PRECISION OF SELENODETIC FRAMES OF REFERENCE

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Abstract. The number of reference points for the fixing of a selenodetic reference frame in the Moon's body is estimated. It is shown that, for this purpose, from 40 to 100 reference points are sufficient. Precision of the selenodetic coordinate transformations from one system to another is also analysed.

Positions of the lunar surface points are assumed to be determined with reference to the dynamical system. Axes of this system should coincide with the Moon's principal axes of inertia. However, practically all frames of reference are fixed in the Moon's body by a number of the selenodetic reference points. Coordinates of these points are inevitably affected by errors of observation, both accidental and systematic. Because of this, the system of different catalogues of reference points coincide neither with the dynamical system, nor one with another (Gavrilov and Kisliuk, 1971; Kisliuk, 1971).

Taking into account all these circumstances we find it important to determine the number and quality of selenodetic reference points, which are necessary for setting up a selenodetic system of definitive validity.

Let ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 be rectangular coordinates of the reference points in catalogues 1 and 2, respectively. It is possible to represent the differences between any two system of catalogues (Kisliuk, 1971) by

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \\ \Delta\zeta \end{pmatrix} = \begin{pmatrix} a & b & c \\ e & f & g \\ m & n & l \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{pmatrix} + \begin{pmatrix} d \\ h \\ k \end{pmatrix}, \tag{1}$$

where $\Delta\xi = \xi_2 - \xi_1$, $\Delta\eta = \eta_2 - \eta_1$, $\Delta\zeta = \zeta_2 - \zeta_1$ and $a, b, c, \ldots, k$ – are the reduction coefficients.

Matrices of these coefficients depend on the systematic differences of the compared catalogues and have the following geometrical properties:

$$A = \begin{pmatrix} a & b & c \\ e & f & g \\ m & n & l \end{pmatrix} = R + D, \tag{2}$$

where

$$R = \begin{pmatrix} 0 & \frac{1}{2}(b-e) & \frac{1}{2}(c-m) \\ -\frac{1}{2}(b-e) & 0 & \frac{1}{2}(g-n) \\ -\frac{1}{2}(c-m) & -\frac{1}{2}(g-n) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \tag{3}$$

– matrix of the relative rotation, and

$$D = \begin{pmatrix} a & \frac{1}{2}(b+e) & \frac{1}{2}(c+m) \\ \frac{1}{2}(b+e) & f & \frac{1}{2}(g+n) \\ \frac{1}{2}(c+m) & \frac{1}{2}(g+n) & l \end{pmatrix} = \begin{pmatrix} a & \mu & \nu \\ \mu & f & \pi \\ \nu & \pi & l \end{pmatrix} \tag{4}$$

– deformations matrix. Matrix

$$B = \begin{pmatrix} d \\ h \\ k \end{pmatrix} \tag{5}$$

denotes coordinates differences of the origins the systems.

The elements of matrices (2)–(5) and variances $\varepsilon^2_{\Delta\xi}$, $\varepsilon^2_{\Delta\eta}$, $\varepsilon^2_{\Delta\zeta}$ could be determined by means of the least-squares fitting of the positions of points (not less than 4) common to the two catalogues. Simultaneously, the following values could be determined:

$$P = \begin{pmatrix} Q_{11}^{-1} & & & \\ & Q_{22}^{-1} & & \\ & & Q_{33}^{-1} & \\ & & & Q_{44}^{-1} \end{pmatrix} \tag{6}$$

– weight matrix, where Q_{ii} – diagonal elements of the inverse matrix composed from the coefficients of normal equations;

$$r = \begin{pmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{21} & 1 & r_{23} & r_{24} \\ r_{31} & r_{32} & 1 & r_{34} \\ r_{41} & r_{42} & r_{43} & 1 \end{pmatrix} \tag{7}$$

– correlation matrix.

