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Planar Rectifiable Curves Are Determined by Their Projections

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Abstract. In 1954 Steinhaus raised the question of whether a rectifiable curve is characterized by its projections. A projection onto a line G at the point p counts the number of points in the set which lie on the line which is perpendicular to G and passes through p. We prove this is so, and give a method to reconstruct a closed connected rectifiable set from its projections.

1. Introduction

A fundamental inverse problem in integral geometry is to determine when sets are determined by its projections. A projection onto a line G at the point p counts the number of points in the set which lie on the line which passes through p and is perpendicular to G. A classical version of this inverse problem, the subject of this paper, is whether (the trace of) a rectifiable curve A is determined by its projections. This problem was raised by Steinhaus [11] in 1954. (Steinhaus actually said "We believe without proof this theorem to be true.") This paper answers the problem in the affirmative by giving a method for reconstructing the curve A from knowledge of the projections of A. Some history and results on the problem appear as problem G8 in [1]. Santaló [9] indicates a proof for rectifiable Jordan curves. Fast [3] independently solved Steinhaus's problem. The results in [3] are actually more general, allowing recovery of closed (not necessarily connected) sets. However, the method of reconstruction presented in this paper is simpler and more direct than that in [3].

In an earlier paper [8] we solved a similar uniqueness and reconstruction problem over the class of sets of finite total absolute curvature, which we call K-sets. However, not all rectifiable curves trace out a set of this type. This paper is based on a reconstruction method different from that of [8], but it is related to it in that both reconstruction methods are founded on differentiability properties of the projections. The reconstruction method of this paper is both more complicated and less stable than the reconstruction method for K-sets, in cases where both methods apply. The two reconstruction methods are independent in the sense that each method applies to some set that the other does not. (K-sets can have multiplicities.)

2. Main Result

Let $\mathbb{H}$ denote the set of lines in the plane and let $\mathbb{H}^{\text{or}}$ denote the set of oriented lines. Let $\mathbb{S}^1$ denote the unit sphere in $\mathbb{R}^2$. An oriented line H^{or} in the plane is represented by $(p, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where θ is the positively oriented unit tangent to the corresponding *line* $\mathcal{O}(H^{\text{or}}) := \{x \in \mathbb{R}^2: \langle x, *\theta \rangle = p\}$, where $*\theta \in \mathbb{S}^1$ denotes the unit vector obtained by rotating θ through (positive) $90°$. Here $\mathcal{O}: \mathbb{H}^{\text{or}} \to \mathbb{H}$ represents the two sheeted covering of $\mathbb{H}$ by $\mathbb{H}^{\text{or}}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$. The measure $d\mu := dp\, d\theta$ is the *kinematic density* on $\mathbb{H}^{\text{or}}$ [9]. The kinematic density is the Haar measure (i.e., the invariant measure) on $\mathbb{H}$ induced by Euclidean motion in the plane. The parameters p and θ serve as local coordinates on $\mathbb{H}$ via $\mathcal{O}$. Hence, the kinematic density on $\mathbb{H}$ is locally expressible as $dp\, d\theta$.

To a set A in the plane one can associate a function $n_A: \mathbb{H} \to \overline{\mathbb{Z}^+}$, defined by $n_A(H) := \mathcal{H}^0(A \cap H)$ ($\mathcal{H}^0$ is the counting measure.) This function is referred to as the *characteristic Crofton function* [6] of the set A. If we fix θ, then the function $n_A(p, \theta)$, defined for $p \in \mathbb{R}$, is the projection of A in the direction θ. For notational simplicity, we use $\tilde{n}_A: \mathbb{H}^{\text{or}} \to \overline{\mathbb{Z}^+}$ to denote $n_A \circ \mathcal{O}$.

The *Favard*, or *integral geometric*, *measure* [4], [5], [7], [10] of a Borel set B is defined as

$$\mathcal{I}(B) := \tfrac{1}{2} \int_{\mathbb{H}} n_B(H)\, d\mu.$$

The functional $\mathcal{I}$ extends to a Borel regular measure. A fundamental theorem of integral geometry is Crofton's formula,

$$\text{length}(c) = \mathcal{I}(c),$$

where c is a curve. More generally, the one-dimensional Hausdorff measure [2], [5], [10], [12], $\mathcal{H}^1$, defined by

$$\mathcal{H}^1(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i): A \subset \bigcup_{i=1}^{\infty} U_i,\ \text{diam}(U_i) \le \delta \right\},$$

agrees with $\mathcal{I}$ on rectifiable sets [5, (3.2.26)], [10].

