Discrete Comput Geom 16:33–54 (1996)

Discrete & Computational
Geometry

An Algorithm To Compute Orders and Ramification Indices of Cyclic Actions on Compact Surfaces, II*

E. Bujalance,[1] A. F. Costa,[1] J. M. Gamboa,[2] and J. Lafuente[3]

[1]Departamento de Matemáticas Fundamentales, Fac. Ciencias,
U.N.E.D., 28040 Madrid, Spain
emilio.bujalance@uned.es
antonio.costa@uned.es

[2]Departamento de Algebra, Fac. C. Matemáticas,
Universidad Complutense, 28040 Madrid, Spain

[3]Departamento de Geometría y Topología. Fac. C. Matemáticas,
Universidad Complutense, 28040 Madrid, Spain

Abstract. In this paper we obtain an effective algorithm to compute all even orders and ramification indices of homeomorphisms of finite order acting on compact surfaces, orientable or not. This completes the case of odd orders, previously studied by the authors.

Introduction

The study of actions of finite groups on compact topological surfaces goes back to Wiman [8], and a partial list of the vast literature on the subject produced in the last 20 years can be seen in the references of [4].

Although most authors restrict themselves to orientation-preserving actions on orientable surfaces without boundary, we get an algorithm to compute all orders and ramification indices of homeomorphisms of finite order acting on compact surfaces, possibly with boundary, orientable or not. Obtaining this information is the first step in determining the homeomorphism groups of surfaces. As is well known, the problem is equivalent to computing the data for automorphisms on compact Klein surfaces, i.e., compact topological surfaces, orientable or not, possibly with boundary, equipped with a dianalytic

* E. Bujalance and A. F. Costa were partially supported by DGICYT PB 92-0716 and CEE-CHRX-CT93-0408. J. M. Gamboa was partially supported by DGICYT PB 92-0498-C02-02 and CEE-CHRX-CT93-0408. J. Lafuente was partially supported by DGICYT PB 92-0220.

structure. Our result is directly applicable to the birational geometry of real or complex algebraic curves, using the functorial correspondence between compact Klein surfaces and projective, smooth, irreducible algebraic curves as stated in [1].

Throughout the paper we only consider actions of even order. The odd case was solved by the authors in [3], but now the situation is more involved. In fact, if f is a homeomorphism of finite order on the surface S, we can attach to the pair (S, f) the numbers $\sigma_1(S, f) = 2$ or 1 according to S being orientable or not, $\sigma_2(S, f) = 2$ or 1 according to the quotient S_f of S under the action of f being orientable or not and, in case S is orientable, $\tau(S, f) = 1$ or -1 according to f being orientation-preserving or not. For fixed g_0 and k_0 and fixed σ_1, σ_2, and τ we want to determine the numbers N such that there exist a surface S of genus g_0 and k_0 boundary components and a homeomorphism f on S of order N such that $\sigma_i(S, f) = \sigma_i$, $\tau(S, f) = \tau$. For odd N it is obvious that $\sigma_1(S, f) = \sigma_2(S, f)$ and if this value is 1, then $\tau(S, f) = 1$, but in the even case only the triple $(\sigma_1, \sigma_2, \tau) = (2, 1, 1)$ cannot occur. As we shall see later, another extra difficulty in the even case is the existence of corner points for the canonical projection $S \to S_f$, that is, ramification points in the boundaries. Thirdly, as is quite obvious, the results of the odd case are required to compute even orders. We freely use them throughout the paper, which is organized as follows: in Section 1 we state precisely the problem and introduce the notation and some auxiliary functions to be used later. A finite number of cases, which are not covered by the general algorithms, are studied in Section 2. The main results are stated and proved in Sections 3–7, according to the orientability character of the surface and its quotient and, dealing with orientable surfaces, the orientation-preserving (resp. reversing) character of the homeomorphism. Some explicit examples are explained in Section 8.

1. Preliminaries

Let S_f be the quotient of the compact surface S under the action of the cyclic group of order N generated by the homeomorphism f on S. The topological data of the canonical covering $\pi_f: S \to S_f$ are the following integers:

(i) The genus g_0 and the number k_0 of boundary components of S.
(ii) The genus g and the number k of boundary components of S_f.
(iii) The ramification R_f of π_f at interior points of S_f. It is a finite set of pairs $R_f = \{(m_1, \mu_1), \dots, (m_r, \mu_r)\}$ where μ_i is a positive integer, $m_1, \dots, m_r$ are distinct integers larger than or equal to two, with the obvious meaning: for every $1 \leq i \leq r$ there are μ_i interior points in S_f over which π_f ramifies with multiplicity m_i.
(iv) The action of π_f on the set of connected components of the boundary ∂S of S is codified by a set of pairs $A_f = \{(l_1, \lambda_1), \dots, (l_p, \lambda_p)\}$ where l_i, λ_i are positive integers, $l_i \neq l_j$ if $i \neq j$, with the following geometrical meaning: for every $j = 1, \dots, p$ there are exactly λ_j blocks of N/l_j boundary components of S which are mapped by π_f onto the same connected component of ∂S_f.
(v) The total number $2c$ of corner points of π_f.

(vi) The integer $\alpha = \alpha(S) = 2$ or 1 according to S being orientable or not. The integer $\sigma = \sigma(S, f) = 2$ or 1 according to S_f being orientable or not. In case $\alpha = 2$, we define $\tau = \tau(S, f) = 1$ or -1 according to f reverses the orientation of S or not.

This notation is fixed throughout the paper.

Definition and Remark 1.1. The ordered list

$$(g_0, k_0, \alpha; g, k, \sigma, \tau; m_1, \dots, m_r, \mu_1, \dots, \mu_r, l_1, \dots, l_p, \lambda_1, \dots, \lambda_p, c)$$

is called the topological data of the projection $\pi_f: S \to S_f$.

For technical reasons it is convenient to introduce two more integers. First, $p_0(S) = p_0 = \alpha g_0 + k_0 - 1$ which, except in case of classical surfaces, i.e., $(k_0, \alpha) = (0, 2)$, is the genus of the canonical double cover of S. Of course, $p_0 \geq 0$ except for $g_0 = k_0 = 0$, i.e., for spheres. Secondly, the deficiency of the covering π_f is the nonnegative integer

$$k' = \begin{cases} k - \sum_{i=1}^{p} \lambda_i & \text{if} \quad \sigma = 1 \quad \text{or} \quad (\alpha, \tau) = (2, 1), \\ k - \sum_{i=1}^{p} \lambda_i - 1 & \text{otherwise.} \end{cases} \tag{1.0}$$

Counting the number of boundary components of S and using the Riemann–Hurwitz formula we get the fundamental equalities

$$\begin{cases} k_0 - c\dfrac{N}{2} = \sum_{i=1}^{p} \dfrac{N}{l_i}\lambda_i, \\ p_0 - c\dfrac{N}{2} - 1 = N\left[\sigma g + k - 2 + \sum_{i=1}^{r}\left(1 - \dfrac{1}{m_i}\right)\mu_i\right]. \end{cases} \tag{1.1}$$

Statement of the Problem. The input of the algorithm is a triple (g_0, k_0, α), where g_0 and k_0 are nonnegative integers and $\alpha = 1$ or 2. We denote by $K(g_0, k_0, \alpha)$ the family of (orientable if $\alpha = 2$, nonorientable if $\alpha = 1$) compact Klein surfaces of genus g_0 whose boundary has k_0 connected components. Without any further reference we use the sets O_α of odd orders of automorphisms acting on surfaces in $K(g_0, k_0, \alpha)$ which were computed in [3]. We define the sets $E_1(\sigma)$, $E_2(-1, \sigma)$, $E_2(1)$, $\sigma = 1, 2$, as follows: $E_1(\sigma)$ is the set of even orders of automorphisms on surfaces in $K(g_0, k_0, 1)$ with an orientable quotient if $\sigma = 2$ and nonorientable if $\sigma = 1$. The sets $E_2(1)$ and $E_2(-1, \sigma)$ are defined in the same way; $E_2(1)$ refers to orientation-preserving automorphisms on orientable surfaces and $E_2(-1, \sigma)$ corresponds to orientation-reversing automorphisms on orientable surfaces, with $\sigma = 2$ for an orientable quotient and $\sigma = 1$ in the other case. Note that the quotient of an orientable surface under the action of an orientation-preserving automorphism is necessarily orientable, and so all possibilities are covered by these five sets. The output of the algorithm consists of these E-sets and, for every even integer N in each of them, the topological data of the covering $S \to S_f$ associated to all automorphisms f of order N acting on surfaces $S \in K(g_0, k_0, \alpha)$ in the prescribed way.

