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The Asymptotic Value of the Circle-Packing Rigidity Constants s_n*

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Abstract. The hexagonal circle-packing rigidity constants s_n are known to satisfy $s_n = O(1/n)$. In this paper it is shown that

$$\lim_{n\to\infty} ns_n = 2\sqrt[3]{2}\Gamma^2(\tfrac{1}{3})/3\Gamma^2(\tfrac{2}{3}) = 4.45165\ldots.$$

Introduction

The hexagonal circle-packing rigidity constants s_n were defined in [RS] as follows. Consider all circle packings in the plane which have the combinatorics of n generations of the regular hexagonal circle packing. Given such a configuration, choose any circle γ of generation 1 and let ρ be the ratio of its radius to the radius of the circle of generation 0. The supremum of $|1 - \rho|$ over all possible choices for γ and all such n generation hexagonal circle-packing configurations is denoted by s_n.

The sequence s_n contains valuable information. The convergence $s_n \to 0$ is a key ingredient for establishing the relationship between circle packings and conformal mappings—that the correspondence between two circle packings with the same combinatorics is close to conformal if the radii are small [RS]. The convergence $s_n \to 0$ also answers purely geometric open questions about circle packings [BFP].

The order of convergence $s_n \to 0$ was found in [H] where it is shown that $s_n = O(1/n)$ (see also [Ah1] and [Ah2]). One consequence of this estimate is that

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in the convergence of circle packings to conformal mappings, the ratio of the radii of corresponding circles converges to the modulus of the derivative of the limiting conformal map. This estimate also gives information on the rate of convergence of circle packings to conformal mappings [H], [R], [DHR].

In this paper we show that

$$\lim_{n\to\infty} ns_n = 2\sqrt[3]{2}\Gamma^2(\tfrac{1}{3})/3\Gamma(\tfrac{2}{3}) = 4.45165\ldots.$$

The outline of the proof is as follows. Let P be a regular hexagon centered at the origin and having sides of unit length. Let H_n be the regular hexagonal circle packing of P by circles of radius $1/2n$. A circle packing H'_n which is combinatorially equivalent to H_n and which has its generation 0 circle normalized to be $\{|w| = 1/2n\}$ is viewed as the image of H_n under a mapping g_n. We show that $\{g_n\}$ is a normal family and that each limit function φ is a normalized conformal mapping of P. The limit of ns_n is shown to be the supremum of $|\varphi''(0)|$ for all such limit functions φ. Explicit circle packings H'_n are constructed so that the corresponding limit function φ is the Koebe function but with domain transferred conformally from the unit disk to P. The classical Bieberbach inequality is then used to show that this limit function yields the desired supremum $\varphi''(0) = 4/R = 4.45165\ldots$, where R is the conformal radius of P.

Preliminary Results

Throughout this paper H_{n+1} (with $n \geq 2$) denotes $n + 1$ generations of the regular hexagonal circle packing in the complex plane $\mathbb{C}$, normalized so that the circles have radii $1/2n$, the generation 0 circle is centered at $0 \in \mathbb{C}$, and $1/2n \in \mathbb{C}$ is the point of tangency between the generation 0 circle and a circle of generation 1. Note that $1 \in \mathbb{C}$ is the center of a circle of generation n.

Similarly, H'_{n+1} denotes any circle-packing configuration which is combinatorially equivalent to H_{n+1} and which is normalized so that

(a) the generation 0 circle has radius $1/2n$,
(b) the generation 0 circle is centered at $0 \in \mathbb{C}$, and
(c) the point $1/2n \in \mathbb{C}$ is a point of tangency between the generation 0 circle and one of the generation 1 circles.

The circles of H'_{n+1} will not, in general, have equal radii, and we explicitly allow the case that a circle (=disk) of generation $n + 1$ is a half-plane.

Given H_{n+1} and an H'_{n+1} configuration, let P_n (resp. P'_n) be the polygon whose boundary consists of line segments joining the centers of pairs of tangent circles of generation n in H_{n+1} (resp. in the H'_{n+1} configuration). Note that $P \equiv P_n$ is a regular hexagon of side length one.

