PHYSICAL LIBRATIONS DUE TO THE THIRD AND FOURTH DEGREE HARMONICS OF THE LUNAR GRAVITY POTENTIAL

DONALD H. ECKHARDT
Air Force Cambridge Research Laboratories, L. G. Hanscom Field, Bedford, Mass., U.S.A.

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Abstract. The existence of third and fourth harmonics of the lunar gravity potential gives rise to sizable lunar physical librations. Using one recent set of potential estimates, the following effects are noted: the mean sub-Earth point is displaced from the earthward principal moment of inertia axis by 168″; the inclination of the lunar equator to the ecliptic is decreased by 14″.5; and a six year period libration in longitude, with amplitude 13″.1, is induced.

1. Introduction

In the development of the theory of the physical librations of the Moon, its dynamical figure has customarily been represented by the second degree harmonics of the lunar gravity potential. The higher degree harmonics have been neglected because they have seemed to be of little significance while their inclusion would have greatly complicated the theory. With the advent of lunar spacecraft, new improved techniques for measuring the librations and the lunar gravity potential have come into existence. Laser retroreflectors placed on the Moon afford a means for resolving librations to better than 0″.1 on the lunar surface. From the Earth, this is equivalent to an angular resolution better than 0″.0005, a thousandfold improvement over the resolution of the standard astronomical instrument for libration measurements, the heliometer. Tracking data from lunar orbiters have provided means for estimating the higher degree harmonics of the lunar gravity potential, quantities which previously were unavailable. Because of these advances it has been recognized that without consideration of the third, and perhaps higher, degree harmonics, present libration theory is inadequate (Mulholland and Silverberg, 1972) so the purpose of this paper is to extend the theory and estimate the higher degree librations.

2. The Dynamical Equations

We define Cartesian selenographic co-ordinates whose axes are the same as those of the Moon's principal moments of inertia. Axis 1 coincides with the axis of the minimum principal moment of inertia, A, and it is earthward; Axis 3 coincides with the axis of the maximum principal moment of inertia, C, and it is positive northward; and Axis 2 coincides with the intermediate principal moment of inertia, B, and it completes an orthogonal right-hand system. Let ω_1, ω_2, ω_3 be the angular velocities of the Moon about its 1, 2, 3 axes. Then the (rigid) Moon's rate of change of angular momentum is given by the vector

$$\mathrm{d}\mathbf{L}/\mathrm{d}t = \begin{bmatrix} A\dot{\omega}_1 + (C - B)\,\omega_2\omega_3 \\ B\dot{\omega}_2 + (A - C)\,\omega_3\omega_1 \\ C\dot{\omega}_3 + (B - A)\,\omega_1\omega_2 \end{bmatrix}, \tag{1}$$

which equals the torque caused by the Earth and the Sun in the Moon's gravitational field.

We consider now the effect of the Earth in the Moon's gravitational field. The gravitational energy of the Earth in the Moon's field may be written

$$V = -\,(GMm/r)\left[1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}(R/r)^n\,Y_{n,m}\right],$$

where G is the gravitational constant; M is the mass of the Moon; m is the mass of the Earth; r is the Earth-Moon distance; R is the mean radius of the Moon; and $r^nY_{n,m}$ is a solid harmonic, $r^nP_n^m(\sin\varphi)\,[C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda]$; $P_{n,m}$ is the associated Legendre polynomial of degree n and order m; φ is the selenographic latitude; λ is the selenographic longitude; and $C_{n,m}$ and $S_{n,m}$ are coefficients which are defined identically as by Michael and Blackshear (1972).

Let the coordinates of the Earth be x, y, z; and let its direction cosines from the Moon be $u_1 = x/r$, $u_2 = y/r$, and $u_3 = z/r$. The surface harmonic, $Y_{n,m}$, may be expressed in terms of the direction cosines by using the formulae

$$\begin{aligned} P_{n,m}(\sin\varphi) = P_{n,m}(u_3) &= \frac{(1-u_3^2)^{m/2}}{2^n n!}\,\frac{\mathrm{d}^{n+m}(u_3^2-1)^n}{\mathrm{d}u_3^{n+m}} \\ \cos m\lambda &= u_3^{-m}\,\mathrm{Re}\,(u_1 + iu_2)^m \\ \sin m\lambda &= u_3^{-m}\,\mathrm{Im}\,(u_1 + iu_2)^m. \end{aligned}$$

Then $r^nY_{n,m}$ may be expressed as a solid harmonic of degree n in x, y, z. Such transformations have already been calculated and tabulated (but with one misprint) by MacMillan, p. 372 (1958).