The algorithms have been implemented in the language C by the fourth author in the operating systems MSDOS and UNIX. All computations can be carried out practically. In Example 8.1 all information concerning automorphisms on orientable surfaces of genus 12 with two boundary components is given. In fact, the use of the computer gave us evidence of the veracity of the following surprising result:

Theorem. (See Theorem 7.2.) *Let* (g_0, k_0, α) *be fixed and let* E *be one of the sets* $E_1(\sigma)$, $E_2(-1, \sigma)$, $E_2(1)$. *If* D *is even divisor of* $N \in E$, *then* $D \in E$.

The starting point to produce the algorithm is to obtain necessary and sufficient conditions of the existence of even cyclic actions in terms of diophantine equations in the parameters of the branching data.

Remarks 1.2. (1) Let g_0, k_0, α, σ, and N be given. The existence of a solution of the system of diophantine equations (1.1) is a necessary, but in general not sufficient as we shall see later, condition for N occurring in the corresponding set E. Note that since N is bounded *a priori* in terms of the input—see 1.4—all variables in (1.1) are bounded by above as a function of (g_0, k_0, α). In particular, $c \leq 2k_0/N$.

(2) Let E be one of the sets $E_1(\sigma)$, $E_2(-1, \sigma)$, $E_2(1)$ and let f be an automorphism of order $N \in E$ on a surface $S \in K(g_0, k_0, \alpha)$ acting in the prescribed way. We can write $N = M \cdot 2^e$ for some $e \geq 1$ and some odd M, and $g = f^{2^e}$, $h = f^M$ are automorphisms on S of orders M and 2^e, respectively. Hence, M being odd, it follows that $M \in O_\alpha$ and $2^e \in E$. Thus, as a general strategy, we look first for the set P of 2-powers occurring in E to determine after what elements of $P \cdot O_\alpha$ actually occur in E. In both steps, Theorem 7.2 quoted above simplifies the computations.

In order to avoid unnecessary repetition, we now introduce some auxiliary functions which are used to obtain sharp upper bounds for orders of automorphisms of the form 2^e.

Definition 1.3. Given integers $k_0 \geq 1$ and $e \geq 1$, the "2-*adic expansion of* k_0 *with respect to* e" is given by the formula

$$k_0 = \sum_{j=0}^{e-1} a_j(e, k_0) \cdot 2^j, \qquad 0 \leq a_j \leq 1 \quad \text{if} \quad j \neq e-1, \quad a_{e-1} \geq 0.$$

For $i = 1, 2$, we define

$$\rho_i : \mathbf{N} \times \mathbf{N} \to \mathbf{N} : (e, k_0) \mapsto a_{e-1}(e, k_0) + i \sum_{j=0}^{e-2} a_j(e, k_0).$$

For $i = 1, 2, 3$ we define $\varepsilon_i \colon \mathbf{N} \times (\mathbf{N} \cup \{0\}) \to \mathbf{N}$ as follows:

$$\varepsilon_1(e, k_0) = \begin{cases} 3, & k_0 = 0, \quad e < 3 \quad \text{or} \quad (e, k_0) = (1, 2), \\ 2^{e-1} - 1, & k_0 = 0, \quad e \geq 3, \\ 2, & (k_0, e) = (1, 1), \\ 2^{e-1}, & k_0 = 1, \quad e > 1, \\ 2^{e-1} + 1, & k_0 = 2, \quad e > 1, \\ 2^e(\rho_1(e, k_0) - 1), & k_0 > 2 \quad \text{is odd}, \\ 2^e(\rho_1(e, k_0) - 1) + 1, & k_0 > 2, \quad k_0 \equiv 2 \pmod 4, \quad e > 1, \\ 2^e \rho_1(e, k_0) - 1, & \text{otherwise}. \end{cases}$$

$$\varepsilon_2(e, k_0) = \begin{cases} 2^{e-1} + 1, & k_0 = 1 \quad \text{or} \quad (k_0, e) = (0, 1), \\ 2^{e-1}, & k_0 = 0, \quad e > 1, \\ k_0, & k_0 > 1, \quad e = 1, \\ 2^{e-1}(\rho_2(e, k_0) - 2) + 1, & k_0 > 1 \text{ is odd}, \quad e > 1, \\ 2^{e-1}\rho_2(e, k_0), & \text{otherwise}. \end{cases}$$

$$\varepsilon_3(e, k_0) = \begin{cases} 3, & k_0 = 0, \quad e \leq 2, \\ 2^{e-1} - 1, & k_0 = 0, \quad e > 2, \\ k_0 + 1, & k_0 \leq 2, \quad e = 1, \\ k_0 - 1, & k_0 > 2, \quad e = 1, \\ 5, & k_0 = e = 2, \\ 2^{e-1} + 1, & k_0 = 2, \quad e > 2, \\ 2^e \left(\rho_1 \left(e - 1, \frac{k_0}{2} \right) - 1 \right), & k_0 > 2, \quad k_0 \equiv 0 \pmod 4, \quad e \geq 2, \\ 2^e \left(\rho_1 \left(e - 1, \frac{k_0}{2} \right) - 1 \right) + 1, & k_0 > 2, \quad k_0 \equiv 2 \pmod 4, \quad e \geq 2, \\ 1, & \text{otherwise}. \end{cases}$$

The first step to produce an algorithm to compute the E-sets above is to obtain sharp upper bounds on the number of their elements. Until recently this has been an unsolved problem in general, but a complete solution has been obtained for prime-powers in E, in case $k_0 > 0$, in [5]. This together with the classical bound of Wiman and the results for $k_0 = 0$ in [2], [6], and [7] provides us with the following:

Proposition 1.4. *Let g_0, k_0, α, σ, and N be given, $p_0 = \alpha g_0 + k_0 - 1$. Let E be one of the sets $E_1(\sigma)$, $E_1(1)$, $E_1(-1, \sigma)$. Then:*

$$N \leq \begin{cases} 2(p_0 + 1) & \text{if} \quad N \in E_1(\sigma) \quad \text{or} \quad N \in E, \quad k_0 > 0, \\ 2(p_0 + 2) & \text{if} \quad N \in E_2(1), \quad k_0 = 0, \\ 2(p_0 + 3) & \text{if} \quad N \in E_2(-1, \sigma), \quad k_0 = 0. \end{cases}$$

Moreover, if $N = 2^e$ for some $e \geq 1$, then

$$p_0 \geq \begin{cases} \varepsilon_2(e, k_0) & \text{if} \quad 2^e \in E_1(\sigma), \\ \varepsilon_1(e, k_0) & \text{if} \quad 2^e \in E_2(1), \\ \varepsilon_3(e, k_0) & \text{if} \quad 2^e \in E_1(-1, 1). \end{cases}$$

2. Even Cyclic Actions on Surfaces with $p_0 \leq 1$

The general algorithm to be developed in the forthcoming sections holds for surfaces admitting the upper half-plane as the universal covering. In this section we deal with surfaces $S \in K(g_0, k_0, \alpha)$ whose universal covering is either the sphere or the euclidean plane, i.e., $p_0 = \alpha g_0 + k_0 - 1 \leq 1$. For every automorphism f on S of finite order, the topological data of the covering $S \to S_f$ satisfy, from (1.1), the inequality

$$\sigma g + k - 2 + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) \mu_i + \frac{c}{2} \leq 0. \tag{2.1}$$

Since c can be written as

$$c = \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \quad \text{for some integers} \quad n_{ij} \geq 2, \tag{2.2}$$

it is easy to obtain by inspection all solutions of (2.1), which are listed in Table 2.1. We use the symbol $*$ when $k = 0$, and nothing in the fifth column indicates the empty set. Except for the annulus, and in this case $c = 0$, the number k of boundary components of the quotient satisfies $k \leq 1$ and so we denote for $k = 1$ the ramification indices in the boundary as $n_1, \ldots, n_{s_1}$.

In Table 2.1, the first 17 cases, i.e., those with $p_0 = 1$, correspond to the well known 17 plane euclidean groups. The strategy now is the following: for each row in Table 2.1, we look for all surfaces S and all automorphisms f of finite order on S such that $p_0(S) \leq 1$ and the topological data of the covering $S \to S_f = M$ are the ones of the chosen row. Let M be the *candidate* to be S_f with these data and let Λ be its fundamental group as orbifold. Then we must search all epimorphisms from Λ onto cyclic groups whose kernel is the fundamental group of some compact surface, the S we are looking for.