We make use of several results from [DHR]. For convenience we restate them in Lemma A below in the form in which they will be needed. Parts (i)–(iii) are from Lemmas 1.2 and 1.3, part (iv) is from Lemma 1.4 of [DHR], and part (v) is from Lemma 1.8 of [DHR]. In [DHR] the possibility that a circle in H'_{n+1} of generation $n + 1$ is a half-plane was not explicitly considered. However, the more general situation is merely a limiting case and so the results of [DHR] apply.

Lemma A. *Given H_{n+1} and an H'_{n+1} configuration, there is a K-quasi-conformal homeomorphism $g_n: P \to P'_n$ with the following properties:*

(i) *$g_n(0) = 0$, K is independent of n, and the subset of P where g_n fails to be conformal has area $< C/n$ for some absolute constant C.*
(ii) *g_n maps circles of generation $\leq n - 1$ onto corresponding circles of the H'_{n+1} configuration, and arcs of circles of generation n onto the corresponding arcs in H'_{n+1}.*
(iii) *If Ω is the closed region bounded by three mutually tangent circles of generation $\leq n - 1$ in H_{n+1} (i.e., an interstice), then the restriction of g_n to Ω is a Möbius transformation which is uniquely determined by the three tangency points on the boundary and their corresponding tangency points in the H'_{n+1} configuration.*
(iv) *There is a constant C such that, for all $n \geq 2$ and all z with $|z| \leq 1/2n$,*

$$|g_n(z) - z| \leq C/n^2.$$

(v) *There are constants C and $\delta > 0$ such that if M is the restriction of g_n to one of the six interstices adjacent to the center circle of H_{n+1}, then, for all $n \geq 2$ and all z with $|z| \leq \delta$,*

$$|g_n(z) - M(z)| \leq C|z|^3 + C/n^2.$$

The hexagonal packing constants s_n are related to the derivative of the Möbius transformation M defined in Lemma A(v). M maps the generation 0 circle of H_{n+1} onto the generation 0 circle of the H'_{n+1} configuration. It also maps the six generation 1 circles of H_{n+1} onto six circles tangent to the generation 0 circle in the H'_{n+1} configuration (two of these six circles will coincide with generation 1 circles in the H'_{n+1} configuration). Consider the radii r of the largest and smallest images under M of the six generation 1 circles of H_{n+1}. Then

$$s_n \leq \sup\{|1 - 2nr|\}, \tag{1}$$

where the supremum is taken over all choices of M for all H'_{n+1} configurations.

These extremal radii r under M can be estimated in terms of $|M(0)|$. The calculation is simplified if we consider the conjugate transformation $T(w) = 2nM(w/2n)$ which leaves $|w| \leq 1$ invariant. If M has the form

$$M(z) = e^{it}(z - \beta)/(4n^2\bar{\beta}z - 1), \tag{2}$$

then T has the form

$$T(w) = e^{it}(w - \alpha)/(\bar{\alpha}w - 1) \qquad (\alpha = 2n\beta). \tag{3}$$

We may assume $|\alpha| < \frac{1}{3}$. Set $\gamma_1 \equiv \{|w| = 1\}$, $\gamma_2 \equiv \{|w| = 3\}$, and $\gamma_2' = T(\gamma_2)$. Then $T(\gamma_1) = \gamma_1$. The largest and smallest circles which are mutually tangent to γ_1 and γ_2' have radii $(1 - |\alpha|)/1 + 3|\alpha|)$ and $(1 + |\alpha|)/1 - 3|\alpha|)$. Thus

$$|1 - 2nr| \leq 4|\alpha| + [\text{terms of higher order in}|\alpha|]. \tag{3.1}$$

By (v) of Lemma A, we have $|\beta| = O(1/n^2)$. Therefore

$$|\alpha| = O(1/n) \tag{4}$$

and we obtain

$$s_n \leq \sup\{4|\alpha|\} + O(1/n^2). \tag{5}$$

Note that the result of [H], that $s_n = O(1/n)$, is a consequence of (4) and (5).
From (2) and (3) we obtain

$$|M'(0)| = 1 - |\alpha|^2, \qquad |M''(0)| = 4n|\alpha|(1 - |\alpha|^2). \tag{6}$$

Formulas (5) and (6) give $ns_n \leq \sup\{|M''(0)|\} + O(1/n)$.