The torque on the Moon caused by the Earth is

$$\begin{aligned} \mathbf{T} &= \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}\begin{bmatrix} -\partial V/\partial x \\ -\partial V/\partial y \\ -\partial V/\partial z \end{bmatrix} = \\ &= (Gm/a^3)\,(MR^2)\sum_{n=2}^{\infty}\sum_{m=0}^{n}(R/a)^{n-2}\,(a/r)^{n+1}\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}\times \\ &\quad\times\begin{bmatrix} r^{-n}\,\partial(r^nY_{n,m})/\partial x \\ r^{-n}\,\partial(r^nY_{n,m})/\partial y \\ r^{-n}\,\partial(r^nY_{n,m})/\partial z \end{bmatrix}, \end{aligned} \tag{2}$$

where a is the mean value of r. Setting $\mathrm{d}\mathbf{L}/\mathrm{d}t = \mathbf{T}$ gives the dynamical equations for the rotation of the Moon due to the Earth. In solving the dynamical equations, it has been customary to truncate all harmonics of degree $n \geqslant 3$ in the development of the lunar gravitational potential. One justification for this simplification, which will be shown to be faulty, is that because the torque caused by harmonics of degree n has a factor $(R/a)^{n-2} = (1/221)^{n-2}$, the effects of the higher harmonics are strongly attenuated. For the second degree we have only the solid harmonics

$$r^2 Y_{2,0} = C_{2,0}\left(-\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + z^2\right)$$

and

$$r^2 Y_{2,2} = C_{2,2}\left(3x^2 - 3y^2\right).$$

The coefficients are related to the moments of inertia by the relations

$$C_{2,0} = (1/2MR^2)(A + B - 2C)$$

and

$$C_{2,2} = (1/4MR^2)(B - A).$$

All other second degree coefficients are zero because the coordinate system has been defined so that the products of inertia vanish. The second degree torque is then

$$\mathbf{T}_2 = (Gm/a^3)(MR^2)(a/r)^3 \begin{bmatrix} (-3C_{2,0} - 6C_{2,2})\, yz/r^2 \\ (3C_{2,0} - 6C_{2,2})\, zx/r^2 \\ 12C_{2,2}\, xy/r^2 \end{bmatrix} =$$

$$= (3Gm/a^3)(a/r)^3 \begin{bmatrix} (C - B)\, u_2 u_3 \\ (A - C)\, u_3 u_1 \\ (B - A)\, u_1 u_2 \end{bmatrix}. \tag{3}$$

Setting $\mathrm{d}\mathbf{L}/\mathrm{d}t = \mathbf{T}_2$, the second degree dynamical equations are

$$\begin{bmatrix} \dot{\omega}_1 + \alpha\omega_2\omega_3 \\ \dot{\omega}_2 - \beta\omega_3\omega_1 \\ \dot{\omega}_3 + \gamma\omega_1\omega_2 \end{bmatrix} = (3Gm/a^3)(a/r)^3 \begin{bmatrix} \alpha u_2 u_3 \\ -\beta u_3 u_1 \\ \gamma u_1 u_2 \end{bmatrix}, \tag{4}$$

where $\beta = (C - A)/B$; $\gamma = (B - A)/C$; and $\alpha = (\beta - \gamma)/(1 - \beta\gamma) = (C - B)/A$.

Equation (4) extended to include the relatively small second degree solar torques, had been solved for various combinations of the second degree libration parameters α, β and γ (Eckhardt, 1970). The factor (a/r) is proportional to the sine parallax from the Lunar Theory, but the values of the direction cosines u_i depend on the Lunar Theory plus the integrals of ω_i so the integration is a complicated procedure involving transformation, linearization and iteration (Eckhardt 1967). The integrals of ω_i are transformed to p_1, p_2 and τ where p_i are the direction cosines of the pole of the ecliptic and τ is departure of the Moon from uniform rotation about Axis 1, referred to the mean node of its equator on the ecliptic. The left and right hand sides of the equations are then adjusted so that linear terms in p_1, p_2 and τ which come from the expansions of $u_1 u_2$, $u_2 u_3$ and $u_3 u_1$ on the right hand sides are included with the differential operators on the left hand sides. The right hand sides of the modified differential equations then contain the dependent variables p_1, p_2 and τ only in quadratic and higher power terms. The equations, with linear differential operators, are then solved iteratively.