According with the data of the chosen row, the group Λ has the following presentation by means of generators and relations (see [4]):

Generators:

$$x_1, \ldots, x_r; e_1, \ldots, e_k; c_{io}, \ldots, c_{is_i}, \qquad i = 1, \ldots, k, \quad \text{and}$$
$$a_1, b_1, \ldots, a_g, b_g \quad \text{if} \quad \sigma = 2 \qquad \text{or} \qquad d_1, \ldots, d_g \quad \text{if} \quad \sigma = 1,$$

Relations:

$$x_l^{m_l} = e_i^{-1} c_{i0} e_i c_{is_i} = c_{i,j}^2 = (c_{i,j-1} c_{i,j})^{m_{ij}} = 1,$$
$$x_1, \ldots, x_r \cdot e_1 \cdots e_k \cdot [a_1, b_1] \cdots [a_g, b_g] = 1 \qquad \text{if} \quad \sigma = 2,$$
$$x_1, \ldots, x_r \cdot e_1 \cdots e_k \cdot d_1^2 \ldots d_g^2 = 1 \qquad \text{if} \quad \sigma = 1. \tag{2.3}$$

Note that the condition $p_0(S) \leq 1$ means that S is one of the following surfaces: sphere, disk, projective plane, torus, annulus, Möbius strip, or Klein bottle. The study of all epimorphisms from such groups Λ onto cyclic groups involves much computation. For a guide to the reader we carefully develop three examples. The complete final result is displayed in Table 2.2.

Table 2.1

Surface $S_f = M$	p_0	g	k	σ	$m_1, \dots, m_r; \mu_1, \dots, \mu_r$	$n_1, \dots, n_s$
Torus	1	1	0	2		*
Annulus	1	0	2	2		
	1	0	1	2	$m_1 = \mu_1 = 2$	
	1	0	1	2	$m_1 = 2; \mu_1 = 1$	2, 2
	1	0	1	2		2, 2, 2, 2
	1	0	1	2	$m_1 = 4; \mu_1 = 1$	2
	1	0	1	2	$m_1 = 3; \mu_1 = 1$	3
	1	0	1	2		2, 4, 4
	1	0	1	2		3, 3, 3
	1	0	1	2		2, 3, 6
	1	0	0	2	$m_1 = 2; \mu_1 = 4$	*
	1	0	0	2	$m_1 = 2; \mu_1 = 1; m_2 = 2; \mu_2 = 4$	*
	1	0	0	2	$m_1 = 3; \mu_1 = 3$	*
	1	0	0	2	$m_1 = 2; m_2 = 3, m_3 = 6; \mu_1 = \mu_2 = \mu_3 = 1$	*
Klein bottle	1	2	0	1		*
Möbius strip	1	1	1	1		
	1	1	0	1	$m_1 = \mu_1 = 2$	*
Sphere	0	0	0	2		*
	0	0	0	2	$m_1 = m; \mu_1 = 2$	*
	0	0	0	2	$m_1 = 2; \mu_1 = 1; m_2 = 3; \mu_2 = 2$	*
	0	0	0	2	$m_1 = 2; m_2 = 3, m_3 = 4; \mu_1 = \mu_2 = \mu_3 = 1$	*
	0	0	0	2	$m_1 = 2; m_2 = 3, m_3 = 5; \mu_1 = \mu_2 = \mu_3 = 1$	*
Disk	0	0	1	2		
	0	0	1	2	$m_1 = m; \mu_1 = 1$	
	0	0	1	2		$n_1 = n_2 = n$
	0	0	1	2		2, 3, 3
	0	0	1	2	$m_1 = 3; \mu_1 = 1$	2
	0	0	1	2		2, 3, 3
	0	0	1	2		2, 3, 4
	0	0	1	2		2, 3, 5
Projective plane	0	1	0	1		*
	0	1	0	1	$m_1 = m; \mu_1 = 1$	*

Example 2.1. (i) We analyze the third row in the Table 2.1. The group Λ has the presentation

$$\Lambda = \langle x_1, x_2, e, c;\ x_1 x_2 e = x_1^2 = x_2^2 = c^2 = 1 \rangle.$$

Let N be an even positive integer and let $\theta\colon \Lambda \to \mathbf{Z}_N$ be an epimorphism onto the cyclic group of order N whose kernel is the fundamental group of some compact surface S. Then

$$2\theta(c) = 0; \qquad \theta(x_1) = \theta(x_2) = \frac{N}{2}, \quad \text{and so} \quad \theta(e) = 0.$$

Hence $N = 2$ since θ is surjective, and two possibilities can occur:

(1) $\theta(c) = 0$. In this case S is orientable with $k_0 = N/(\text{order of } \theta(e)) = 2$ boundary components with genus $g_0 = 0$, because $2g_0 + k_0 - 1 = p_0 \leq 1$. Consequently S is an annulus and the projection $\pi_f\colon S \to S_f = M$ associated to the induced automorphism f on S maps both components of ∂S onto the unique one of ∂M. From Corollary 3.2.3 in [4], f preserves the orientation of S.

Table 2.2 ($p_0 \leq 1$)

Surface	p_0	g_0	k_0	g	k	σ	E	(m_i, μ_i)	(l_j, λ_j)	c
Even cyclic actions on nonorientable surfaces: $E_1(\sigma)$										
Klein surface	1	2	0	2	2	2	N		*	
	1	2	0	0	1	2	2	$m_1 = 2, \mu_1 = 2$	*	
	1	2	0	1	1	1	N		*	
	1	2	0	1	0	1	2	$m_1 = 2, \mu_1 = 2$	*	
Möbius strip	1	1	1	0	2	2	N		$l_1 = N, \lambda_1 = 2$	
	1	1	1	0	1	2	2	$m_1 = 2, \mu_1 = 1$	$l_1 = 2, \lambda_1 = 1$	
Projective plane	0	1	0	0	1	1	N	$m_1 = N, \mu_1 = 1$	*	
Even orientation-preserving cyclic actions on orientable surfaces: $E_2(1)$										
Torus	1	1	0	1	0	2	N		*	
	1	1	0	0	0	2	2	$m_1 = 2, \mu_1 = 1; m_2 = 2, \mu_2 = 4$	*	
	1	1	0	0	0	2	6	$m_1 = 2, m_2 = 3, m_3 = 6, \mu_1 = \mu_2 = \mu_3 = 1$	*	
Annulus	1	0	2	0	2	2	N		$l_1 = N/2, \lambda_1 = 2$	
	1	0	2	0	1	2	2	$m_1 = 2, \mu_1 = 2$	$l_1 = 1, \lambda_1 = 1$	
Sphere	0	0	0	0	0	2	N	$m_1 = N, \mu_1 = 2$	*	
Disk	0	0	1	0	1	2	N	$m_1 = N, \mu_1 = 1$	$l_1 = N, \lambda_1 = 1$	
Even orientation-reversing cyclic actions on orientable surfaces: $E_2(-1, \sigma)$										
Torus	1	1	0	0	2	2	$N, N/2$ odd		*	*
	1	1	0	1	1	1	$N, N/2$ odd		*	*
	1	1	0	2	0	1	N		*	*
Annulus	1	0	2	0	2	2	$N, N/2$ odd		$l_1 = N/2, \lambda_1 = 2$	*
	1	0	2	0	1	2	2		*	2
	1	0	2	1	1	1	N		$l_1 = N/2, \lambda_1 = 2$	*
Sphere	0	0	0	0	1	2	$N, N/2$ odd	$m_1 = N/2, \mu_1 = 1$	*	*
	0	0	0	1	0	1	N	$m_1 = N/2, \mu_1 = 1$	*	*
	0	0	0	1	0	1	2		*	*
Disk	0	0	1	0	1	2	2		*	1

(2) $\theta(c) \neq 0$. Consequently S is nonorientable with $k_0 = 0$ boundary components, because $x_1 \cdot c \in \ker\theta$, and since $1 = p_0 = g_0 + k_0 - 1$, S has genus $g_0 = 2$, i.e., S is a Klein bottle.

Both possibilities actually occur, as is shown by the epimorphisms induced by the assignments:

$$1 = \theta(x_1) = \theta(x_2); \qquad \theta(e) = 0; \qquad \theta(c) = 0 \text{ or } 1.$$

(ii) The fundamental group as an orbifold of the Möbius strip M occurring in the 16th row of Table 2.1 is presented as

$$\Lambda = \langle d, e, c; ed^2 = c^2 = 1\rangle.$$

Let $\theta: \Lambda \to \mathbf{Z}_N$ be an epimorhism whose kernel Γ is the fundamental group, as an orbifold, of a surface $S \in K(g_0, k_0, \alpha)$ with $\alpha g_0 + k_0 - 1 = p_0(S) = 1$. We should distinguish several subcases:

(1) $c \in \Gamma$. Then $x = \theta(d)$ generates $\mathbf{Z}_N$, since $\theta(e) = -2\theta(d)$, and so $\theta(e) = -2x$ has order $N/2$, which implies $k_0 = 2$. Therefore $\alpha = 2$, $g_0 = 0$, i.e., S is an annulus. Since M is nonorientable, the induced automorphism f reverses the orientation of S, and the covering $\pi_f: S \to S_f = M$ maps the boundary components of the annulus to the one component of M.