Lemma B. *The hexagonal packing constants s_n satisfy*

$$ns_n \leq \sup\{|M''(0)|\} + O(1/n),$$

where the supremum is taken over all Möbius transformations M corresponding to generation 1 interstices of H'_{n+1} configurations.

Let us denote the interstice of H_{n+1} which has vertices $\{1/2n, e^{i\pi/3}/2n, (1 + e^{i\pi/3})/2n)\}$ by Ω_{n+1}, and the generation 1 circle of H'_{n+1} which passes through $1/2n$ by c'_{n+1}. Suppose we have a sequence M_n of Möbius transformations such as M above, and such that M_n corresponds to the generation 1 interstice Ω_{n+1}. Assume further that $M'_n(0) \to 1$ and $M''_n(0) \to C > 0$. It then follows that $n\alpha_n = (C/4) + o(1)$ and so $\arg \alpha_n \to 0$. In this case the argument which established (3.1) can be sharpened by observing that γ_1 and γ_2' are furthest apart in the direction of α, and therefore

$$n|1 - 2n \text{ radius}(c'_{n+1})| = C + O(1/n).$$

This proves

Lemma C. *If the Möbius transformation M_n corresponds to the generation 1 interstice Ω_{n+1} of H_{n+1} and if $M_n'(0) \to 1$ and $M_n''(0) \to C > 0$, then*

$$ns_n \geq C + O(1/n).$$

Schlict Functions

Let $\{g_n: n \geq 2\}$ be a sequence of the quasi-conformal mappings in Lemma A, and let M_n be a Möbius transformation which is the restriction of g_n to a generation 1 interstice.

We first show that the mappings g_n are equicontinuous on the unit hexagon P. It is clear that they are equicontinuous on the punctured unit hexagon $P - \{0\}$ since they omit 0 and ∞ [LV, Theorem II.5.1]. To show that they are equicontinuous at 0 we use the fact that $s_n = O(1/n)$. Two adjacent circles of generation $\leq m$ in H'_{n+1} have radii whose ratio is $\leq 1 + s_k$ where $k = n - m$. Since the generation 0 circle of H'_{n+1} has radius $1/2n$, every point w in a circle of generation $\leq m$ in H'_{n+1} satisfies

$$\begin{aligned}|w| &\leq (1/2n)(1 + (1 + s_k) + \cdots + (1 + s_k)^m) \\ &= ((1 + s_k)^{m+1} - 1)/2ns_k \\ &\leq O(k/n)(\exp(O((m + 1)/k)) - 1).\end{aligned}$$

Thus if $|z| \leq d < \frac{1}{2}$, then $|g_n(z)|$ can be estimated from above by the modulus of points w in circles of H'_{n+1} of generation $\leq m = [2dn]$. In the above inequality for $|w|$, $(m + 1)/k \leq 2d/(1 - d)$ and $k/n \leq 1$; thus $|g_n(z)| \to 0$ uniformly in n as $d \to 0$.

By the equicontinuity, any subsequence of g_n contains a subsequence which converges uniformly on compacta of P. By (i) of Lemma A, the limit mapping is conformal or else constant. Let $\{g_{n_i}\}$ be a convergent subsequence and denote the limit mapping by φ. Clearly, $\varphi(0) = 0$. We find that $|\varphi'(0)| = 1$ (see Lemma C) and therefore φ is nonconstant.