Let us consider first the precision of the transformation (1) as a whole. The variance of the function of dependent values could be presented in the form (Kemnitz, 1970)

$$\begin{aligned} \sigma^2_{\Delta\xi} &= c^T\sigma_x r \sigma_x c\,, \\ \sigma^2_{\Delta\eta} &= c^T\sigma_y r \sigma_y c\,, \\ \sigma^2_{\Delta\zeta} &= c^T\sigma_z r \sigma_z c\,, \end{aligned} \tag{8}$$

where C^T is the transpose of

$$c = \begin{pmatrix} \xi \\ \eta \\ \zeta \\ 1 \end{pmatrix},$$

$$\sigma_x = \begin{pmatrix} \sigma_a & & & \\ & \sigma_b & & \\ & & \sigma_c & \\ & & & \sigma_d \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} \sigma_e & & & \\ & \sigma_f & & \\ & & \sigma_g & \\ & & & \sigma_h \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} \sigma_m & & & \\ & \sigma_n & & \\ & & \sigma_l & \\ & & & \sigma_k \end{pmatrix} \tag{9}$$

are diagonal matrices of standard errors of the coefficients of transformation (1).

Formulae (8) could be written in expanded form as

$$\sigma_{\Delta\xi}^2 = \xi^2\sigma_a^2 + \eta^2\sigma_b^2 + \zeta^2\sigma_c^2 + \sigma_d^2 + 2\xi\eta\sigma_a\sigma_b r_{12} + \\ + 2\xi\zeta\sigma_a\sigma_c r_{13} + 2\xi\sigma_a\sigma_d r_{14} + 2\eta\zeta\sigma_b\sigma_c r_{23} + \\ + 2\eta\sigma_b\sigma_d r_{24} + 2\zeta\sigma_c\sigma_d r_{34}. \tag{10}$$

Analogous expressions for $\sigma_{\Delta\eta}^2$ and $\sigma_{\Delta\zeta}^2$ could be obtained by using σ_e, σ_f, σ_g, σ_h and σ_m, σ_n, σ_l, σ_k respectively.

If reference points are more or less regularly distributed on the Moon's surface, reduction coefficients in Equations (1) are determined most reliably, and Formula (10) could be simplified. For the pair of catalogues of Tucson (Arthur and Bates, 1968) and AMS (Army Map Service, 1964) with number of common points $n=60$, we can determine the weight matrix

$$p = \begin{pmatrix} 9.35 & & & \\ & 13.50 & & \\ & & 1.80 & \\ & & & 2.94 \end{pmatrix} \tag{11}$$

and correlation matrix

$$r = \begin{pmatrix} 1 & -0.02 & -0.10 & 0.09 \\ & 1 & -0.12 & 0.13 \\ & & 1 & -0.97 \\ & & & 1 \end{pmatrix}. \tag{12}$$

Elements of matrices A and B, standard errors (9) and errors of unit weight for each coordinate are presented in Table I. All date in this and the next tables are expressed in terms of the lunar radius.

TABLE I

The elements of matrixes A and B, standard errors, and errors of unit weight for each coordinate.

a e m	b f n	c g l	d h k	$\varepsilon_{\Delta\xi}$ $\varepsilon_{\Delta\eta}$ $\varepsilon_{\Delta\zeta}$
		$\times 10^{-5}$		
-41 ± 6	37 ± 5	0 ± 14	-16 ± 11	± 0.00018
-44 ± 6	-39 ± 5	-17 ± 13	12 ± 10	± 0.00017
-14 ± 42	-143 ± 38	7 ± 100	-160 ± 78	± 0.00134

Taking into account the above data we come to the conclusion that

$$r_{12} = r_{13} = r_{14} = r_{24} = 0, \qquad r_{34} = -0.97 \approx -1. \tag{13}$$

The expression (10) can now be rewritten as

$$\sigma_{\Delta\xi}^2 = \xi^2\sigma_a^2 + \eta^2\sigma_b^2 + \zeta^2\sigma_c^2 + \sigma_a^2 - 2\zeta\sigma_c\sigma_d . \tag{14}$$