Steinhaus defines the following distance function for plane sets:

$$d_{\mathcal{I}}(A_1, A_2) = \int_{\mathbb{H}} |n_{A_1}(H) - n_{A_2}(H)|\, d\mu.$$

The problem Steinhaus [11] raised was whether $d_{\mathcal{I}}$ is a metric on $\mathcal{H}^1$ equivalence classes. Obviously, $d_{\mathcal{I}}(\cdot, \cdot) \ge 0$ and $d_{\mathcal{I}}$ satisfies the triangle inequality. Thus, Steinhaus' question is whether $d_{\mathcal{I}}(A_1, A_2) = 0$ implies $\mathcal{H}^1(A_1 \triangle A_2) = 0$, where $\triangle$ denotes symmetric difference. There are entirely nonrectifiable sets of positive and finite $\mathcal{H}^1$ measure. Such sets have zero Favard measure [2], [5]. Every set of finite $\mathcal{H}^1$ measure uniquely decomposes, modulo sets of $\mathcal{H}^1$ measure zero, into a rectifiable part and entirely nonrectifiable part. Thus, apart from the issue of multiplicity, the most general version of Steinhaus's problem concerns rectifiable sets.

In this paper we prove the following.

Theorem 2.1. *If A_1 and A_2 are closed connected rectifiable sets, then $d_{\mathcal{I}}(A_1, A_2) = 0$ if and only if $\mathcal{H}^1(A_1 \Delta A_2) = 0$.*

One direction is obvious: $d_{\mathcal{I}}(A_1, A_2) = 0$ if $\mathcal{H}^1(A_1 \Delta A_2) = 0$ since $\mathcal{I}$ and $\mathcal{H}^1$ agree on rectifiable sets. We prove the opposite implication by giving a procedure to reconstruct A from any measurable function $\hat{n}$ satisfying $\int_{\mathbb{H}} |n_A(H) - \hat{n}(H)|\, d\mu = 0$, i.e., from any representative of the $L^1(\mathbb{H})$ equivalence class of n_A.

In [8] it is shown that the characteristic Crofton function of sets of finite total curvature is a function of bounded variation. The theory of functions of bounded variation is exploited there as the basis of the reconstruction. For less regular sets, however, the characteristic Crofton function has a less accessible structure and a different approach is required.

Gelfand and Smirnov [6] define the concept of a *Crofton density*. An *even 1-density* in $\mathbb{R}^2$ is a real-valued function $\varphi(x, u)$ of $x \in \mathbb{R}^2$ and a vector u in the tangent space at x such that

$$\varphi(x, au) = |a|\varphi(x, u)$$

for all $a \in \mathbb{R}$. (To simplify the notation we identify the tangent space to x in $\mathbb{R}^2$ with $\mathbb{R}^2$ in the canonical way, i.e., $u \in \mathbb{R}^2$.) Even densities can be integrated over manifolds without orientation [6]. An even 1-density φ is a *Crofton 1-density* in $\mathbb{R}^2$ if the integral of φ over a manifold can be represented as the integral of the characteristic Crofton function of the manifold with respect to a measure that is absolutely continuous with respect to the kinematic density. The reconstruction method developed for K-sets in [8] used certain simple Crofton densities. The reconstruction method of this paper considers instead more complicated Crofton densities and shows that the underlying set can be recovered from evaluation of the integrals of these Crofton densities over the set.

3. Rectifiable Sets

A *(parametrized plane) curve* is a continuous function c: $[a, b] \to \mathbb{R}^2$ that is nonconstant on any open subinterval. The *trace* of a curve c is the point set image of c, i.e.,

$$\mathrm{tr}(c) := \{x : c^{-1}(x) \neq \emptyset\}.$$

The *length* of a curve is defined by

$$L(c) := \sup\left\{\sum_{i=1}^{n} |c(t_i) - c(t_{i-1})| : a \leq t_0 < t_1 < \cdots < t_n \leq b\right\}.$$

For C^1 curves, $L(c)$ is equal to $\int_a^b |\dot{x}(t)|\, dt$, i.e., the length of c according to the classical definition.

If $L(c) < \infty$, then c is a *rectifiable* curve. A set $E \subset \mathbb{R}^2$ satisfying $\mathcal{H}^1(E) < \infty$ is *rectifiable* if $\mathcal{H}^1(E \backslash \bigcup_{i=1}^{\infty} \operatorname{tr} c_i) = 0$ for some countable collection of rectifiable curves $\{c_i\}$.

Let $B_\rho(x)$ denote the open ball of radius ρ centered at x, and let B_ρ abbreviate $B_\rho(0)$. Let $E \subset \mathbb{R}^n$. Assuming it exists, the one-dimensional density of E at x is

$$\Theta(E, x) := \lim_{\rho \to 0+} \frac{\mathcal{H}^1(B_\rho(x) \cap E)}{2\rho}.$$

Points $x \in \mathbb{R}^n$ such that $\Theta(E, x) = 1$ are *regular* points of E. If E is rectifiable, then $\mathcal{H}^1$-almost every point in E is regular [2]. Rectifiable sets also have tangents (in a suitably weak sense) $\mathcal{H}^1$-almost everywhere. (See [2] or (3.2.19) of [5].) Thus, even 1-densities can be integrated (with respect to $\mathcal{H}^1$) over rectifiable sets.