From (2) and the fact that $|\beta| = O(1/n^2)$, it follows that the second and third derivatives of M_{n_i} are uniformly bounded in a neighborhood of the origin. Thus there are constants C, $\delta > 0$ such that, for all $|z| < \delta$,

$$|M_{n_i}(z) - [M_{n_i}(0) + M'_{n_i}(0)z + (M''_{n_i}(0)/2)z^2]| < C|z|^3. \tag{7}$$

Combine inequality (7) with part (v) of Lemma A, use the fact that $|\beta| = |M_n(0)| = O(1/n^2)$ from (4), and obtain

$$\begin{aligned}&|g_{n_i}(z) - [M'_{n_i}(0)z + (M''_{n_i}(0)/2)z^2]| \\ &\quad < |g_{n_i}(z) - M_{n_i}(z)| + |M_{n_i}(z) - [M_{n_i}(0) + M'_{n_i}(0)z + (M''_{n_i}(0)/2)z^2]| + |M_{n_i}(0)| \\ &\quad < C|z|^3 + C/n_i^2 + C|z|^3 + O(1/n_i^2) \\ &\quad < C|z|^3 + C/n_i^2. \end{aligned} \tag{8}$$

We also have $|\varphi(z) - [\varphi'(0)z + (\varphi''(0)/2)z^2]| < C|z|^3$ for $|z| < \delta$. Together with (8) this gives

$$|[g_{n_i}(z) - \varphi(z)] + [\varphi'(0) - M'_{n_i}(0)]z + [\varphi''(0) - M''_{n_i}(0)]z^2/2| < C|z|^3 + C/n_i^2. \quad (9)$$

If we fix $0 < |z| < \delta$ and let $n_i \to \infty$ we conclude from (9) that $M'_{n_i}(0)$ converges to $\varphi'(0)$ and $M''_{n_i}(0)$ converges to $\varphi''(0)$. (This result is closely related to the convergence of the first and second derivatives of circle-packing approximations to the Riemann mapping [DHR]. We do not appeal directly to that result for the present case, however, since the contexts differ slightly—mappings from a variable bounded region to the disk in one case, and mappings from a fixed hexagon to variable regions which are not necessarily bounded in the other case). By (6), $|M'_{n_i}(0)|$ converges to 1. Therefore $|\varphi'(0)| = 1$ and we have

Lemma D. *Let $\{g_n : n \geq 2\}$ be a sequence of the mappings in Lemma A. Any subsequence contains a subsequence $\{g_{n_i}\}$ which converges uniformly on compacta of P to a conformal mapping φ which satisfies $\varphi(0) = 0$, $|\varphi'(0)| = 1$, and $\varphi''(0) = \lim M''_{n_i}(0)$.*

Let h be the conformal mapping of the unit disk $\mathbb{D}$ onto P with $h(0) = 0$ and $h'(0) > 0$; the quantity $R \equiv h'(0)$ is called *the conformal radius* of P. The value of R is (see p. 196 of [N] or p. 411 of [He])

$$R = 3\sqrt[3]{4}\,\Gamma(\tfrac{2}{3})/\Gamma^2(\tfrac{1}{3}) = 0.89854\ldots \quad (10)$$

If φ is a conformal mapping of P satisfying $\varphi(0) = 0$, $|\varphi'(0)| = 1$, then $f(z) \equiv \varphi(h(z))/(R\varphi'(0)) = z + a_2 z^2 + \cdots$ is a normalized schlict function on $\mathbb{D}$. Bieberbach's inequality $|a_2| \leq 2$ holds with equality only for rotations of the Koebe function (e.g., Theorem 1.5 of [P]).

Since $h: \mathbb{D} \to P$ is an odd function, $h''(0) = 0$. Hence $f''(0) = R\varphi''(0)/\varphi'(0)$. The Bieberbach inequality yields

$$|\varphi''(0)| \leq 4/R. \quad (11)$$

Apply Lemma D to subsequences where $|M''_{n_i}(0)|$ has modulus approaching its supremum. Lemma B and inequality (11) then imply

Lemma E. $\limsup\{ns_n\} \leq 4/R$.

In the next section we prove the following lemma by constructing circle-packing approximations to the Koebe function.