(2) $c \notin \Gamma$. Now $y = \theta(c)$ has order 2 and $\{y, x = \theta(d)\}$ generate $\mathbf{Z}_N$. If $q =$ order of x, $N = \text{l.c.m.}\{2, q\}$ and two possibilities occur:

(2.1) $q \neq N$. Then $q = N/2$ must be odd, and also $\theta(e)$ has order q. In particular, $d^q \in \Gamma$ and, q being odd, S is nonorientable with $k_0 = 0$ because $c \notin \Gamma$. Thus $g_0 = 2$, i.e., S is a Klein bottle.

(2.2) $q = N$. Again $k_0 = 0$ and $d^{N/2} \cdot c \in \Gamma$. Thus S is orientable if and only if $N/2$ is odd. On the other hand, $\alpha g_0 = 2$ and we conclude that S is a torus if $N/2$ is odd, and S is a Klein bottle if $N/2$ is even.

These three possibilities actually occur. It is enough to consider the epimorphisms $\Lambda \to \mathbf{Z}_N$ induced by assignments:

$$\begin{aligned}
&\theta_1(d) = 1, \quad \theta_1(e) = N-2, \quad \theta_1(c) = 0 && \text{in case (ii(1))},\\
&\theta_2(d) = 2, \quad \theta_2(e) = N-4, \quad \theta_2(c) = \frac{N}{2} && \text{in case (ii(2.1))},\\
&\theta_3(d) = 1, \quad \theta_3(e) = N-2, \quad \theta_3(c) = \frac{N}{2} && \text{in case (ii(2.2))}.
\end{aligned}$$

(iii) Our last example corresponds to the last row in Table 2.1.

We deal with an orbifold M whose fundamental group is $\Lambda = \langle d, x; xd^2 = x^m = 1\rangle$. Every epimorphism $\theta: \Lambda \to \mathbf{Z}_N$ with even N whose kernel is the fundamental group of some $S \in K(g_0, k_0, \alpha)$, with $p_0(S) \leq 0$, maps x to an element of order m and so $\theta(d)$ generates $\mathbf{Z}_N$. Hence $m = N/2$, because $2\theta(d) = -\theta(x)$, and, N being even, S is orientable. By (1.1),

$$2g_0 + k_0 - 2 = N\left(\sigma g + k - 2 + 1 - \frac{1}{m}\right) = -2,$$

i.e., $g_0 = k_0 = 0$ and S is a sphere. The induced automorphism on S reverses its orientation because M is nonorientable. Of course, the assignment $\theta(d) = 1, \theta(x) = N - 2$ shows that such an epimorphism actually exists.

3. The Algorithm To Compute Even Cyclic Actions on Nonorientable Surfaces with an Orientable Quotient

Now that the study of cyclic actions on the *exceptional* surfaces is complete, we begin the algorithmic part of the paper. In this and the forthcoming sections g_0 and k_0 are fixed nonnegative integers and $\alpha = 1$, which in particular implies $g_0 \geq 1$. We always assume $p_0 \geq 2$ which in this section means $g_0 + k_0 \geq 3$. The key result to compute $E_1(2)$, i.e, even orders of automorphisms acting on surfaces in $K(g_0, k_0, 1)$ with an orientable quotient, is the next proposition which follows from Theorem 3.1.6 in [4] for $k_0 > 0$ and Theorem 3.5 in [2] for $k_0 = 0$. Recently, Yokoyama [9]–[11] has obtained a similar result to Proposition 3.1 and the analogous Propositions 4.1, 5.2, 6.1, and 7.1 in the forthcoming sections.

Proposition 3.1. *A positive even integer N occurs in $E_1(2)$ if and only if there exist nonnegative integers g, r, p, k', c, positive divisors $m_1, \ldots, m_r, l_1, \ldots, l_p$ of N, $m_i \neq m_j, l_i \neq l_j$ if $i \neq j, m_i \geq 2$, and positive integers $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$, called the multiplicites of $m_1, \ldots, m_r, l_1, \ldots, l_p$, respectively, such that:*

(1) $k_0 - c(N/2) = \sum_{i=1}^{p} (N/l_i)\lambda_i$.
(2) $p_0 - c(N/2) - 1 = N(2g + k' + \lambda - 1) + \sum_{i=1}^{r} (N - N/m_i)\mu_i, \lambda = \sum_{i=1}^{p} \lambda_i$.
(3) *If $g = k' = 0$, then $N =$ l.c.m. $(m_1, \ldots, m_r, l_1, \ldots, l_p)$.*

Remarks 3.2. (1) The first two conditions are nothing other than the fundamental equalities in (1.1). The variables in the statement have the geometric meaning of Section 1. In particular k' is the deficiency as defined in (1.0). So, the number k of connected components of the boundary of the quotient is $k = k' + \lambda + 1$.

(2) It does not seem obvious from the definition that all even divisors of elements in $E_1(2)$ belong to $E_1(2)$ too. Let f be an automorphism on the surface S of order N with orientable quotient S_f. The *natural* candidate to realize N/d as order of an automorphism on a surface of the same topological type is f^d, but there is no reason for S_{f^d} to be orientable too, if d is even. However, from Proposition 3.1. we get:

Corollary 3.3. *All even divisors of $N \in E_1(2)$ also occur in $E_1(2)$.*

Proof. Let D be an even divisor of N. If the quotient $d = N/D$ is odd, the result is obvious since, for every automorphism f of order N on $S \in K(g_0, k_0, 1)$ with orientable quotient S_f, the automorphism f^d on S has order D and S_{f^d} is orientable. Hence we can assume that $N = 2D$ and let $g_N, r, p, k'_N, c_N, m_1, \ldots, m_r, l_1, \ldots, l_p, \mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$ be a solution of the equations in Proposition 3.1 for the integer N. We change the notation slightly as follows: we write

$$M_N = \{m_1, \dots, m_r\}; \qquad L_N = \{l_1, \dots, l_p\}$$

and define, for all $m \in M_N$ and $l \in L_N$ their multiplicities

$$\mu_N(m) = \mu_i \quad \text{if} \quad m = m_i; \qquad \lambda_N(l) = \lambda_i \quad \text{if} \quad l = l_i.$$

We define

$$\mu_N = \sum_{m \in M_N,} \mu_N(m); \qquad \lambda_N = \sum_{l \in L_N,} \lambda_N(l),$$

$$\mu'_N = \sum_{m \in M_N, (M/m)\text{odd}} \mu_N(m); \qquad \lambda'_N = \sum_{l \in L_N, (N/l)\text{odd}} \lambda_N(l),$$

and we now find a solution of the equations in Proposition 3.1 for the integer D. Note that $\mu'_N + \lambda'_N \neq 0$ if $g_N = k'_N = 0$ by condition (3), and so $h = 2(2g_N + k'_N) - 1 + \mu'_N + \lambda'_N \geq 0$. Hence there exist nonnegative integers g_D and k'_D such that $2g_D + k'_D = h$.

We choose $c_D = 2c_N$ and, for every divisors m and l of D with $m \geq 2$, we define

$$\mu_D(m) = \begin{cases} 2\mu_N(m) & \text{if} \quad D/m \text{ is even,} \\ 2\mu_N(m) + \mu_N(2m) & \text{if} \quad D/m \text{ is odd,} \end{cases}$$

$$\lambda_D(l) = \begin{cases} 2\lambda_N(l) & \text{if} \quad D/l \text{ is even,} \\ 2\lambda_N(l) + \lambda_N(2l) & \text{if} \quad D/l \text{ is odd,} \end{cases}$$

where the functions μ_N and λ_N are extended to be equal to zero outside M_N and L_N, respectively. To be coherent with the statement of Proposition 3.1 we take $M_D = \{2 \leq m \text{ divisor of } D: \mu_D(m) > 0\}$ and $L_D = \{l \text{ divisor of } D: \lambda_D(l) > 0\}$. We claim that g_D, k'_D, c_D and the sets M_D, L_D with multiplicities μ_D and λ_D are a solution of the equations in Proposition 3.1 for the integer D. In fact, it is easily checked that

$$\sum_{l \in L_D} \frac{D}{l} \lambda_D(l) = \sum_{l \in L_N} \frac{N}{l} \lambda_N(l),$$

which implies that condition (1) is fulfilled, and analogously

$$\sum_{m \in M_D} \frac{D}{m} \mu_D(m) = \sum_{m \in M_N} \frac{N}{m} \mu_N(m),$$

which together with $2\mu_N - \mu'_N = \mu_D$; $2\lambda_N - \lambda'_N = \lambda_D$, where

$$\mu_D = \sum_{m \in M_D} \mu_D(m), \qquad \lambda_D = \sum_{l \in L_D} \lambda_D(l),$$

prove that condition (2) also is satisfied by our choice of g_D, k'_D, and c_D.

