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Polytopes that Fill $\mathbb{R}^n$ and Scissors Congruence

J. C. Lagarias[1] and D. Moews[2]

[1] AT & T Bell Laboratories, 600 Mountain Avenue,
Murray Hill, NJ 07974, USA
jcl@research.att.com

[2] University of California, Department of Mathematics,
Berkeley, CA 94720, USA
dmoews@xraysgi.ims.uconn.edu

Abstract. Suppose that P is a (not necessarily convex) polytope in $\mathbb{R}^n$ that can fill $\mathbb{R}^n$ with congruent copies of itself. Then, except for its volume, all its classical Dehn invariants for Euclidean scissors congruence must be zero. In particular, in dimensions up to 4, any such P is Euclidean scissors congruent to an n-cube. An analogous result holds in all dimensions for translation scissors congruence.

1. Introduction

The problem of characterizing polyhedra that can fill $\mathbb{R}^3$ with congruent copies is complicated even for the simplest case of tetrahedra. In 1896 a tetrahedral space filler was found by Hill [17]. In 1923 Sommerville [32], [33] listed four space-filling tetrahedra, and claimed this was the complete set. This was later shown wrong by the discovery of other space-filling tetrahedra, including three infinite families found in 1974 by Goldberg [11]. The set of all tetrahedral space fillers is, to our knowledge, still not completely classified.

In 1900 Hilbert [16] in his eighteenth problem raised the question: "Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible." This question was answered in 1928 by Reinhardt [28], who found a nonconvex space-filling polyhedron in $\mathbb{R}^3$ that is not a fundamental domain of any discrete subgroup of Euclidean motions. In 1935 Heesch found a nonconvex polygon that tiles $\mathbb{R}^2$ and is not a fundamental domain, and later Kershner [23] found a convex polygon that tiles $\mathbb{R}^2$ but is not a fundamental domain. These and more recent results are surveyed in Milnor [27] and in Bezdek and Kuperberg [2].

More generally, there are polyhedra that can fill $\mathbb{R}^3$ with congruent copies but only via complicated tilings. In 1988 Schmitt [30] found a polyhedron P that can tile $\mathbb{R}^3$ with uncountably many different face-to-face tilings, for which all face-to-face tilings are aperiodic. Conway and Danzer then found an infinite family of eight-sided convex polyhedra (biprisms) having this property, see [5]. According to Danzer, the method of [5] extends to prove that biprisms exist for which *all* tilings of $\mathbb{R}^3$ are aperiodic.

This paper gives a necessary condition for a polyhedron (three-dimensional polytope) to be space filling; more generally we give a necessary condition for a polytope in $\mathbb{R}^n$ to be space filling. This necessary condition arises in connection with Hilbert's third problem. Call polyhedra P and Q *Euclidean scissors congruent* if P can be cut up into a finite number of pieces by plane cuts and the pieces reassembled (using Euclidean motions) to make the polyhedron Q. Hilbert's third problem asked if all three-dimensional polyhedra of the same volume are Euclidean scissors congruent.[1] It was immediately solved in the negative by Dehn [6], [7], who showed among other results that a regular tetrahedron is not Euclidean scissors congruent to a cube. Dehn actually derived invariants which gave necessary conditions for two (sets of) polytopes to be Euclidean scissors congruent in $\mathbb{R}^n$.

We show that a necessary condition for a space-filling polyhedron is:

Theorem 1. *Any (convex or nonconvex) polyhedron P that can fill $\mathbb{R}^3$ with congruent copies is Euclidean scissors congruent to a cube.*

In this result the space-filling tiling of $\mathbb{R}^3$ by copies of the polyhedron P need not be face-to-face. We remark that in 1943 Sydler [35] showed that a necessary and sufficient condition for a polyhedron P in $\mathbb{R}^3$ to be scissors congruent to a cube is that P be scissors congruent to a finite set of smaller polyhedra all similar to itself. Later Sydler [36] and Goldberg [9], [10] studied tetrahedra that are scissors congruent to a cube.

The proof of Theorem 1 is given in Section 2. It uses a result of Sydler [37] giving necessary and sufficient conditions for Euclidean scissors congruence in $\mathbb{R}^3$, which are equality of volumes and of a single codimension 2 Dehn invariant $\Delta(P)$ defined in Section 2. All cubes Q have $\Delta(Q) = 0$, and the point of the proof is to show that $\Delta(P) = 0$.