From (11), (12) and (13) we shall have

$$\begin{gathered} Q_{11} = Q_{22}, \qquad \varepsilon_{\Delta\xi} = \varepsilon_{\Delta\eta}, \\ \sigma_a = \sigma_b = \sigma_e = \sigma_f, \quad \sigma_c = \sigma_g, \quad \sigma_d = \sigma_h, \quad \sigma_m = \sigma_n \end{gathered} \tag{15}$$

and, therefore,

$$\sigma_{\Delta\xi}^2 = \sigma_{\Delta\eta}^2 . \tag{16}$$

Let us, furthermore, use the following notations

$$\begin{aligned} \sigma_1 &= \tfrac{1}{4}(\sigma_a + \sigma_b + \sigma_e + \sigma_f) = \varepsilon_{\Delta\xi}\sqrt{Q_{11}}, \\ \sigma_2 &= \tfrac{1}{2}(\sigma_c + \sigma_g) = \varepsilon_{\Delta\xi}\sqrt{Q_{33}}, \\ \sigma_3 &= \tfrac{1}{2}(\sigma_d + \sigma_h) = \varepsilon_{\Delta\xi}\sqrt{Q_{44}}, \\ \sigma_4 &= \tfrac{1}{2}(\sigma_m + \sigma_n) = \varepsilon_{\Delta\zeta}\sqrt{Q_{11}}, \\ \sigma_5 &= \sigma_l = \varepsilon_{\Delta\zeta}\sqrt{Q_{33}}, \\ \sigma_6 &= \sigma_k = \varepsilon_{\Delta\zeta}\sqrt{Q_{44}} \end{aligned} \tag{17}$$

and

$$\begin{aligned} A &= \sigma_2^2 - \sigma_1^2, \\ B &= -2\sigma_2\sigma_3, \\ C &= \sigma_1^2 + \sigma_3^2, \\ D &= \sigma_5^2 - \sigma_4^2, \\ E &= -2\sigma_5\sigma_6, \\ F &= \sigma_4^2 + \sigma_6^2 . \end{aligned} \tag{18}$$

From (16), (17), (18) and if we take into account that $\xi^2 + \eta^2 + \zeta^2 = 1$, formulae for $\sigma_{\Delta\xi}^2$ and ${}^2_{\Delta\zeta}$ can be written in the forms

$$\begin{aligned} \sigma_{\Delta\xi}^2 &= A\zeta^2 + B\zeta + C, \\ \sigma_{\Delta\zeta}^2 &= D\zeta^2 + E\zeta + F; \end{aligned} \tag{19}$$

and

$$\begin{aligned} \sigma_{\Delta\xi}^2 &= \varepsilon_{\Delta\xi}^2[(Q_{33} - Q_{11})\zeta^2 - 2\sqrt{Q_{33}Q_{44}}\,\zeta + (Q_{11} + Q_{44})], \\ \sigma_{\Delta\zeta}^2 &= \varepsilon_{\Delta\zeta}^2[(Q_{33} - Q_{11})\zeta^2 - 2\sqrt{Q_{33}Q_{44}}\,\zeta + (Q_{11} + Q_{44})]. \end{aligned} \tag{20}$$

Finally, we have

$$\begin{aligned} \sigma_{\Delta\xi} &= \varepsilon_{\Delta\xi}\sqrt{Q}, \\ \sigma_{\Delta\zeta} &= \varepsilon_{\Delta\zeta}\sqrt{Q}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} Q &= Q_1\zeta^2 + Q_2\zeta + Q_3, \\ Q_1 &= Q_{33} - Q_{11}, \\ Q_2 &= -2\sqrt{Q_{33}Q_{44}}, \\ Q_3 &= Q_{11} + Q_{44}. \end{aligned} \tag{22}$$

Thus, systematic errors of coordinates transformation when using Formulae (1) can be estimated from (19) or (21) and (22).

Since, $\zeta = \sin K$, where K is the angular distance from the centre of the lunar disk, the quantity Q is a weight coefficient. This coefficient depends on the number of the points which are common to both systems, and which were used for determination of elements of matrix A and B, as well as on the position of the point whose coordinates are transformed.