In this paper the symbol ν always denotes a unit normal to a rectifiable set A. A rectifiable set A is *oriented* by ($\mathcal{H}^1$-measurably) selecting a unit normal vector at each point of A possessing a tangent line. For an even 1-density φ, we define

$$\int_A \varphi := \int_A \varphi(x, \nu(x))\, d\mathcal{H}^1(x).$$

4. The Reconstruction

The reconstruction has two distinct steps. The first step finds all straight line segments in A. The second step yields a dense subset of that part of A not containing any straight line segments. Both steps rely on using back-projection to define Crofton densities.

If $\theta_1 \neq -\theta_2$, then let $\angle(\theta_1, \theta_2)$ denote the (signed) angle of minimal magnitude through which θ_1 can be rotated to make it coincide with θ_2. For $x \in \mathbb{R}^2 \backslash 0$, let $\vec{x}$ denote the projection of x on $\mathbb{S}^1$, i.e., $\vec{x} = x/|x|$.

4.1. *Weighted Back Projection*

For any $x \in \mathbb{R}^2$, the back projection of a rectifiable set A at x is essentially the total angle subtended by A at x. Here we consider weighted back projections.

Let T_z denote translation by z in the plane. For any 1-density φ, the 1-density $T_z\varphi$ is given by $T_z\varphi(x, u) = \varphi(x - z, dT_z(u)) = \varphi(x - z, u)$.

Let $\omega \in C^\infty(\mathbb{S}^1)$ be an even function, i.e., $\omega(\theta) = \omega(-\theta)$. For each such ω, we define a 1-density φ^ω as follows: set $\varphi^\omega(0, u) = 0$ and if $x \neq 0$, then set

$$\varphi^\omega(x, u) := \omega(\vec{x}) \frac{|\langle u, \vec{x} \rangle|}{|x|}.$$

For each $x, h, u \in \mathbb{R}^2$ let $s(x, u, h) \in \{1, 0, -1\}$ be defined as follows:

$$s(x, u, h) := \begin{cases} \operatorname{sgn}\langle x, u \rangle, & \langle x, u \rangle \neq 0, \\ \operatorname{sgn}\langle -h, u \rangle, & \langle x, u \rangle = 0. \end{cases}$$

For $x \neq 0$ and $u \in \mathbb{R}^2$, we define

$$D_h\varphi^\omega(x, u) := s(x, u, h)|x|^{-2}(\omega(\vec{x})(2\langle u, \vec{x} \rangle \langle h, \vec{x} \rangle - \langle u, h \rangle) + \dot{\omega}(\vec{x})\langle u, \vec{x} \rangle \langle *h, \vec{x} \rangle),$$

where $\dot{\omega}$ denotes the derivative of ω on $\mathbb{S}^1$ in the usual sense. For $x, y \in \mathbb{R}^2$, let

$$\mathcal{P}(x, y) := \{z \in \mathbb{R}^2 \colon \langle z, y\rangle\langle z, x\rangle < 0\}.$$

The following regularity estimate is an important device in our reconstruction.

Lemma 4.1. *Let $x \in \mathbb{R}^2\backslash 0$ and assume $|h| \leq 0.1\,|x|$. Then*

$$T_h\varphi^\omega(x, u) - \varphi^\omega(x, u) = D_h\varphi^\omega(x, u) + e(\omega, h, x, u),$$

where, if $u \in \mathcal{P}(x - h, x)$, then $e(\omega, h, x, u)$ satisfies

$$|e(\omega, h, x, u)| \leq \frac{|u||h|}{|x|^3}\left(5|x|\,\|\omega\|_\infty + |h|\,\|\dot{\omega}\|_\infty\right),$$

and, if $u \notin \mathcal{P}(x - h, x)$, then $e(\omega, h, x, u)$ satisfies

$$|e(\omega, h, x, u)| \leq 6\frac{|u||h|^2}{|x|^3}\left(\|\omega\|_\infty + \|\dot{\omega}\|_\infty + |x|\,\|\ddot{\omega}\|_\infty\right).$$

Proof. The proof is elementary and is given in the Appendix. □

Observe that $\int_A T_x\varphi^\omega$ is continuous on $\mathbb{R}^2\backslash\overline{A}$ as a function of x (here $\overline{A}$ denotes the topological closure of A.) There may be points $x \in \overline{A}$ at which $\int_A T_x\varphi^\omega$ does not have a continuous extension. Rather than concern ourselves with these points, we smooth φ^ω. For any $x, u \in \mathbb{R}^2$ and $\rho > 0$ define

$$\varphi^\omega_\rho(x, u) := \frac{1}{\pi\rho^2}\int_{B_\rho(x)} \varphi^\omega(y, u)\,dy.$$

Although φ^ω is not a Crofton density (see below), φ^ω_ρ is.