Lemma F. *For each $n \geq 2$ there is an H'_{n+1} configuration (to be denoted K'_{n+1}) such that the maps $\{g_n\}$ (to be denoted k_n) associated to these configurations by Lemma A converge uniformly on compact subsets to the conformal mapping φ (to* *be denoted k) of the unit hexagon P onto the complex plane minus the slit* $\{-\infty \leq x \leq -R/4\}$ *with* $\varphi(0) = 0$, $\varphi'(0) = 1$, *and* $\varphi''(0) = 4/R$.

If we apply Lemmas C and D to the configurations K'_{n+1} of Lemma F we conclude that $\liminf\{ns_n\} \geq 4/R$. We therefore have

Theorem. *The hexagonal packing constants* $\{s_n\}$ *satisfy*

$$\lim ns_n = 4/R = 2\sqrt[3]{2}\Gamma^2(\tfrac{1}{3})/3\Gamma(\tfrac{2}{3}).$$

The Koebe Packings

For the proof of Lemma F we construct a family of circle packings which are discrete analogs of the Koebe function. Let HL_n consist of those points of the hexagonal lattice

$$\{(a/n) + (b/n)e^{i\pi/3}: a, b \in \mathbb{Z}\}$$

which are contained in the closed unit hexagon P. HL_n determines a triangulation of P by equilateral triangles. The vertices of the triangulation are the lattice points of HL_n. We modify this triangulation to obtain a decomposition of the entire 2-sphere.

For each vertex v on the boundary of HL_n other than the two vertices at $\pm 1 \in \mathbb{C}$, add an edge joining v to its complex conjugate $\bar{v}$. The resulting complex yields a decomposition of the 2-sphere into triangles and quadrilaterals (Fig. 1). By the theorem of Koebe–Andreev–Thurston (for a statement see Andreev's theorem in [RS]; we have since learned of the earlier proof in [K]) there are circle packings of the 2-sphere with combinatorics determined by this decomposition. We wish to single out those circle packings which respect the symmetry $v \to \bar{v}$ of HL_n. To that end we add a vertex to the interior of each quadrilateral, connect it to all four vertices of the quadrilateral, and thereby obtain a triangulation of the 2-sphere. The circle packings which realize this triangulation are related to each other by Möbius transformations. We ignore the circles that correspond to the added vertices and call the resulting circle packings *allowable*. Figure 2 shows an allowable realization for $n = 2$ (the circle of generation is labeled 0; the six circles of generation 1 are labeled 1.1, 1.2, ..., 1.6; the remaining twelve circles of generation 2 are labeled 2.1, 2.2, ..., 2.12).

For a fixed n, the allowable circle-packing realizations are related to each other by linear fractional transformations. We now select a particular realization as follows. We require that the circle (disk) that corresponds to the vertex of HL_n at $1 \in \mathbb{C}$ should be a right half-plane, the circle that corresponds to the vertex of HL_n at $-1 \in \mathbb{C}$ should have its diameter on the real axis with left-hand endpoint at $-\frac{1}{4}$, and the circle which corresponds to the vertex of H_n at 0 should be centered at the origin (we show later that its diameter is $O(1/n)$). This particular allowable circle packing is denoted $\mathscr{K}_n$. Figures 3(a)–(c) show $\mathscr{K}_2$ at various scales.

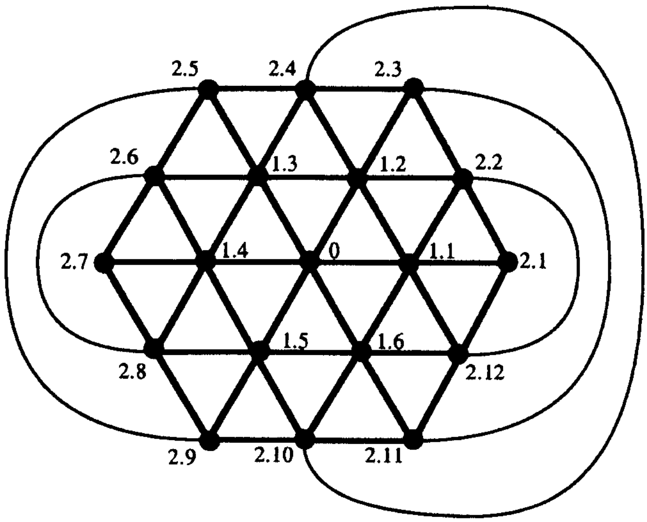

Fig. 1. The decomposition for $n = 2$.