In Figure 1 values of $\sqrt{Q}$ are presented for different numbers of common points

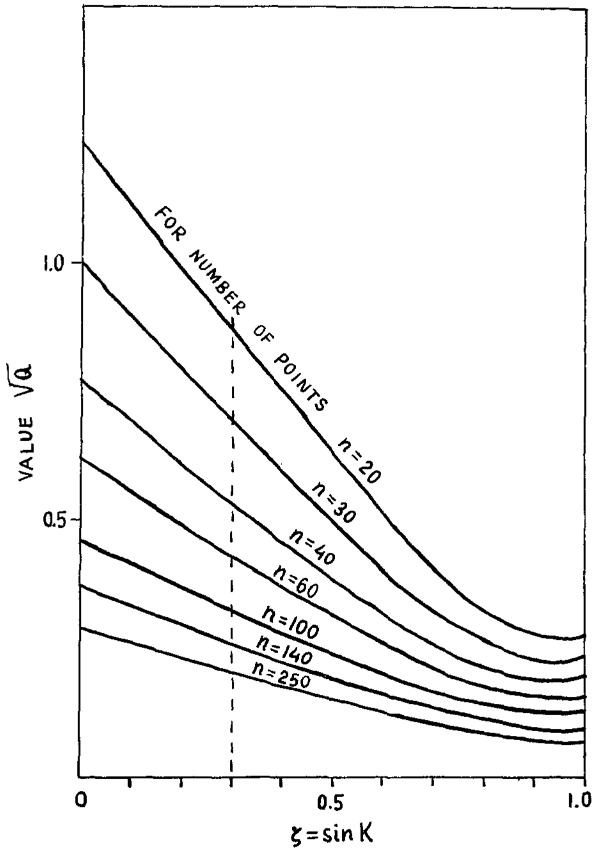


Fig. 1.

versus the value $\zeta = \sin K$. The curves are drawn for 20, 40, 60, 80, 100, 120, 140 and 250 common points. They are parabolas, minima of which are situated at $\zeta = 0.92$ ($K = 20°$). Therefore, the lines of equal errors $\sigma_{\Delta\xi}(\sigma_{\Delta\eta})$ and $\sigma_{\Delta\zeta}$ should be concentric circles on the lunar disk.

The selenodetic positions of reference points are determined from the Earth only for $\zeta \geqslant 0.3 R_{☾}$ ($K \leqslant 70°$). This limit is marked in Figure 1 by broken line.

Now we consider the precision of elements, which characterize the differences of two selenodetic systems. From (3) and (4) one can write

$$\sigma_\alpha = \sigma_\mu ,$$
$$\sigma_\beta = \sigma_\nu ,$$
$$\sigma_\gamma = \sigma_\pi ,$$

and if we take account of (15),

$$\sigma_\beta = \sigma_\gamma , \qquad \sigma_a = \sigma_f , \qquad \sigma_d = \sigma_h ;$$

so that we have to estimate only 6 values among 12

$$\sigma_d, \sigma_k, \sigma_\alpha, \sigma_\beta, \sigma_a, \sigma_l .$$

From (17) and taking into account (3) we obtain the following formulae

$$\sigma_d = \varepsilon_{\Delta\xi}\sqrt{Q_{44}} , \tag{24}$$
$$\sigma_k = \varepsilon_{\Delta\zeta}\sqrt{Q_{44}} , \tag{25}$$
$$\sigma_\alpha = \frac{1}{\sqrt{2}}\sigma_a = \frac{1}{\sqrt{2}}\varepsilon_{\Delta\xi}\sqrt{Q_{11}} , \tag{26}$$
$$\sigma_\beta = \tfrac{1}{2}\sqrt{\sigma_c^2 + \sigma_m^2} = \tfrac{1}{2}\sqrt{\varepsilon_{\Delta\xi}^2 Q_{33} + \varepsilon_{\Delta\zeta} Q_{11}} , \tag{27}$$
$$\sigma_a = \varepsilon_{\Delta\xi}\sqrt{Q_{11}} , \tag{28}$$
$$\sigma_l = \varepsilon_{\Delta\zeta}\sqrt{Q_{33}} . \tag{29}$$

In Figure 2 values of weight coefficients $Q_{11} = Q_{22}$, Q_{33} and Q_{44} are presented versus the number of common reference points, which we used for solving Equations (1).

TABLE II

Weight and mean errors of coordinates.