Lemma 4.2. *For every $\rho > 0$, φ^ω_ρ is a Crofton density.*

Proof. If $\operatorname{dist}(0, A) > 0$, then polar projection of $\mathbb{R}^2\backslash 0$ onto the unit circle ($x \to x/|x|$) is a Lipschitz function on A. In this case we can apply the area–coarea formula of geometric measure theory [5, (3.2.22)], [7] to obtain

$$\int_A \varphi^\omega = \tfrac{1}{2}\int_{\mathbb{S}^1} \omega(\theta)\,\tilde{n}_A(0, \theta)\,d\theta. \tag{4.1}$$

Equation (4.1) shows that $\int_A \varphi^\omega$ is just the ω-weighted back projection of A at the origin. Strictly speaking, φ^ω is not a Crofton density because the required measure is singular with respect to the kinematic density.

Let $\eta \in C^\infty(\mathbb{R})$ be a nonnegative function with range $[0, 1]$ and support in $[-1, 1]$ satisfying $\eta(t) = 1$ for $|t| \leq \frac{1}{2}$. For $r > 0$, define $\varphi^{\omega,r}(x, u) := (1 - \eta\,(r|x|))\varphi^\omega(x, u)$ and $\varphi^{\omega,r}_\rho := (1/\pi\rho^2)\int_{B_\rho(x)} \varphi^{\omega,r}(y, u)\,dy$. Furthermore, let $n_{A,r}(H) := \int_{A\cap H} (1-\eta(r|x|))\,d\mathcal{H}^0(x)$. This function can be interpreted as the characteristic Crofton function of the set A, where the multiplicity of A at $x \in A$ has been set to $1 - \eta(r|x|)$. Since $\int_A \varphi^{\omega,r} = \int_{A\backslash B_{1/(2r)}} \varphi^{\omega,r}$, we can again apply the area–coarea formula (even if $0 \in A$) to obtain

$$\int_A \varphi^{\omega,r} = \tfrac{1}{2}\int_{\mathbb{S}^1} \omega(\theta)\,\tilde{n}_{A,r}(0,\theta)\,d\theta.$$

It follows from the Fubini theorem that

$$\int_A \varphi_\rho^{\omega,r} = \frac{1}{2\pi\rho^2}\int_{B_\rho}\int_{\mathbb{S}^1} \omega(\theta)\,\tilde{n}_{T_y A,r}(0,\theta)\,d\theta\,dy.$$

A straightforward application of the monotone convergence theorem (let $r \to \infty$) and a change of variables yields

$$\begin{aligned}\int_A \varphi_\rho^{\omega} &= \frac{1}{2\pi\rho^2}\int_{B_\rho}\int_{\mathbb{S}^1} \omega(\theta)\,\tilde{n}_{T_y A}(0,\theta)\,d\theta\,dy \\ &= \frac{1}{\pi\rho^2}\int_{-\rho}^{\rho}\int_{\mathbb{S}^1} \omega(\theta)\,(\rho^2 - p^2)^{1/2}\tilde{n}_A(p,\theta)\,d\theta\,dp.\end{aligned}$$ □

4.2. *Recovering Straight Line Segments*

To recover line segments we set $\omega(\theta) = 1$. Henceforth in this section we drop ω from our notation. Thus, $\varphi = \varphi^\omega$ where $\omega \equiv 1$.

Assume now that A is a rectifiable set with bounded support, i.e., $\overline{A} \subset B_R$ for some $R < \infty$. It is clear that a suitable R can be determined from the L^1 equivalence class of n_A since $n_A(H) = 0$ for all H such that $B_R \cap H = \emptyset$. Thus, there is no loss in generality in assuming a fixed R.

Let $x \in \mathbb{R}^2\backslash\overline{B_R}$ and define $L_x(A) := \{y \in A\colon \langle \nu(y), y - x\rangle = 0\}$. We conclude from (4.1) that $\int_A T_x\varphi = \lim_{\rho\to 0+}\int_A T_x(\varphi_\rho)$, hence $\int_A T_x\varphi$ is uniquely determined by the L^1 equivalence of n_A. Now, for any $|x| > R$, we have

$$\int_A T_x D_h\varphi = \int_{T_{-x}A} D_h\varphi.$$

Since $\bigcap_{\varepsilon>0}\{y \in T_{-x}A\colon \nu(y) \in \mathcal{P}(y - \varepsilon h, y)\} = \emptyset$, we have

$$\lim_{\varepsilon\to 0+} \mathcal{H}^1\{y \in T_{-x}A\colon \nu(y) \in \mathcal{P}(y - \varepsilon h, y)\} = 0$$

and we can apply Lemma 4.1 to obtain

$$\int_A T_x D_h\varphi = \lim_{\varepsilon\to 0+} \varepsilon^{-1}\left(\int_A T_{x+\varepsilon h}\varphi - \int_A T_x\varphi\right).$$