3. Third and Fourth Degree Libration Solutions

We now write the dynamical equations for the perturbations δp_1, δp_2 and $\delta\tau$ which are induced by the Earth because of the third and fourth degree harmonics of the lunar gravitational potential. We use the linearized differential operators mentioned in the previous paragraph. The equations are

$$\begin{bmatrix} \delta\ddot{p}_1 + 4\beta\delta p_1 - (1-\beta)\,\delta\dot{p}_2 \\ \delta\ddot{p}_2 + \alpha\delta p_2 + (1-\alpha)\,\delta\dot{p}_1 \\ \delta\ddot{\tau} + 2.955\,(Gm/a^3)\,\gamma\delta\tau \end{bmatrix} = (Gm/a^3) \sum_{n=3}^{4} \sum_{m=0}^{n} (R/a)^{n-2}\,(a/r)^{n+1} \times$$
$$\times \begin{bmatrix} -\,(MR^2/B)\;p_3 r^{-n}\{x\partial(r^nY_{n,m})/\partial z - z\partial(r^nY_{n,m})/\partial x\} \\ (MR^2/A)\;p_3 r^{-n}\{z\partial(r^nY_{n,m})/\partial y - y\partial(r^nY_{n,m})/\partial z\} \\ (MR^2/C)\quad r^{-n}\{y\partial(r^nY_{n,m})/\partial x - x\partial(r^nY_{n,m})/\partial y\} \end{bmatrix}.$$

The unit of time is taken as one tropical month, so $(Gm/a^3)=0.9905$. Michael and Blackshear have shown that MR^2/B, MR^2/A and MR^2/C do not depart substantially from 2.5, which would be the case for a uniform density spherical Moon; all three of these ratios are taken as 2.5 exactly. For R/a, we take 1738 km/384400 km $=0.004521$. The variable (a/r) is the sine parallax scaled so that its mean value is 1; it is available from the Lunar Theory in a table which presents a Fourier expansion in terms of the Delaunay arguments, l, l', F and D (Eckert *et al.*, 1954). For various sets of the second degree libration parameters, p_3 and u_i (that is x/r, y/r and z/r) are available in similar tables as solutions of the unperturbed dynamical equations; as an example, expansions of $u_2=y/r$ and $u_3=z/r$ are given in Tables IIb and IIc of Eckhardt (1970).

For $\beta=0.00063$, $\gamma=0.00022$ and harmonic coefficients $C_{n,m}$ and $S_{n,m}$ of Michael and Blackshear, the linearized perturbation equations have been solved using computer algorithms for the multiplication and addition of the long Fourier expansions involved, and for the term by term integration of the equations. The solutions, which are presented in Tables I and II, were calculated separately for each solid harmonic so that the contribution of each individual term, having a coefficient from the sets C_n^m or S_n^m, is displayed. Only terms with amplitudes of 0.″10 or larger are included. If solutions are required for any other sets of coefficients, Tables I and II may be appropriately modified by a simple rescaling. A rescaling can also be used to modify the tables if solutions are required for values of MR^2/A, MR^2/B and MR^2/C other than 2.5 because, no matter how much these ratios may differ from 2.5, they do not differ by much from each other, so they can all be taken with the same value without significantly degrading the solutions.