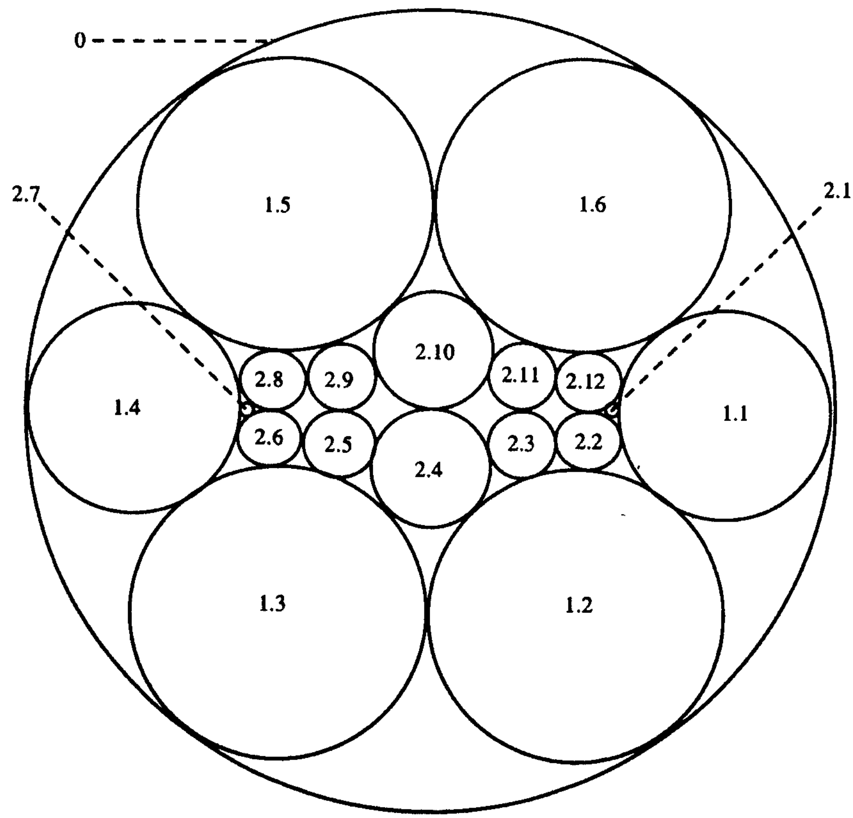

Fig. 2. Allowable realizations for $n = 2$.