Finally, assume $g_D = k'_D = 0$. Then $g_N = k'_N = 0$, $\mu'_N + \lambda'_N = 0$ and so $N = \text{l.c.m.}(M_N \cup L_N)$ and without loss of generality we can suppose that all l in L_N divide D and there is a unique $q \in M_N$ with N/q odd, $\mu_N(q) = 1$. Now, if $q' = q/2$ we know that $\mu_D(q') = 2\mu_N(q') + \mu_N(q) \geq 1$, i.e., $q' \in M_D$. Since M_D and L_D contain all elements x in M_N (resp. L_N) with even N/x, and $q' \in M_D$ it follows that $D = \text{l.c.m.}(M_D \cup L_D)$ as desired. □

Comment. As observed in Remarks 1.2, to develop the algorithm we first compute the set $P_1(2)$ of 2-powers occurring in $E_1(2)$. For $N = 2^e, e \geq 1$, its divisors are $2^i, i = 0, \dots, e$, and we change the notation in Proposition 3.1 by writing

$$y_j = \begin{cases} \lambda_{j'} & \text{if } \ l_{j'} = 2^j, \\ 0 & \text{if } \ 2^j \notin \{l_1, \dots, l_p\}, \end{cases}$$

$$x_j = \begin{cases} \mu_{j'} & \text{if } \ m_{j'} = 2^j, \\ 0 & \text{if } \ 2^j \notin \{m_1, \dots, m_r\}. \end{cases}$$

This way, the third condition in Proposition 3.1 says $x_e + y_e \geq 1$ if $g = k' = 0$, and so we get

Corollary 3.4. *For $e \geq 1, 2^e$ occurs in $E_1(2)$ if and only if there exist nonnegative integers $g, k', c', x_1, \dots, x_e, y_0, \dots, y_e$ such that:*

(1) $k_0 - c2^{e-1} = \sum_{i=0}^{e} y_i 2^{e-i}$.
(2) $p_0 - c2^{e-1} - 1 = 2^e(2g + k' + \lambda - 1) + \sum_{i=1}^{e}(2^e - 2^{e-i})x_i, \ \lambda = \sum_{i=0}^{e} y_i$.
(3) *If $g = k' = 0$, then $x_e + y_e \geq 1$.*

Now, by using Propositions 1.4, 3.1, and 3.3 we get the desired algorithm:

Step 1. Determine the set $X = \{e \geq 1 : 2^e \leq 2(p_0 + 1)\}$ and its maximum e_X.

Step 2. Determine the set $Y = \{1 \leq e \leq e_X : \varepsilon_2(e, k_0) \leq p_0\}$ and its maximum e_Y.

Step 3. Determine the set

$$P_1(2) = \{2^e : 1 \leq e \leq e_Y : \text{ the equations in Corollary 3.4 admit a solution}\}.$$

Step 4. Determine the set of candidates

$$C_1(2) = \{2^e M : 2^e \in P_1(2), M \in O_1, 2^e M \leq 2(p_0 + 1)\}.$$

Step 5. Determine the set

$$E_1(2) = \{N \in C_1(2) : \text{ the equations in Proposition 3.1 admit a solution}\}.$$

Once $E_1(2)$ is known, the final step is

Step 6. FIND SOLUTIONS. This means that for every $N \in E_1(2)$ the computer calculates all solutions

$$(g, k', m_1, \dots, m_r, \mu_1, \dots, \mu_r, l_1, \dots, l_p, \lambda_1, \dots, \lambda_p)$$

of the equations in Proposition 3.1. This gives the topological data of the covering $S \to S_f$ for all automorphisms f of order N acting on surfaces $S \in K(g_0, k_0, 1)$ with S_f orientable, taking into account that the number k of boundary components of S_f is, by (1.0), $k = k' + 1 + \sum_{i=1}^{p} \lambda_i$. In the next sections we just write FIND SOLUTIONS.

Remarks 3.5. (1) As previously observed, the variable c in Proposition 3.1 is bounded above by $2k_0/N$. Thus, the implemented version of our algorithm treats it as a discrete parameter with values $c = 0, 1, \ldots$ up to the integral part of $2k_0/N$ instead of an unknown and solve many more equations with one variable less.

(2) In our previous paper [3] we explained how the computer handled these diophantine equations with nonnegative unknowns and so we do not repeat it here. It should be mentioned that knowing the precise distribution of the corner points the computer also gives, by the same procedure, all possible decompositions of $2c = \sum \gamma_i c_i$ for even positive integers c_i with $0 < \sum \gamma_i \leq k'$. This way it is known how many corner points appear on each boundary component of the quotient.

(3) To compute the set $P_1(2)$ in Step 3 we begin with a set of candidates $\{2^e : 1 \leq e \leq e_Y\}$ and we know after Corollary 3.3 that the divisors of elements in $P_1(2)$ occur in $P_1(2)$ also. Hence, the efficiency of the algorithm seems to be optimum if we begin the test of Step 3 with the *intermediate* exponent e_m = integral part of $e_Y/2$ and then going *up or down* according to whether 2^{e_m} does or does not belong to $P_1(2)$. In the same way, for every odd $M \in O_1$ and the maximum exponent e_M such that $2^{e_M} \in P_1(2)$ and $M2^{e_M} \leq 2(p_0+1)$, the algorithm seems to be more efficient if we begin the test of Step 5 with $M2^{e_{m'}}$, with $e_{m'}$ = integral part of $e_M/2$.

These remarks remain valid for the forthcoming sections.

4. The Algorithm To Compute Even Cyclic Actions on Nonorientable Surfaces, with a Nonorientable Quotient

As in the preceding section, $g_0 \geq 1$ and $k_0 \geq 0$ are fixed integers and $p_0 = g_0 + k_0 - 1 \geq 2$. The analogous result to Proposition 3.1 concerning nonorientable quotient follows from Theorem 3.1.8 in [4] for $k_0 > 0$ and Theorem 3.6 in [2] for $k_0 = 0$:

Proposition 4.1. *A positive even integer N occurs in $E_1(1)$ if and only if there exist nonnegative integers $g \geq 1, r, p, k', c$, positive divisors $m_1, \ldots, m_r, l_1, \ldots, l_p$ of N, $m_i \neq m_j, l_i \neq l_j$ if $i \neq j$, $m_i \geq 2$, and positive integers $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$ such that:*

(1) $k_0 - c(N/2) = \sum_{i=1}^{p}(N/l_i)\lambda_i$.
(2) $p_0 - c(N/2) - 1 = N(g + k' + \lambda - 2) + \sum_{i=1}^{r}(N - (N/m_i))\mu_i, \lambda = \sum_{i=1}^{p} \lambda_i$.
(3) *We define*

$$\lambda'_N = \sum_{(N/l_i)\text{odd}} \lambda_i; \qquad \lambda''_N = \sum_{N/l_i \notin 4\mathbf{Z}} \lambda_i,$$
$$\mu'_N = \sum_{(N/l_i)\text{odd}} \mu_i; \qquad \mu''_N = \sum_{N/l_i \notin 4\mathbf{Z}} \mu_i.$$

(3.1) *If $k' = 0$, then $c = 0$ and $\lambda'_N + \mu'_N$ is even. If, moreover, $g = 1$, then $N =$ l.c.m.$(m_1, \ldots, m_r, l_1, \ldots, l_p)$.*
(3.2) *If $k' = \lambda'_N + \mu'_N = 0$ and $g = 2$ and $N \in 4\mathbf{Z}$, then $\lambda''_N + \mu''_N$ is odd.*

As in the last section we obtain

Corollary 4.2. *All even divisors of $N \in E_1(1)$, also occur in $E_1(1)$.*

Proof. As in Corollary 3.3 it suffices to study the case of $D = N/2$. We denote by g_N, k'_N, and c_N the values of g, k', and c realizing N as an element of $E_1(1)$ and define

$$\epsilon = \begin{cases} 1 & \text{if } g_N = 1, \quad \lambda'_N = \mu'_N = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now choose $g_D = 2(g_N - 1) + \lambda'_N + \mu'_N + \epsilon \geq 1$, $k'_D = 2k'_N - \epsilon$, $c_D = 2c_N$ and define M_D, L_D and the multiplicities μ_D and λ_D as in Corollary 3.3. The same computations of Corollary 3.3 show that these values fulfill conditions (1) and (2) of Proposition 4.1 for the integer D, and we must just check that the restrictions of (3.1), (3.2) are also satisfied. First, if $k'_D = 0$, then $k'_N = 0$, hence $c_N = 0$ and so $c_D = 0$. Also

$$\begin{aligned} \lambda'_D + \mu'_D &= \sum_{l \in L_D, (D/l) \text{odd}} \lambda_D(l) + \sum_{m \in M_D, (D/m) \text{odd}} \mu_D(m) \\ &= \sum_{l \in L_D, N/l \notin 4\mathbf{Z}} (2\lambda_N(l) + \lambda_N(2l)) + \sum_{m \in M_D, N/m \notin 4\mathbf{Z}} (2\mu_N(m) + \mu_N(2m)) \\ &= \lambda'_N + \mu'_N \pmod 2 = 0 \pmod 2. \end{aligned}$$

If, moreover, $g_D = 1$, also $g_N = 1$ and so l.c.m.$(M_N \cup L_N) = N$. Hence $\lambda'_N + \mu'_N = 1$ and as in Corollary 3.3 we conclude that $D =$ l.c.m.$(M_D \cup L_D)$.