Thus, $\nabla_x \int_A T_x\varphi$ exists if and only if $\mathcal{H}^1(L_x(A)) = 0$. More precisely,

$$\int_A D_h T_x\varphi + \int_A D_{-h}T_x\varphi = 2\int_{L_x(A)} \frac{|\langle \nu, h\rangle|}{|y - x|^2}\,d\mathcal{H}^1(y).$$

Since $\mathcal{H}^1(A) < \infty$, there are at most countably many lines H for which $\mathcal{H}^1(H \cap A)$ is positive. It is now evident that this set of lines is determined by $\int_A T_x\varphi$, $x \in \mathbb{R}^2 \backslash B_R$, and that if $x \in H$, then $\int_{H\cap A} |y-x|^{-2}\, d\mathcal{H}^1(y)$ is determined. By differentiating with respect to x on H, we can also determine $\int_{H\cap A} |y-x|^{-(2+i)}\, d\mathcal{H}^1(y)$ for $i \in \mathbb{Z}^+$. From the Stone–Weierstrass theorem and the fact that $\operatorname{dist}(x, A) > 0$, we see that $\int_{H\cap A} f(|y-x|)\, d\mathcal{H}^1(y)$ is determined for each $f \in C(\mathbb{R})$. It follows from the general theory of rectifiable sets that $A \cap H$ is uniquely determined up to a set of $\mathcal{H}^1$ measure zero.

Let L_A denote the closure of the set $\{y\colon \Theta(A \cap H, y) = 1\}$ for some $H \in \mathbb{H}$. Obviously, $L_A \subset A$ when A is closed. We have proved the following.

Theorem 4.3. *If A is a rectifiable set of compact support, then L_A is uniquely determined by the L^1 equivalence class of n_A.*

4.3. *Recovering Curved Segments*

In this section we prove Theorem 2.1. The key idea is to study the regularity of n_A near lines which are, in some sense, tangent to A.

Let $\tilde{\omega} \in C^\infty(\mathbb{R})$ be a nonnegative even function with support in $(-1, 1)$ satisfying $\int_{\mathbb{R}} \tilde{\omega}(x)\, dx = \frac{1}{2}$ and $\tilde{\omega}(x) = 0$ in some neighborhood of zero. For any $0 < \varepsilon \le 1$ and $\psi \in \mathbb{S}^1$, we define an even function $\omega_{\varepsilon,\psi} \in C^\infty(\mathbb{S}^1)$ by

$$\omega_{\varepsilon,\psi}(\theta) := \varepsilon^{-1/2}\tilde{\omega}(\varepsilon^{-1/2}\arcsin\langle \theta, \psi\rangle),$$

where arcsin: $[-1, 1] \to (-\pi/2, \pi/2]$. We have, $\int_{\mathbb{S}^1} \omega_{\varepsilon,\psi}(\theta)\, d\theta = 1$. Let

$$Q_A(x) := \sup_{\psi\in\mathbb{S}^1, s>0} \limsup_{\varepsilon\to 0+} \left| \int_A T_{x+\varepsilon s\psi}\varphi_\varepsilon^{\omega_\varepsilon,\psi} - \int_A T_{x-\varepsilon s\psi}\varphi_\varepsilon^{\omega_\varepsilon,\psi} \right|.$$

The function Q_A is carefully designed to test for a particular regularity property of n_A. Since $T_z\varphi_\varepsilon^{\omega_\varepsilon,\psi}$ is a Crofton density for any z, Q_A is determined by the L^1 equivalence class of n_A.

Lemma 4.4. *Let A be a rectifiable set. If $\operatorname{dist}(x, A) > 0$, then $Q_A(x) = 0$.*

Proof. Fix x, s, and ψ. Without loss of generality, we can assume $x = 0$. Let δ denote $\min(\operatorname{dist}(0, A), 1)$ and assume $\delta > 0$. Define

$$A_\varepsilon := \{y \in A\colon \omega_{\varepsilon,\psi}(\vec{y}) \ne 0\}.$$

Since $\bigcap_{\xi>0}(\bigcup_{\varepsilon<\xi} A_\varepsilon) = \emptyset$, we have $\lim_{\varepsilon\to 0+} \mathcal{H}^1(A_\varepsilon) = 0$. Let $z \in B_\varepsilon$ be arbitrary and let h denote $z + \varepsilon s\psi$. Using the notation of Lemma 4.1 we write

$$\begin{aligned} &T_h\varphi^{\omega_{\varepsilon,\psi}}(y, u) - T_{-h}\varphi^{\omega_{\varepsilon,\psi}}(y, u) \\ &\qquad = D_h\varphi^{\omega_{\varepsilon,\psi}}(y, u) - D_{-h}\varphi^{\omega_{\varepsilon,\psi}}(y, u) + e(\omega_{\varepsilon,\psi}, h, y, u) - e(\omega_{\varepsilon,\psi}, -h, y, u). \end{aligned}$$