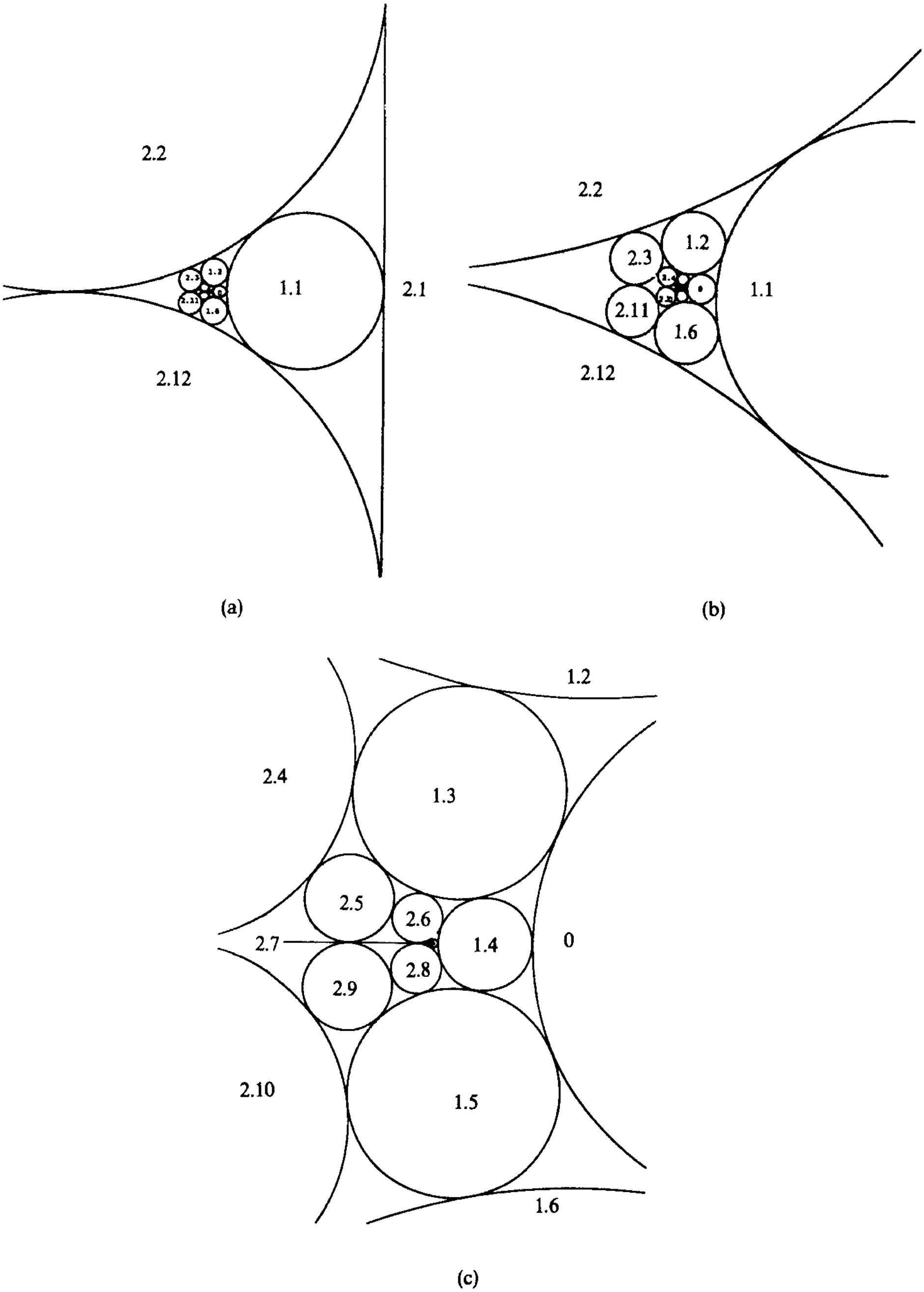

Fig. 3. $\mathscr{K}_2$ at various scales.

$\mathcal{K}_{n+1}$ fails to be an H'_{n+1} circle-packing configuration only because the radius r_n of the generation 0 circle may be unequal to $1/2n$. Let K'_{n+1} be the image of $\mathcal{K}_{n+1}$ under the mapping $z \to z/2nr_n$. Then K'_{n+1} is an H'_{n+1} configuration and Lemma A applies. The mapping denoted generically by g_n in Lemma A(i) will, for this configuration K'_{n+1}, be denoted k_n. We wish to show that k_n converges to a conformal mapping k of P onto the region $W \equiv \mathbb{C} - \{-\infty \le z \le -R/4\}$.

We work with $\mathcal{K}_{n+1}$ rather than K'_{n+1}. Let $\mathcal{Q}_n$ be the polygon formed by joining the centers of the generation n circles of $\mathcal{K}_{n+1}$. Let $\mathscr{k}_n: P \to \mathcal{Q}_n$ be the mapping $\mathscr{k}_n(z) = 2nr_nk_n(z)$. We show that $\mathcal{Q}_n$ converges to the Koebe region $\mathscr{W} \equiv \mathbb{C} - \{z: -\infty \le z \le -\frac{1}{4}\}$ in the sense of Carathéodory domain convergence; namely, given $\mathscr{W}_0 \subset\subset \mathscr{W}$ we have $\mathscr{W}_0 \subset \mathcal{Q}_n \subset \mathscr{W}$ for all sufficiently large n. It will then follow that $\mathscr{k}_n$ must converge to a conformal map $\mathscr{k}$ of P onto $\mathscr{W}$.