TABLE I

Third-degree librations

Harmonic coefficient	δp_1					δp_2					$\delta\tau$				
		l	l'	F	D		l	l'	F	D		l	l'	F	D
$C_{3,0} = 2.8440 \times 10^{-5}$	Cosine series					Sine series									
	37″.86	0	0	0	0										
	0″.55	1	0	0	0	0″.55	1	0	0	0					
$C_{3,1} = 2.4153 \times 10^{-5}$	Sine series					Cosine series					Sine series				
	− 17″.21	0	0	1	0	−17″.15	0	0	1	0	0″.40	2	0	−2	0
	− 0″.42	1	0	−1	0	0″.34	1	0	−1	0					
$S_{3,1} = 2.0808 \times 10^{-5}$	Cosine series					Sine series					Cosine series				
	1″.31	0	0	1	0	− 1″.32	0	0	1	0	108″.45	0	0	0	0
						0″.38	1	0	−1	0	− 0″.14	2	0	−2	0
$C_{3,2} = 7.6323 \times 10^{-6}$	Cosine series					Sine series					Cosine series				
	−101″.72	0	0	0	0						— 13″.76	1	0	−1	0
	− 1″.45	1	0	0	0	− 1″.52	1	0	0	0					
$S_{3,2} = 2.2711 \times 10^{-6}$	Sine series					Cosine series					Sine series				
						0″.19	0	0	0	0	− 1″.84	1	0	−1	0
	− 0″.90	1	0	0	0	− 0″.91	1	0	0	0					
$C_{3,3} = 1.4112 \times 10^{-6}$	Sine series					Cosine series					Sine series				
	2″.69	0	0	1	0	2″.68	0	0	1	0	0″.31	0	1	0	0
	− 0″.15	1	0	−1	0						− 2″.09	2	0	−2	0
$S_{3,3} = 3.1126 \times 10^{-7}$	Cosine series					Sine series					Cosine series				
	0″.59	0	0	1	0	− 0″.59	0	0	1	0	48″.77	0	0	0	0
											0″.46	2	0	−2	0

The solutions are also somewhat dependent on the choice of β and γ. Considering the uncertainties in the current estimates of these parameters and the resulting uncertainties in the solutions, the only likely solution modifications of any consequence are those due to a probable change in γ from 0.00022. The modified τ solutions may be found by multiplying each tabulated solution coefficient by the factor $(0.00064394\text{-}\nu^2)/(2.927\gamma\text{-}\nu^2)$ where ν is the rate of change of the corresponding argument in revolutions per tropical month.

TABLE II

Fourth–degree librations

Harmonic coefficients causing no significant librations

$C_{4,0} = 3.4688 \times 10^{-5}$
$S_{4,1} = -8.5100 \times 10^{-6}$
$C_{4,2} = -2.5444 \times 10^{-6}$
$S_{4,3} = -2.7683 \times 10^{-6}$
$C_{4,4} = -5.3982 \times 10^{-6}$

All terms are cosines

Harmonic coefficient	δp_1					$\delta\tau$				
		l	l'	F	D		l	l'	F	D
$C_{4,1} = -1.9892 \times 10^{-5}$	$-0''.59$	0	0	0	0					
$S_{4,2} = -4.1656 \times 10^{-6}$						$-0''.96$	0	0	0	0
$C_{4,3} = -5.6321 \times 10^{-7}$	$0''.23$	0	0	0	0	$0''.72$	1	0	-1	0
$S_{4,1} = 1.1411 \times 10^{-7}$						$-0''.74$	0	0	0	0

The more prominent third and fourth degree librations are as follows:

(1) The $C_{3,0}$, $C_{3,2}$, $C_{4,1}$ and $C_{4,3}$ harmonics cause a constant displacement of $-64''.1$ in p_1; and the $S_{3,1}$, $S_{3,3}$, $S_{4,2}$ and $S_{4,4}$ harmonics cause a constant displacement of $-155''.5$ in τ. This is equivalent to a shift in the mean sub-Earth point from the earthward principal moment of inertia axis of $64''.1$ north and $155''.5$ east.

(2) The $C_{3,1}$ and $C_{3,3}$ harmonics cause terms of $-14''.5 \sin F$ in p_1 and $-14''.5 \cos F$ in p_2. This is equivalent to a change in the inclination of the Moon's equator to the ecliptic, I, of $-14''.5$.

(3) The $C_{3,2}$, $C_{4,3}$ and $S_{3,2}$ harmonics cause a significant six year motion in τ of $-13''.0 \cos(l-F) - 1''.8 \sin(l-F)$.

(4) The $S_{3,1}$ and $S_{3,3}$ harmonics cause terms of $1''.9 \cos F$ in p_1 and $-1''.9 \sin F$ in p_2. If σ is the libration in the longitude of the node of the Moon's equator on the ecliptic, then this is equivalent to a change in $I\sigma$ of $-1''.9$.