Secondly, we are going to prove that conditions $k'_D = \lambda'_D + \mu'_D = 0$, $g_D = 2$, and $D \in 4\mathbf{Z}$ never occur, and therefore the restriction (3.2) is trivially satisfied. Otherwise we would have $k'_N = 0, \lambda'_N = \mu'_N = 0, \lambda_N(l) = \mu_N(m) = 0$ for all $l \in L_D$ and $m \in M_D$ with N/l and $N/m \notin 4\mathbf{Z}$, $2 = g_D = 2(g_N - 1) + \epsilon$ and, since $D \in 4\mathbf{Z}$, also $N \in 4\mathbf{Z}$. Thus, on the one hand, $k'_N = \lambda'_N + \mu'_N = 0$, $g_N = 2$, $N \in 4\mathbf{Z}$, and this implies $\lambda''_N + \mu''_N$ is odd. However, since each $\lambda_N(l) = \mu_N(m) = 0$ for $N/l, N/m \notin 4\mathbf{Z}$, this means $\lambda''_N + \mu''_N = 0$, which is absurd. □

With the same "change of variables" of Corollary 3.4 we now obtain:

Corollary 4.3. *For $e \geq 1, 2^e$ occurs in $E_1(1)$ if and only if there exist nonnegative integers $g \geq 1, k', c, x_1, \ldots, x_e, y_0, \ldots, y_e$ such that:*

(1) $k_0 - c2^{e-1} = \sum_{i=0}^{e} y_i 2^{e-i}$.
(2) $p_0 - c2^{e-1} - 1 = 2^e(g + k' + \lambda - 2) + \sum_{i=0}^{e} (2^e - 2^{e-i})x_i$, $\lambda = \sum_{i=0}^{e} y_i$.
(3) *If $k' = 0$, then $c = 0$ and $x_e + y_e$ is even. If, moreover, $g = 1$, then $x_e + y_e > 0$.*
(4) *If $k' = 0 = x_e + y_e$ and $g = 2$ and $e > 1$, then $x_{e-1} + y_{e-1}$ is odd.*

If we define $P_1(1)$ as the set of those 2-powers occurring in $E_1(1)$, the algorithm to determine $E_1(1)$ and the corresponding topological data is the following:

Steps 1 *and* 2. As in Section 3.

Step 3. Determine the set

$$P_1(1) = \{2^e : 1 \leq e \leq e_Y : \text{ the equations in Corollary 4.3 admit a solution}\}.$$

Step 4. Determine the set of candidates:

$$C_1(1) = \{2^e M : 2^e \in P_1(1), M \in O_1, 2^e M \leq 2(p_0 + 1)\}.$$

Step 5. Determine the set

$$E_1(1) = \{N \in C_1(1) : \text{ the equations in Proposition 4.1 admit a solution}\}.$$

Step 6. FIND SOLUTIONS.

Note that according to (1.0) for each solution of the equations of Proposition 4.1, the number of boundary components of each quotient is $k = k' + \sum_{i=1}^{p} \lambda_i$.

5. The Algorithm To Compute Orientation-Preserving Even Cyclic Actions on Orientable Surfaces

As is well known, the quotient S_f of an orientable surface S under the action of the group generated by an orientation-preserving automorphism f of finite order, is orientable. Moreover, from Corollary 3.2.3 of [4], the deficiency k' of the covering $S \to S_f$ is zero, and in particular there are no corner points. So it is simpler to study the topological data of the covering in this case, and the diophantine equations we must solve to decide if $N \in E_2(1)$ have fewer unknowns than in the preceding sections. On the other hand, it is obvious than f^2 preserves the orientation of S and the quotient S_{f^2} is orientable. Hence, the analogy in this case to Corollaries 3.4 and 4.3 is obviously true. To understand the condition $N \in E_2(1)$ better we first need the following:

Definition 5.1.

(i) A family $\{q_1, \ldots, q_s\}$ of (nonnecessarily distinct) natural numbers satisfies the *elimination property* if either every $q_i = 1$ or each of them divides the l.c.m. of the others.

(ii) Let $p_1, \ldots, p_s$ be distinct natural numbers and let $\rho_1, \ldots, \rho_s$ be positive integers. The pairs $\{(p_1, \rho_1), \ldots, (p_s, \rho_s)\}$ satisfy the *elimination property* if this is so for the family

$$\{p_1, \overset{\rho_1}{\ldots}, p_1, \ldots, p_s, \overset{\rho_s}{\ldots}, p_s\}.$$

For fixed $g_0 \geq 0, k_0 \geq 0$ with $p_0 = 2g_0 + k_0 - 1 \geq 2$ we get from Theorem 3.1.5 and Corollary 3.2.3 in [4], for $k_0 > 0$, and [7], for $k_0 = 0$, the following:

Proposition 5.2. *A positive even integer N occurs in $E_2(1)$ if and only if there exist nonnegative integers g, r, p, positive divisors $m_1, \ldots, m_r, l_1, \ldots, l_p$ of N, $m_i \neq m_j, l_i \neq l_j$, if $i \neq j$, $m_i \geq 2$, and positive integers $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$ such that:*

(1) $k_0 = \sum_{i=1}^{p}(N/l_i)\lambda_i$.
(2) $p_0 - 1 = N(2g + k - 2) + \sum_{i=1}^{r}(N - (N/m_i))\mu_i$, $k = \sum_{i=1}^{p}\lambda_i$.
(3) *The pairs* $\{(m_1, \mu_1), \ldots, (m_r, \mu_r), (l_1, \lambda_1), \ldots, (l_p, \lambda_p)\}$ *satisfy the elimination property.*
(4) *Let* $M = \text{l.c.m.}\{m_1, \ldots, m_r, l_1, \ldots, l_p\}$ *and* $\mu = \sum_{(M/m_i)\text{odd}}\mu_i$. *If* $g = 0$, *then* $M = N$, *and if* $k_0 = 0$, *then* μ *is even.*

As in the previous sections we characterize from Proposition 5.2 the set $P_2(1)$ of 2-powers occurring in $E_2(1)$:

Corollary 5.3. *If* $e \geq 1$, *the number* 2^e *occurs in* $P_2(1)$ *if and only if there exist nonnegative integers* $g, x_1, \ldots, x_e, y_0, \ldots, y_e$ *such that:*

(1) $k_0 = \sum_{i=0}^{e} y_i 2^{e-i}$.
(2) $p_0 - 1 = 2^e(2g + k - 2) + \sum_{i=1}^{e}(2^e - 2^{e-i})x_i$, $k = \sum_{i=0}^{e} y_i$.
(3) *Let* $\Lambda = \{1 \leq i \leq e : x_i + y_i \neq 0\}$ *and define*

$$d = \begin{cases} \max \Lambda & \text{if } \Lambda \text{ is nonempty,} \\ 0 & \text{otherwise.} \end{cases}$$

Then either $d = 0$ *or* $x_d + y_d \geq 2$.
(4) *If* $g = 0$, *then* $x_e + y_e \geq 1$. *If* $k_0 = 0$, *then* x_d *is even.*

In terms of the set O_2 of odd orders of automorphisms acting on surfaces in $k(g_0, k_0, 2)$, which was calculated in [3], we get the following algorithm to compute $E_2(1)$ and the topological data:

Step 1. Determine the set

$$X = \left\{ e \geq 1, 2^e \leq \begin{cases} 2(p_0 + 1) & \text{if } k_0 > 0 \\ 2(p_0 + 2) & \text{if } k_0 = 0 \end{cases} \right\}$$

and its maximum e_X.

Step 2. Determine the set $Y = \{1 \leq e \leq e_X : \varepsilon(e, k_0) \leq p_0\}$ and its maximum e_Y.

Step 3. Determine the set

$$P_2(1) = \{2^e : 1 \leq e \leq e_Y : \text{ the equations in Corollary 5.3 admit a solution}\}.$$

Step 4. Determine the set of candidates:

$$C_2(1) = \left\{ 2^e M : 2^e \in P_2(1), M \in O_2, 2^e M \leq \begin{cases} 2(p_0 + 1) & \text{if } k_0 > 0 \\ 2(p_0 + 2) & \text{if } k_0 = 0 \end{cases} \right\}.$$

Step 5. Determine the set

$$E_2(1) = \{N \in C_2(1) : \text{ the equations in Proposition 5.2 admit a solution}\}.$$

Step 6. FIND SOLUTIONS.

Recall that the variable k in Proposition 5.2 is the number of boundary components of the quotient, which is orientable, and that there are no corner points in this case.

