and Computation, 1994, Boca Raton, FL.
2. R. L. Graham, B. L. Rothschild, and E. G. Straus, Are there $n + 2$ points in E^n with pairwise odd integral distances? *Amer. Math. Monthly* **81** (1974), 21–25.
3. H. Harborth, A. Kemnitz, and M. Möller, An upper bound for the minimum diameter of integral point sets, *Discrete Comput. Geom.* **9** (1993), 427–432.
4. M. Rosenfield, Oral communication, Conference on Graph Theory, Combinatorics, and Computation, 1994, Boca Raton, FL.
5. J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

Received February 27, 1995, and in revised form August 31, 1995.