The following lemma is a spherical metric version of the Length–Area Lemma of [RS]. A sequence of circles $\gamma_1, \gamma_2, \ldots, \gamma_p$ from $\mathcal{K}_{n+1}$ is called a *cross-cut chain* if γ_1 and γ_p are of generation $n+1$ and each circle other than the first is tangent to the preceding one.

Lemma G. *Suppose there are disjoint cross-cut chains of circles from $\mathcal{K}_{n+1}$ of combinatorial length $m_1, m_2, m_3, \ldots, m_k$ which together with the ray $[-\infty, -R/4]$ separate a circle γ from $0 \in \mathbb{C}$. Then the spherical metric radius ρ of γ satisfies $\rho \le C(m_1^{-1} + m_2^{-1} + m_3^{-1} + \cdots + m_k^{-1})^{-1/2}$ for an absolute constant C.*

Proof. Let the jth cross-cut chain consist of circles whose radii in the spherical metric are ρ_{ji}, $1 \le i \le m_j$. Then, by the Schwarz inequality,

$$\left(\sum_i \rho_{ji}\right)^2 \le m_j \sum_i \rho_{ji}^2.$$

Let $s_j = 2\sum_i r_{ji}$ be the spherical length of the jth chain. We obtain

$$s_j^2 m_j^{-1} \le 4 \sum_i \rho_{ji}^2,$$

$$\sum_j s_j^2 m_j^{-1} \le 4 \sum_{ji} \rho_{ji}^2.$$

In the spherical metric with curvature 1, the area a and radius r of a circle are related by $a = 2\pi(1 - \cos r)$; therefore $\pi r^2 < C_1 a$ for an absolute constant C_1. In the last term above we have $\sum \rho_{ji}^2 < 1/\pi(4\pi C_1)$ and so

$$\sum_j s_j^2 m_j^{-1} \le 16 C_1.$$

Thus $s = \min\{s_1, s_2, \ldots, s_k\}$ satisfies

$$s^2 \le 16C_1(m_1^{-1} + m_2^{-1} + m_3^{-1} + \cdots + m_k^{-1})^{-1}.$$

We can find a Jordan curve of length $< 2s$ which separates the circle γ from the origin and which meets $[-\infty, -R/4]$. If s is sufficiently small, then γ must have a diameter less than s. This proves Lemma G. $\square$

A circle of generation n or $n + 1$ in H_{n+1} can be separated from the generation 0 circle by disjoint cross-cut chains of combinatorial lengths not exceeding 7, $10, \ldots, 3n + 1$. Therefore Lemma G allows us to conclude that each circle γ of generation n or $n + 1$ in $\mathscr{K}_{n+1}$ has a spherical metric radius

$$\rho < C(7^{-1} + 10^{-1} + \cdots + (3n + 1)^{-1})^{-1/2} \to 0.$$

Therefore, $\mathscr{Q}_n$ converges to $\mathscr{W}$ in the sense of Carathéodory. Therefore $\mathscr{k}_n$ converges to a conformal map $\mathscr{k}$ of P onto $\mathscr{W} \equiv \mathbb{C} - \{z\colon -\infty \le z \le -\frac{1}{4}\}$ with $\mathscr{k}(0) = 0$.

Recall from (6) that the conformal map $h\colon \mathbb{D} \to P$ has $h(0) = 0$ and $h'(0) = R$. Since $\mathscr{k} \circ h$ is the Koebe function with derivative 1 at $0 \in \mathbb{C}$ we must have $\mathscr{k}'(0) = 1/R$. Since $\mathscr{k}'(0) = \lim \mathscr{k}_n'(0) = \lim 2nr_n$, we conclude that $k = \lim k_n$ has $k'(0) = 1$ and maps P onto $W \equiv \mathbb{C} - \{-\infty \le z \le -R/4\}$. Since $(\mathscr{k} \circ h)''(0) = 4$, $k''(0) = 4/R$. This completes the proof of Lemma F. $\square$

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