The basic argument of Section 2 generalizes to a higher-dimensional result concerning the vanishing of various Dehn invariants in $\mathbb{R}^n$. Dehn's invariants for

[1] The definition of scissors congruence given here is called *equidecomposability*. Hilbert allowed sets of polyhedra and actually asked for *equicomplementability*, also called *stable scissors congruence*, which asserts that a finite set of auxiliary polyhedra $L_1, \ldots, L_k$ exist such that $P, L_1, \ldots, L_k$ can be separately cut up and reassembled to make $Q, L_1, \ldots, L_k$. The concepts of equicomplementability and equidecomposability were shown to be equivalent for Euclidean motions in $\mathbb{R}^2$ by Hilbert [15], for Euclidean motions in $\mathbb{R}^n$ by Hadwiger [12], and for other groups by Zylev [40]. Hilbert asked specifically if two regular tetrahedra are stably scissors congruent to a regular tetrahedron of twice the volume, and Dehn showed they were not.

polytopes in $\mathbb{R}^n$ were originally defined in terms of solvability in integers of certain homogeneous linear equations with coefficients involving metric quantities attached to P. They have since been recast in various more abstract forms, either as additive functionals on an algebra of all polytopes with Minkowski sum as an operation, or in a dual form as elements of certain tensor product spaces, see [20], [3], and [29]. Dehn invariants give necessary conditions, while Hadwiger [13, Satz 8, p. 58] has shown that a necessary and sufficient condition for scissors congruence in $\mathbb{R}^n$ is the equivalence of all "Jessen-content functionals," which are a generalization of Dehn invariants. However, a complete set of such functionals is not explicitly known in dimensions $n \geq 5$. We prove a result for the *classical total Euclidean Dehn invariant* of Sah [29], which essentially encodes a set of Dehn invariants $\Delta_{n,j}(P)$ with $1 \leq j \leq n$. (An exact definition appears in Section 3.) For polytopes P in $\mathbb{R}^n$, $\Delta_{n,n-j}(P) = 0$ whenever j is odd or $j = n$, so there are $[(n+1)/2]$ such invariants that are nontrivial. We also note that $\Delta_{n,n}(P) = \mathrm{vol}_n(P)$ and that $\Delta_{3,1}(P)$ essentially coincides with $\Delta(P)$ above, as explained in Section 3.

Theorem 2. *Any (convex or nonconvex) polytope that can fill $\mathbb{R}^n$ with congruent copies of itself has $\Delta_{n,j}(P) = 0$ for $1 \leq j \leq n-1$.*

Theorem 2 logically includes Theorem 1 but we give a separate proof since it requires more machinery.

Jessen [20] showed that in $\mathbb{R}^4$ equality of volume $\Delta_{4,4}$ and of the Dehn invariant $\Delta_{4,2}$ is also a sufficient condition for Euclidean scissors congruence,[2] so we obtain:

Corollary 2a. *Any (convex or nonconvex) polytope P that can fill $\mathbb{R}^4$ with congruent copies of itself is Euclidean scissors congruent to a 4-cube.*

It remains an open question whether equality of total Dehn invariants is a sufficient condition for Euclidean scissors congruence in $\mathbb{R}^n$ for $n \geq 5$. Regardless of whether this is so, we expect that the analogue of Theorem 1 is valid in all dimensions.

In Section 4 we give some generalizations. We show that analogous theorems hold for translation scissors congruence. We also show that information on Dehn invariants can be obtained for finite sets of polytopes that give tilings of $\mathbb{R}^n$, which may potentially be useful as a method of proving nonperiodicity of tilings by certain tile sets.

2. Dehn Invariants for Polytopes that Tile $\mathbb{R}^3$

In this section we follow the framework of Jessen [19] for Dehn invariants in $\mathbb{R}^3$.