From the estimates of Lemma 4.1 we conclude that there are constants $C_1 = C_1(s, \delta) <$

∞ and $C_2 = C_2(s, \delta) < \infty$ such that if $\varepsilon(1+s) < 0.1\delta$, then

$$\left|\int_A T_h\varphi^{\omega_{\varepsilon,\psi}} - \int_A T_{-h}\varphi^{\omega_{\varepsilon,\psi}}\right|$$
$$\leq C_1(\delta, s)(\varepsilon^{1/2}\|\tilde{\omega}\|_\infty + \|\dot{\tilde{\omega}}\|_\infty)\mathcal{H}^1(A_\varepsilon)$$
$$+ C_2(\delta, s)\varepsilon^{1/2}(\|\tilde{\omega}\|_\infty + \|\dot{\tilde{\omega}}\|_\infty + \|\ddot{\tilde{\omega}}\|_\infty)\mathcal{H}^1(A).$$

The lemma follows easily from this. □

The remainder of the proof rests on establishing that $Q_A > 0$ on a dense subset of the curved part of A. Our approach relies on certain properties of convex sets and functions on $\mathbb{S}^1$ of bounded variation.

The total variation $V_{\mathbb{S}^1}(g)$ of a real function $g \in L^1(\mathbb{S}^1)$ is defined as the supremum of $\{\sum_{i=0}^n |g(\theta_i) - g(\theta_{i-1})|\}$, where $\theta_0, \theta_1, ..., \theta_n = \theta_{-1}$ is an ordered cycle of Lebesgue points [12] of g in $\mathbb{S}^1$. If $V_{\mathbb{S}^1}(g) < \infty$, then we say $g \in BV(\mathbb{S}^1)$. Standard theory on BV functions implies that if $g \in BV(\mathbb{S}^1)$, then $\dot{g}$ exists almost everywhere in $\mathbb{S}^1$.

Let D be a closed bounded convex set in $\mathbb{R}^2$. For each $\theta \in \mathbb{S}^1$ define

$$f(\theta) := \sup\{\langle x, \theta\rangle : x \in D\}.$$

Proposition 4.5. *The function $f(\theta)$ is absolutely continuous and $\dot{f} \in BV(\mathbb{S}^1)$. Furthermore, if $\dot{f}(\theta)$ exists, then $x_e(\theta) := f(\theta)\theta + \dot{f}(\theta) *\theta$ is the unique (extreme) point of D satisfying $f(\theta) = \langle x_e(\theta), \theta\rangle$.*

Proof. Let $\overline{f} := \sup_\theta f(\theta) = \sup_{x\in D}|x|$ and assume $\overline{f} > 0$. Let $\theta_1, \theta_2 \in \mathbb{S}^1$ and assume that $f(\theta_2) \geq f(\theta_1)$. We can find $x \in \partial D$ such that $\langle x, \theta_2\rangle = f(\theta_2)$. It follows that $f(\theta_1) \geq \langle x, \theta_1\rangle$ so $f(\theta_2) - f(\theta_1) \leq |x||\theta_2 - \theta_1| \leq \overline{f}|\angle(\theta_2, \theta_1)|$. Thus, f is absolutely continuous. A simple argument, which we omit, shows that f has left and right derivatives, $\dot{f}(\theta-)$ and $\dot{f}(\theta+)$ respectively, for all θ. Furthermore,

$$\{x \in D\colon \langle x, \theta\rangle = f(\theta)\} = \{f(\theta)\theta + \beta *\theta\colon \beta \in [\dot{f}(\theta-), \dot{f}(\theta+)]\}.$$

Therefore, if $\dot{f}(\theta)$ exists, then $\partial D \cap H_\theta$ is a singleton. Moreover, it follows easily that $\dot{f} \in BV(\mathbb{S}^1)$. □

From Proposition 4.5, we conclude that $\ddot{f}$ exists for almost all $\theta \in \mathbb{S}^1$. Using this fact, we obtain the following.

Lemma 4.6. *For almost all $\theta \in \mathbb{S}^1$, there exists a unique extreme point $x_e(\theta)$ of D, satisfying $f(\theta) = \langle x_e, \theta\rangle$, and a real number $K(\theta) < \infty$ such that*

$$T_{x_e+s\varepsilon\theta}\varphi_\varepsilon^{\omega_{\varepsilon,\theta}}(y, \cdot) \equiv 0$$

for all $y \in D$ whenever $s \geq K$ and $\varepsilon \leq K^{-1}$.