6. The Algorithm To Compute Orientation-Reversing Even Cyclic Actions on Orientable Surfaces with an Orientable Quotient

We fix nonnegative integers g_0, k_0 such that $p_0 = 2g_0 + k_0 - 1 \geq 2$ and let f be an automorphism of finite even order N on a surface $S \in K(g_0, k_0, 2)$ which reverses the orientation of S and such that the quotient S_f is orientable. Then it follows from Corollary 3.2.3 in [4] that $N/2$ is odd. Thus in this section we do not need to compute the maximum 2-power occurring in $E_2(-1, 2)$ and it follows that the set of candidates is

$$C_2(-1,2) = \left\{2M: M \in O_2, 2M \leq \begin{cases} 2(p_0+1) & \text{if } k_0 > 0 \\ 2(p_0+3) & \text{if } k_0 = 0 \end{cases}\right\},$$

where O_2 has the meaning of Section 5. We must just decide what elements of $C_2(-1, 2)$ are actually in $E_2(-1, 2)$ and to do that we use the following result, which is a consequence of Theorem 3.1.5 and Corollary 3.2.3 in [4] for $k_0 > 0$ and of [6] for $k_0 = 0$.

Proposition 6.1. *Let M be odd. Then $N = 2M$ occurs in $E_2(-1, 2)$ if and only if there exist nonnegative integers g, r, p, k', c, positive divisors $m_1, \ldots, m_r, l_1, \ldots, l_p$ of M, $m_i \neq m_j, l_i \neq l_j$ if $i \neq j, m_i \geq 3$, and positive integers $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$ such that*:

(1) $k_0 - cM = 2\sum_{i=1}^{p}(M/l_i)\lambda_i$.
(2) $p_0 - cM - 1 = 2M(2g + k' + \lambda - 1) + 2\sum_{i=1}^{r}(M - (M/m_i))\mu_i, \lambda = \sum_{i=1}^{p}\lambda_i$.
(3) *If* $g = k' = 0$, *then* $M = \text{l.c.m.}\{m_1, \ldots, m_r, l_1, \ldots, l_p\} = 1$.

Example 6.2. In particular, for $M = 1$, we get that $2 \in E_2(-1, 2)$. In fact, to solve the equations above, necessarily $r = 0$, each $l_i = 1$, and condition (3) is trivially fulfilled. Hence only the solvability of the system of equations

$$k_0 - c = 2\lambda; \qquad p_0 - c - 1 = 2(2g + k' + \lambda - 1)$$

must be proved, and since $p_0 = 2g_0 + k_0 - 1$, this is equivalent to $k_0 = c + 2\lambda$; $g_0 = 2g + k'$, which clearly admits solutions.

We describe the algorithm to compute $E_2(-1, 2)$ and the topological data:

Step 1. Determine the set

$$C_2(-1,2) = \left\{2M: M \in O_2, 2M \leq \begin{cases} 2(p_0+1) & \text{if } k_0 > 0 \\ 2(p_0+3) & \text{if } k_0 = 0 \end{cases}\right\}.$$

Step 2. Determine the set

$$E_2(-1,2) = \{2M \in C_2(-1,2)\text{: for this } M \text{ the system of equations in Proposition 6.1 admit a solution}\}.$$

Step 3. FIND SOLUTIONS.

As in the previous section we recall that, for each of the solutions, the number of boundary components of the quotient is $k = k' + \lambda + 1$.

7. The Algorithm To Compute Orientation-Reversing Even Cyclic Actions on Orientable Surfaces with a Nonorientable Quotient

Using Theorem 3.1.9 in [4] for the case $k_0 > 0$ and [6] for $k_0 = 0$ we characterize the elements in $E_2(-1, 1)$ for fixed $g_0 \geq 0, k_0 \geq 0$ with $p_0 = 2g_0 + k_0 - 1 \geq 2$, in the following way:

Proposition 7.1. *A positive even integer N occurs in $E_2(-1, 1)$ if and only if there exist nonnegative integers $g \geq 1, r, p, k', c$, positive divisors $m_1, \ldots, m_r, l_1, \ldots, l_p$ of $N/2$, $m_i \neq m_j, l_i \neq l_j$ if $i \neq j, m_i \geq 2$, and positive integers $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_p$ such that:*

(1) $k_0 - c(N/2) = \sum_{i=1}^{p}(N/l_i)\lambda_i$.
(2) $p_0 - c(N/2) - 1 = N(g + k' + \lambda_N - 2) + \sum_{i=1}^{r}(N - (N/m_i))\mu_i$, $\lambda_N = \sum_{i=1}^{p}\lambda_i$.
(3) *If $k' = 0$, then $c = 0$. If $k' = 0$ and $g = 1$, then $N/2 =$ l.c.m.$(m_1, \ldots, m_r, l_1, \ldots, l_p)$.*
(4) *We define*

$$\lambda'_N = \sum_{N/l_i \notin 4\mathbf{Z}} \lambda_i \quad \textit{and} \quad \mu'_N = \sum_{N/l_i \notin 4\mathbf{Z}} \mu_i.$$

If $N \in 4\mathbf{Z}$, then $k' = 0$ and $\lambda'_N + \mu'_N + g$ is even.

As a consequence we can prove the most striking result:

Theorem 7.2. *Let (g_0, k_0, α) be given with $p_0 = \alpha g_0 + k_0 - 1 \geq 2$, and let E be one of the sets $E_1(\sigma), E_2(-1, \sigma), E_2(1), \sigma = 1, 2$. All even divisors of elements in E also occur in E.*

Proof. We already proved the theorem in case $E = E_1(\sigma)$, and it is obviously true for $E = E_2(1)$ or $E_2(-1, 2)$, in the last case because $E_2(-1, 2)$ does not contain multiples of 4. Thus we are just concerned with $E = E_2(-1, 1)$ and it is enough to prove that if $N \in E$ is a multiple of 4, then $D = N/2 \in E$. By Proposition 7.1 there exist $g_N \geq 1$, two subsets M_N and L_N of divisors of $N/2$, with $m \geq 2$ if $m \in M_N$, and for every $m \in M_N, l \in L_N$ some multiplicity $\mu_N(m), \lambda_N(l) > 0$, such that if $\lambda_N = \sum_{l \in L_N} \lambda_N(l)$,

$$k_0 = \sum_{l \in L_N} \frac{N}{l}\lambda_N(l); \qquad p_0 - 1 = N(g_N + \lambda_N - 2) + \sum_{m \in M_N}\left(N - \frac{N}{m}\right)\mu_N(m)$$

with $\lambda'_N + \mu'_N + g_N$ even and, if $g_N = 1$, $N/2 =$ l.c.m.$(M_N \cup L_N)$.

In particular the last condition implies that

$$g_D = 2(g_N - 1) + \lambda'_N + \mu'_N \geq 1,$$

and for all divisors m, l of $D/2$, $m \geq 2$, we define the new multiplicities

$$\mu_D(m) = \begin{cases} 2\mu_N(m) & \text{if } \dfrac{D}{m} \in 4\mathbf{Z}, \\ 2\mu_N(m) + \mu_N(2m) & \text{if } \dfrac{D}{m} \notin 4\mathbf{Z}, \end{cases}$$

$$\lambda_D(l) = \begin{cases} 2\lambda_N(l) & \text{if } \dfrac{D}{l} \in 4\mathbf{Z}, \\ 2\lambda_N(l) + \lambda_N(2l) & \text{if } \dfrac{D}{l} \notin 4\mathbf{Z}. \end{cases}$$

We take $M_D = \{m \text{ divisor of } D/2\colon \mu_D(m) > 0\}$ and $L_D = \{m \text{ divisor of } D/2\colon \lambda_D(l) > 0\}$.

It is straightforward to check the equalities:

$$k_0 = \sum_{l \in L_D} \frac{D}{l}\lambda_D(l); \qquad p_0 - 1 = D(g_D + \lambda_D - 2) + \sum_{m \in M_D} \left(D - \frac{D}{m}\right)\mu_D(m),$$

where $\lambda_D = \sum_{l \in L_D} \lambda_D(l)$.

Moreover, we have chosen $k' = 0$, because $N \in 4\mathbf{Z}$ and if $g_D = 1$ it means $g_N = 1$ and $\lambda'_N + \mu'_N = 1$. Thus $N/2 =$ l.c.m.$(M_N \cup L_N)$ and without loss of generality we can assume that each $N/l \in 4\mathbf{Z}$ for $l \in L_N$ and there is a unique $q \in M_N$ such that $N/q \notin 4\mathbf{Z}$, $\mu_N(q) = 1$.

Of course $q' = q/2$ divides $D/2$ and $\mu_D(q') = 2\mu_D(q') + \mu_D(q) \geq 1$. This together with the equality $N/2 =$ l.c.m.$(M_N \cup L_N)$ and the fact that M_D (resp. L_D) contains all elements x in M_N (resp. L_N) with even N/x, implies that $D/2 =$ l.c.m.$(M_D \cup L_D)$.