Weight	$\varepsilon_{\Delta\xi} = \varepsilon_{\Delta\eta}$	$\varepsilon_{\Delta\zeta}$
	$\times 10^{-5}$	
5	10	60
1	25	135

Results presented above can be used for estimation of the precision of coordinate transformation in particular cases. Let us consider selenodetic catalogues of two precision classes – excellent and ordinary. Therefore, three cases of coordinate transformation can be considered. In Table II weight and mean errors of coordinates in both catalogues (excellent–weight 5, ordinary–weight 1) are presented.

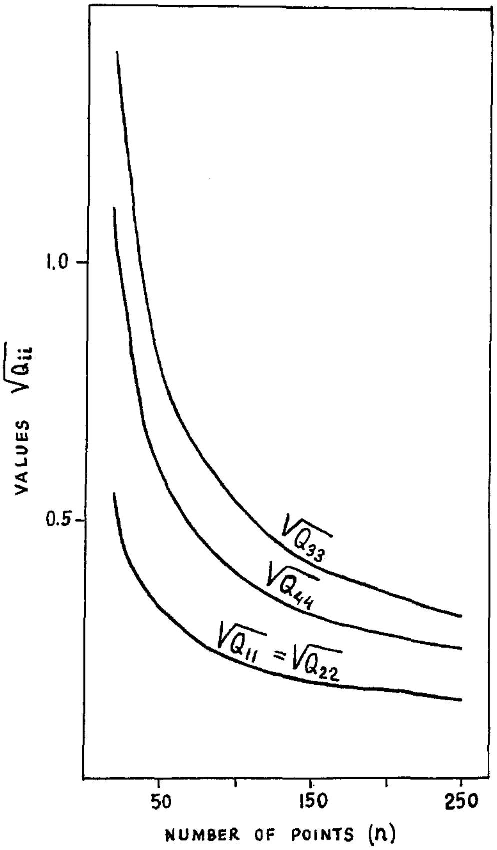


Fig. 2.

Characteristics of coordinates transformations for all three cases are given in Table III. In Figures 3, 4, 5, and 6 are shown the precision of elements transformations (1) as function of the number of common reference points.

The final results are given in Table IV. These data characterize the errors which are inevitable during coordinates transformations from one system to another when 60, 100 and 250 common reference points are used. Here are presented the maximal errors, estimated for the limit case $\zeta = 0.3\ R_{☾}$.

These considered errors refer to the pairs of selenodetic catalogues. The precision of the realisation of selenodetic reference frame based on one catalogue should be multiplied by $2^{-1/2}$ for the cases of equal-precision catalogues ($5 \div 5$; $1 \div 1$).

Data presented in Table IV and Figures 1–6 show that from 40 to 100 reference points on the Moon are sufficient for the setting up of the selenodetic frame of reference with precision of the coordinates used. This number of points is also sufficient for the international list of standard reference points on the front side of the Moon, which are intended for fixing of a standard selenodetic frame of reference in the Moon's body, and attach to it present and future measurements.

TABLE III

Accidental errors of coordinate transformations.

Weights	$\varepsilon_{\Delta\xi} = \varepsilon_{\Delta\eta}$	$\varepsilon_{\Delta\zeta}$
	$\times 10^{-5}$	
$5 \div 5$	15	80
$5 \div 1$	25	135
$1 \div 1$	35	180

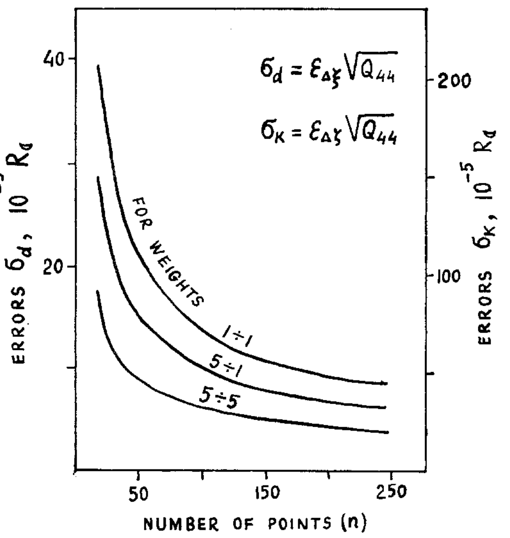


Fig. 3.