Proof. By Proposition 4.5, $\dot{f}$ and $\ddot{f}$ exist $\mathcal{H}^1$-almost everywhere in $\mathbb{S}^1$. Let θ be a point where $\dot{f}$ and $\ddot{f}$ exist, and let $\delta \in (0, 1)$ be small enough so that if $|\angle(\psi, \theta)| \le \delta$, then

$$f(\psi) \le f(\theta) + \dot{f}(\theta)\angle(\theta, \psi) + (1 + |\ddot{f}(\theta)|)\,[\angle(\theta, \psi)]^2. \tag{4.2}$$

By Proposition 4.5, $x_e = f(\theta)\theta + \dot{f}(\theta) *\theta$ is the unique point in D such that $f(\theta) = \langle x_e, \theta \rangle$. It follows that if $|\angle(\psi, \theta)| \le \delta$, then

$$\begin{aligned} \langle x_e, \psi \rangle - f(\psi) \ge\ & f(\theta)(\langle \theta, \psi \rangle - 1) + \dot{f}(\theta)(\langle *\theta, \psi \rangle - \angle(\theta, \psi)) \\ & -(1 + |\ddot{f}(\theta)|)\,[\angle(\theta, \psi)]^2 \\ \ge\ & -(|x_e| + 1 + |\ddot{f}(\theta)|)\,[\angle(\theta, \psi)]^2. \end{aligned} \tag{4.3}$$

Let $K(\theta) = \max\{\delta^{-2}, 2(|x_e| + 2 + |\ddot{f}(\theta)|)\}$ and assume $s \ge K$ and $\varepsilon \le K^{-1}$. Let $z \in B_\varepsilon$ and $y \in \mathbb{R}^2 \backslash (x_e + s\varepsilon\theta + z)$ be such that $T_{x_e + s\varepsilon\theta + z}\varphi^{\omega_{\varepsilon,\theta}}(y, u) \ne 0$ for some $u \in T_y$. We show that this implies $y \notin D$, which proves the lemma.

Since $T_{x_e + s\varepsilon\theta + z}\varphi^{\omega_{\varepsilon,\theta}}(y, u) \ne 0$, the line through y and $x_e + s\varepsilon\theta + z$ has a unit normal ψ that satisfies $|\angle(\theta, \psi)| \le \varepsilon^{\frac{1}{2}} \le \delta$. Note that $\langle \theta, \psi \rangle > \frac{1}{2}$. Using this and (4.3) we obtain

$$\begin{aligned} \langle y, \psi \rangle - f(\psi) &= \langle x_e + s\varepsilon\theta + z, \psi \rangle - f(\psi) \\ &\ge s\varepsilon\langle \theta, \psi \rangle - \varepsilon(|x_e| + 2 + |\ddot{f}(\theta)|) \\ &> 0 \end{aligned}$$

and, therefore, $y \notin D$. □

We are now ready to provė the main technical lemma.

Lemma 4.7. *If A is a closed connected rectifiable set, then the set $\{Q_A \ge 1\}$ is dense in $A \backslash L_A$.*

Proof. Let $y \in A$ be a regular point of A disjoint from L_A. Since A is connected and $\mathcal{H}^1(B_\rho(y) \cap A) \ge \int_0^r \mathcal{H}^0(\partial B_\rho(y) \cap A)\, d\rho$, we can conclude that $\mathcal{H}^0(\partial B_\rho(y) \cap A) \in \{1, 2\}$ for infinitely many arbitrarily small ρ. Let D_ρ denote the closed convex hull of $A \cap B_\rho(y)$ and assume $\mathcal{H}^0(\partial B_\rho(y) \cap A) \in \{1, 2\}$. Since y is disjoint from L_A, it follows that D_ρ has a nonempty interior. Thus, there exists an open subset $E \in \mathbb{S}^1$ such that if $\theta \in E$ and $x \in D_\rho$ maximizes $\langle x, \theta \rangle$ over D_ρ, then for $x \notin \partial B_\rho$. By Lemma 4.6 we can find $\theta \in E$ such that there is a unique extreme point x of D_ρ with exterior normal θ. Furthermore, there is a finite constant $1 < K < \infty$ such that if $\varepsilon < K^{-1}$, then $\int_{A \cap B_\rho(y)} T_{x + \varepsilon K\theta}\varphi_\varepsilon^{\omega_{\varepsilon,\theta}} = 0$. Since x is an extreme point of D_ρ, we have $x \in A$. Since A is connected, there exists a path inside $A \cap D_\rho$ from x to some $x' \in \partial B_\rho \cap D_\rho$. (Connected and pathwise connected are the same for rectifiable sets in the plane [2].) Since $\langle x, \theta \rangle > \langle x', \theta \rangle$ it follows that if ε is sufficiently small, and $T_{x - \varepsilon K\theta + z}\varphi^{\omega_{\varepsilon,\theta}}(w, u) \ne 0$, where $z \in B_\varepsilon$, then the line H through $x - \varepsilon K\theta + z$ and w intersects the line segment joining x to x'. Hence, H intersects the path from x to x' in $A \cap D_\rho$. We conclude

$$\int_{A \cap B_\rho(y)} T_{x - \varepsilon K\theta}\varphi_\varepsilon^{\omega_{\varepsilon,\theta}} \ge 1.$$