(5) The $C_{3,1}$, $S_{3,1}$, $C_{3,3}$ and $S_{3,3}$ harmonics cause a three year motion in τ of $-1''.7 \sin(2l-2F) + 0''.3 \cos(2l-2F)$. Because this term is near resonance, it is sensitive to the choice of γ. The solution to the undamped linearized differential equator governing τ has a pole at $\gamma = 0.0002126$, and the term is only significant near this pole.

4. Discussion

The net displacement of 168″ between the mean sub-Earth point and the earthward principal moment of inertia axis appears reasonable in magnitude because it is smaller than $R/a = 1/221 = 933''$. A similar naive order of magnitude comparison of the second degree physical librations and the periodic third degree (physical) librations seems to indicate that the calculated third degree librations are too large; but a considerate comparison indicates that the orders of magnitude of the third degree librations are indeed reasonable.

The librations of the Moon are conventionally divided into two kinds: optical librations and physical librations. If we assume that the Moon rotates strictly in accordance with Cassini's laws but we fully take into account the orbit of the Moon about the Earth, the optical librations may be specified as the consequent motions in τ, p_1 and p_2. (We exclude consideration of the topocentric optical librations.) Because the rotation of the Moon is not completely described by Cassini's laws, its librations are not completely described by the optical librations; all remaining motions of τ, p_1 and p_2 are then, by definition, the physical librations. In some ways this dichotomy is misleading for it implies that the optical librations are only apparent and do not represent the true rotation of the Moon which is forced on it by the Earth. The mean motion of the node of the lunar equator on the ecliptic is contained in Cassini's laws and it is a physical effect. It is explained dynamically as a precession of the Moon caused by the Earth in the second degree lunar gravity field. It may be written as $p_1 = I \sin F$ and $p_2 = I \cos F$. This is a second degree libration, so the amplitude of the third degree libration in p_1 and p_2 with the same argument should be, at most, of the order of magnitude $(R/a)\, I = 5520''/221 = 25''$. It is; and it is the largest third degree libration in p_1 and p_2. Therefore the third degree librations in p_1 and p_2 are quite reasonable in magnitude even though they are almost as large as the second degree physical librations ($\sim 100''$) in p_1 and p_2.

The mean monthly rotation of the Moon is contained in Cassini's laws, but unlike the motion of the node of the equator, the rotation is not continuously forced by the Earth. The effect of the Earth on the longitudinal rotation of the Moon is to produce torques which rather delicately keep the mean periods of lunar rotation and revolution synchronized, and which cause the physical librations in longitude. The largest second degree libration in τ is approximately $90'' \sin l$, so the amplitude of the third degree libration in τ with the same argument should be, at most, of the order of magnitude $90''/221 = 0''.41$. It is; but other third degree τ terms have much larger amplitudes – those with arguments $l - F$ and $2l - 2F$. Because these are long period terms, the small torques have time to accumulate in effect and force significant librations. These terms are relatively small in the second degree librations because torque terms dependent on the argument F are introduced principally through the factor u_3; and u_3 is not a factor in the second degree torque about the polar axis of the Moon (last line, Equation (3)).

Expressing the second degree librations in Fourier expansions in terms of the

Delaunay arguments, the expansions for p_1 and τ are sine series and the expansion for p_2 is a cosine series. For the third and higher degree librations, the Fourier expansions for p_1, p_2, and τ are mixed sine and cosine series which may be particularly important if we are interested in studying the phase of a particular harmonic. Specifically, the τ term with argument $2l-2F$ is near resonance, so its phase is sensitive to the dissipation parameter, Q, of the interior of the Moon (Eckhardt and Dieter, 1971). To the third degree with our current model ($\beta=0.00063$, $\gamma=0.00022$ and the harmonic coefficients of Michael and Blackshear) this term is $30\overset{''}{.}4 \sin(2l-2F)+0\overset{''}{.}3 \cos(2l-2F)$. The term has a 0.01 rad phase shift from its phase if truncated at the second degree. This phase shift is somewhat sensitive to our choice of γ, but it is even more sensitive to the third-degree harmonic coefficients used, especially $S_{3,1}$ and $S_{3,3}$. The dissipation phase shift should be reckoned from the third-degree libration term, so any estimate of Q from an observed phase shift depends on the adopted values of $S_{3,1}$ and $S_{3,3}$.

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