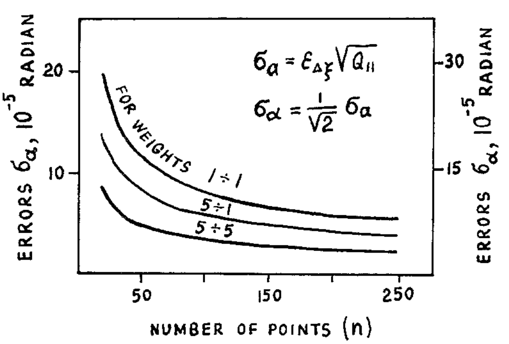
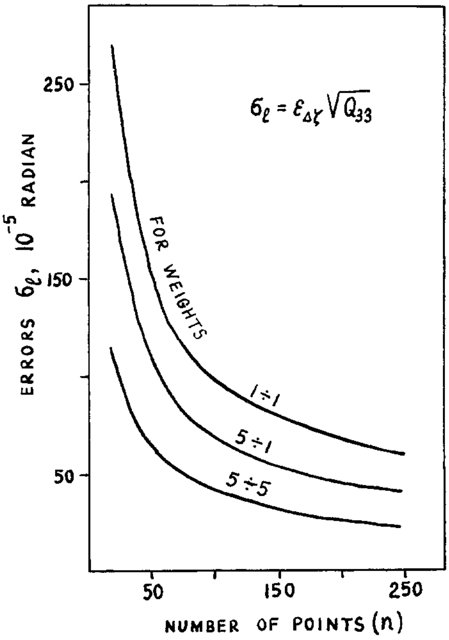
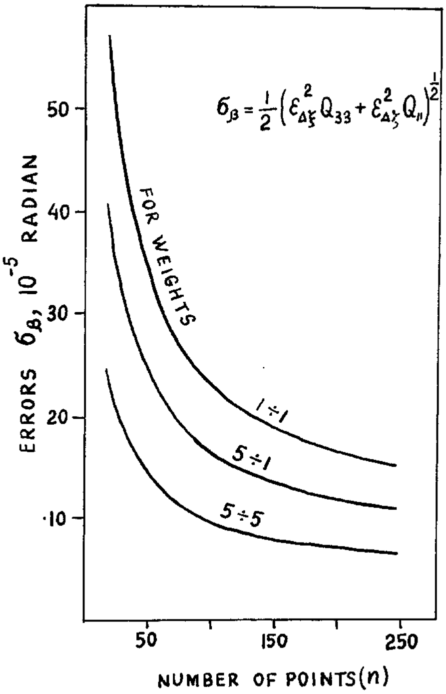


Fig. 4.



$\sigma_\ell = \varepsilon_{\Delta\zeta}\sqrt{Q_{33}}$
ERRORS σ_ℓ, 10^{-5} RADIAN
250
150
50
FOR WEIGHTS
1÷1
5÷1
5÷5
50
150
250
NUMBER OF POINTS (n)

Fig. 5.



$\sigma_\beta = \frac{1}{2}\left(\varepsilon^2_{\Delta\xi} Q_{33} + \varepsilon^2_{\Delta\zeta} Q_{11}\right)^{\frac{1}{2}}$
ERRORS σ_β, 10^{-5} RADIAN
50
40
30
20
·10
FOR WEIGHTS
1÷1
5÷1
5÷5
50
150
250
NUMBER OF POINTS(n)

Fig. 6.

TABLE IV

Systematic errors of coordinate transformations

η	Weights	$\sqrt{Q}$	$\sigma_{\Delta\xi} = \sigma_{\Delta\eta}$ $\times 10^{-5}$	$\sigma_{\Delta\zeta}$ $\times 10^{-5}$
	$5 \div 5$		6	30
60	$5 \div 1$	0.43	11	60
	$1 \div 1$		15	80
	$5 \div 5$		5	25
100	$5 \div 1$	0.32	8	45
	$1 \div 1$		11	60
	$5 \div 5$		3	20
250	$5 \div 1$	0.21	5	30
	$1 \div 1$		7	40

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