From Lemma 4.4, we deduce $Q_{A\setminus B_\rho(y)}(x) = 0$ and we now obtain $Q_A(x) \geq 1$. Since ρ is arbitrarily small, we conclude that y is in the topological closure of $\{Q_A \geq 1\}$. □

By Lemmas 4.4 and 4.7, we see that if A is a closed connected rectifiable set, then $A = \overline{\{Q_A \geq 1\}} \cup L_A$. Since $Q_A(x)$ is determined by the L^1 equivalence class of n_A, we see, by Theorem 4.3, that the proof of Theorem 2.1 is complete.

Appendix

Proof of Lemma 4.1. We assume throughout that $|h| \leq 0.1|x|$.

Assume first that $u \in \mathcal{P}(x-h, x)$. Then

$$|\langle x, u\rangle| \leq |u| \left|\left\langle *\frac{x-h}{|x-h|}, x\right\rangle\right| \leq |u||h|$$

and, similarly, $|\langle x-h, u\rangle| \leq |u||h|$. We now obtain the bounds

$$\begin{aligned} |T_h\varphi^\omega(x,u) - \varphi^\omega(x,u)| &\leq \|\omega\|_\infty \left(\frac{|\langle x-h, u\rangle|}{|x-h|^2} + \frac{|\langle x, u\rangle|}{|x|^2}\right) \\ &\leq 3\|\omega\|_\infty |u||h||x|^{-2}, \\ |D_h\varphi^\omega(x,u)| &\leq 2\|\omega\|_\infty |u||h||x|^{-2} + \|\dot{\omega}\|_\infty |u||h|^2|x|^{-3}. \end{aligned}$$

Combining these bounds yields the claimed estimate.

Henceforth, we assume $u \notin \mathcal{P}(x-h, x)$. Since $\varphi^\omega(x,u)$ is an even 1-density, we can assume that $s(x,u,h) = 1$ without loss of generality. We now have

$$\begin{aligned} T_h\varphi^\omega(x,u) - \varphi^\omega(x,u) &= [T_h\omega(\vec{x}) - \omega(\vec{x})][T_h(|x|^{-2}\langle x,u\rangle) - |x|^{-2}\langle x,u\rangle] \\ &\quad + [T_h\omega(\vec{x}) - \omega(\vec{x})]|x|^{-2}\langle x,u\rangle \\ &\quad + \omega(\vec{x})[T_h(|x|^{-2}\langle x,u\rangle) - |x|^{-2}\langle x,u\rangle]. \end{aligned} \tag{A.1}$$

The first term in (A.1) can be bounded above by $6\|\dot{\omega}\|_\infty|h|^2|u||x|^{-3}$. Since $\|\dot{\omega}\|_\infty \leq \pi\|\ddot{\omega}\|_\infty$, we obtain the estimate

$$\begin{aligned} &\left|T_h\omega(\vec{x}) - \omega(\vec{x}) - \dot{\omega}(\vec{x})\frac{\langle x, *h\rangle}{|x|^2}\right| \\ &\quad \leq \|\dot{\omega}\|_\infty \left|\frac{\langle \vec{x}, *h\rangle}{|x|} - \arcsin\left(\frac{\langle \vec{x}, *h\rangle}{|x-h|}\right)\right| + \|\ddot{\omega}\|_\infty \arcsin^2\left(\frac{\langle \vec{x}, *h\rangle}{|x-h|}\right) \\ &\quad \leq 6|x|^{-2}\|\ddot{\omega}\|_\infty|h|^2. \end{aligned} \tag{A.2}$$

Let

$$F(x,u,h) := \frac{2\langle \vec{x}, u\rangle\langle \vec{x}, h\rangle - \langle u, h\rangle}{|x|^2}.$$

Then

$$T_h\left(\frac{\langle x,u\rangle}{|x|^2}\right) - \left(\frac{\langle x,u\rangle}{|x|^2}\right) = F(x,u,h)\frac{|x|^2}{|x-h|^2} - \frac{|h|^2\langle x,u\rangle}{|x|^2|x-h|^2}.$$

Since $|F(x,u,h)| \le 2|u||h||x|^{-2}$, we obtain

$$\left| T_h\left(\frac{\langle x,u\rangle}{|x|^2}\right) - \frac{\langle x,u\rangle}{|x|^2} - F(x,u,h)\right| \le 6\frac{|h|^2}{|x|^3}. \tag{A.3}$$

Substituting (A.3) and (A.2) into (A.1), the desired result is obtained. □

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