To finish assume that $D \in 4\mathbf{Z}$. We must check that $\lambda'_D + \mu'_D + g_D$ is even. However, $\lambda'_D + \mu'_D = \lambda'_N + \mu'_N \pmod 2$, therefore, $\lambda'_D + \mu'_D + g_D = 2(\lambda'_N + \mu'_N) \pmod 2$ as desired. □

Another consequence of Proposition 7.1 is the following:

Corollary 7.3.

(i) *If* k_0 *is odd,* $E_2(-1, 1)$ *does not contain multiples of* 4.
(ii) *As in Section* 6, $2 \in E_2(-1, 1)$.

Proof. (i) If $N \in 4\mathbf{Z}$ it follows from Proposition 7.1 that $k' = 0$ and so $c = 0$. Substituting in the first equation, $k_0 = \sum_{i=1}^{p} (N/l_i)\lambda_i$ must be even since each l_i divides $N/2$.

(ii) For $N = 2$ in Proposition 7.1 we should have $r = 0$, each $l_i = 1$, and conditions (3) and (4) hold trivially. Thus everything reduces to studing the system $k_0 = c + 2\lambda$; $p_0 - c - 1 = 2(g + k' + \lambda - 2)$, $g \geq 1$, with $c = 0$ if $k' = 0$ which clearly admits solutions. □

Corollary 7.4. *Let* $e \geq 1$ *be an integer. Then* $2^e \in E_2(-1, 1)$ *if and only if there exist nonnegative integers* $g \geq 1, x_1, \dots, x_{e-1}, y_0, \dots, y_{e-1}$ *such that*:

(1) $k_0 = \sum_{i=0}^{e-1} y_i 2^{e-i}$.
(2) $p_0 - 1 = 2^e(g + \lambda - 2) + \sum_{i=1}^{e-1}(2^e - 2^{e-i})x_i$, $\lambda = \sum_{i=0}^{e-1} y_i$.
(3) $x_{e-1} + y_{e-1} + g$ *is even.*

To decide if an even integer N occurs in $E_2(-1,1)$ requires two different approaches according to the parity of $N/2$. Hence we introduce the subsets

$$F_2 = 4\mathbf{Z} \cap E_2(-1,1); \qquad T_2(-1,1) = E_2(-1,1) - F_2(-1,1)$$

and we compute them separately. We also denote by $P_2(-1,1)$ the set of 2-powers occurring in $E_2(-1,1)$. We can describe the algorithm to compute $E_2(-1,1)$ as follows:

Step 1. For odd k_0 define $X = \{1\}$ and for even k_0 determine the set

$$X = \left\{ e \geq 1, 2^e \leq \begin{cases} 2(p_0+1) & \text{if} \quad k_0 > 0 \\ 2(p_0+3) & \text{if} \quad k_0 = 0 \end{cases} \right\}$$

and its maximum e_X.

Step 2. Determine the set $Y = \{1 \leq e \leq e_X : \varepsilon_3(e, k_0) \leq p_0\}$ and its maximum e_Y.

Step 3. Determine the set

$$P_2(-1,1) = \{2^e : 1 \leq e \leq e_Y : \text{ the equations in Corollary 7.4 admit a solution}\}.$$

Step 4. Determine the set of candidates:

$$A_2(-1,1) = \left\{ 2M : M \in O_2, M \leq \begin{cases} p_0+1 & \text{if} \quad k_0 > 0 \\ p_0+3 & \text{if} \quad k_0 = 0 \end{cases} \right\}$$

and

$$B_2(-1,1) = \left\{ 2^e M : 2^e \in P_2(-1,1), M \in O_2, 2^{e-1}M \leq \begin{cases} (p_0+1) & \text{if} \quad k_0 > 0 \\ (p_0+3) & \text{if} \quad k_0 = 0 \end{cases} \right\}.$$

Step 5. Determine the set

$$F_2(-1,1) = \{N \in B_2(-1,1) : \text{ the equations in Corollary 7.4 admit a solution with } k' = c = 0, \lambda'_N + \mu'_N + g \text{ even}\}.$$

Step 6. Determine the set

$$T_2(-1,1) = \{N \in A_2(-1,1) : \text{ equations (1)–(3) in Corollary 7.4 admit a solution}\}.$$

Step 7. Determine the set $E_2(-1,1) = F_2(-1,1) \cup T_2(-1,1)$.

Step 8. FIND SOLUTIONS.

8. Final Remarks and Examples

Let E be one of the sets $E_1(\sigma)$, $E_2(-1,\sigma)$, or $E_2(1)$ for a given input (g_0, k_0, α) and $O = O_\alpha$. As we have explained, the strategy to determine E consists of first producing a finite set of candidates $C = \{M2^e: M \in O_\alpha\}$ and then subjecting elements in C to some tests. This leads us to an unsolved question about how to minimize the complexity of the algorithm: in what order must the elements in C be tested to achieve rapid computation? One should take into account that, from Theorem 7.2, even divisors of elements in E belong to E too. We should thank the computer, "who" after managing a lot of examples gave us practical evidence of the validity of Theorem 7.2.

Example 8.1. Using the algorithm of [3] to compute the odd orders of automorphisms of an orientable surface of genus 12 with two boundary components we obtain that the orders are: 3, 5, 7, 9, 13, 15, 25, 27. By using the algorithm just presented we have the following even orders:

Even orders of automorphisms preserving the orientation: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24, 26, 28, 30, 32, 36, 48, 50.

Even orders of automorphisms reversing the orientation: 2, 4, 6, 8, 10, 12, 14, 16, 18, 24, 26, 28, 30, 32, 36, 48.

Note that there are automorphisms of order 27, but not of order 54 and so, with the notations above, $2O_2 \not\subset E_2$.

We would need many pages to list all topological data of the involved automorphisms and so we do not reproduce them here. For example, all automorphisms of order 18 reverse the orientation and we have:

1. Orientable quotient of genus 0 and with two boundary components. Two branching points with ramification indices 3. If f_1 is such an automorphism then f_1^9 has a fixed curve and the action of f_1 on the boundaries permutes the two components and f_1^2 acts on each component as a rotation of order 9.
2. Nonorientable quotient of genus 1 with one boundary component. Two branching points with ramification indices 3. If f_2 is such an automorphism then f_2^9 has no fixed points and the action of f_2 on the boundaries permutes the two components and f_1^2 acts on each component as a rotation of order 9.

Note that f_1 and f_2 cannot be realized as the restrictions of rigid motions on surfaces embedded in R^3 and there are different topological classes of automorphisms in each type.

Acknowledgment

The authors wish to thank the referees for their helpful suggestions.

References

1. Alling, N. L., Greenleaf, N. *Foundations of the Theory of Klein Surfaces*. Lecture Notes in Mathematics, vol. 219, Springer-Verlag, Berlin, 1971.
2. Bujalance, E. Cyclic groups of automorphisms of compact nonorientable surfaces without boundary. *Pacific J. Math.*, **109** (1983), 279–289.
3. Bujalance, E., Costa, A. F., Gamboa, J. M., Lafuente, J. An algorithm to compute odd orders and ramification indices of cyclic actions on compact surfaces. *Discrete Comput. Geom.*, **12** (1994), 451–464.

4. Bujalance, E., Etayo, J. J., Gamboa, J. M., Gromadzki, J. M. *Automorphism Groups of Compact Bordered Klein Surfaces*. Lecture Notes in Mathematics, vol. 1439, Springer-Verlag, Berlin, 1990.
5. Bujalance, E., Gamboa, J. M., Maclachlan, C. Minimum topological genus of compact bordered Klein surfaces admitting a prime-power automorphism. *Glasgow Math. J.*, **37** (1995), 221–232.
6. Etayo, J. J. Non-orientable automorphisms of Riemann surfaces. *Arch. Math. J.*, **45** (1985), 374–384.
7. Harvey, W. J. Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math. Oxford Ser.* (2), **17** (1966), 86–97.
8. Wiman, A. Über die hyperelliptischen Kurven und diejenigen vom Geschlecht $p = 3$, welche eindeutige Transformationen in sich besitzen. *Bihang till Kolgl. Svenka Vetenskaps-Akademiens Handlingar*, **21**, 1, no. 3 (1895), 41 pp.
9. Yokoyama, K. Classification of periodic maps on compact surfaces, I. *Tokyo J. Math.*, **6** (1983), 75–94.
10. Yokoyama, K. Classification of periodic maps on compact surfaces, II. *Tokyo J. Math.*, **7** (1984), 249–285.
11. Yokoyama, K. Complete classification of periodic maps on compact surfaces. *Tokyo J. Math.*, **15** (1992), 247–249.

Received August 31, 1994, *and in revised form October* 18, 1995.