Proof of Theorem 1. The Dehn–Sydler theorem [19, Theorem 2] states that two

[2] See also Theorem 29 of [3].

polyhedra P and Q in $\mathbb{R}^3$ are scissors-congruent if and only if $\mathrm{Vol}(P) = \mathrm{Vol}(Q)$ and $\Delta(P) = \Delta(Q)$, where $\Delta(P) \in \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$ is a Dehn invariant defined by

$$\Delta(P) = \sum_{e \text{ edge of } P} l(e) \otimes \alpha(e), \tag{2.1}$$

where $l(e)$ is the length of e and $\alpha(e)$ is the dihedral angle measured between the two faces of P incident on the edge e. The space $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$ has the structure of a vector space over $\mathbb{R}$ (obtained using its first factor) of uncountably many dimensions. The invariant $\Delta(P)$ is additive, i.e.,

$$\Delta(P_1 + P_2) = \Delta(P_1) + \Delta(P_2). \tag{2.2}$$

It is preserved under Euclidean motions, and vanishes on cubes (more generally on prisms).

The idea of the proof is simple. Suppose that P tiles $\mathbb{R}^3$ using Euclidean motions. Consider the set of copies of P in this tiling which intersect the open ball $B(r)$ of radius r around $\mathbf{0}$ in $\mathbb{R}^3$: together they form a (possibly nonconvex) polytope Q_r that contains the ball $B(r)$. We compute the invariant $\Delta(Q_r)$ in two ways. One way uses the additivity property (2.2) and gives

$$\Delta(Q_r) = n_r \Delta(P), \tag{2.3}$$

where n_r is the number of copies of P in Q_r. As $r \to \infty$ the quantity n_r grows proportionally to the volume of $B(r)$, i.e., it grows like $\Omega(r^3)$. The other way uses formula (2.1), which is computed using the boundary of Q_r. From it we can show that $\Delta(Q_r)$ grows at most proportionally to the surface area of $B(r)$, i.e., it grows like $O(r^2)$. We get a contradiction for large r unless $\Delta(P) = 0$.

To make this argument rigorous requires some extra details bounding the size of $\Delta(P)$ computed using (2.1), because the tensor product construction introduces relations over $\mathbb{Q}$. A key observation is that the set of dihedral angles that can occur in any Q_r is drawn from a fixed finite set $\mathscr{S}(P)$, independent of r. This occurs because any such dihedral angle comes from juxtaposed copies of P, so must be a sum of dihedral angles of P, possibly together with π, that adds up to less than 2π. (The term π occurs because the tiling need not be face-to-face, so that an edge of one copy of P may cross a face of another copy of P.) Consequently the terms in (2.1) for $\Delta(Q_r)$ all lie in a fixed finite-dimensional $\mathbb{R}$-subspace $V(P)$ of $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$ generated by $\{1 \otimes \alpha \colon \alpha \in \mathscr{S}(P)\}$. Now choose a subset $\mathscr{S}^+(P)$ such that $\{1 \otimes \alpha \colon \alpha \in \mathscr{S}^+(P)\}$ is an $\mathbb{R}$-basis of $V(P)$; thus for each angle $\alpha_i \in \mathscr{S}(P) - \mathscr{S}^+(P)$ there is a unique expression

$$1 \otimes \alpha_i := \sum_{\alpha_j \in \mathscr{S}^+(P)} c_{ij}(1 \otimes \alpha_j). \tag{2.4}$$

(We are removing all the $\mathbb{Q}$-linear dependencies among the elements of $\mathscr{S}(P)$ when we do this.) Given an arbitrary element X of $V(P)$, with

$$X := \sum_{\alpha \in \mathscr{S}^+(P)} t_\alpha (1 \otimes \alpha),$$

define its length $\|X\|$ by

$$\|X\|^2 = \sum_{\alpha \in \mathscr{S}^+(P)} t_\alpha^2.$$

Now formula (2.2) gives

$$\|\Delta(Q_r)\| = n_r \|\Delta(P)\| \geq \frac{\mathrm{Vol}(B(r))}{\mathrm{Vol}(P)} \|\Delta(P)\|,$$

whence

$$\|\Delta(Q_r)\| > C_0 r^3 \|\Delta(P)\| \qquad \text{as} \quad r \to \infty, \tag{2.5}$$

where C_0 is a positive constant depending on P. Next, using the expression for $\Delta(Q_r)$ based on (2.1), since each $\alpha(e) \in \mathscr{S}(R)$ we have

$$\begin{aligned} \|\Delta(Q_r)\| &= \left\| \sum_{e \text{ edge of } Q_r} l(e) \otimes \alpha(e) \right\| \\ &\leq C_1 \sum_{e \text{ edge of } Q_r} |l(e)|, \end{aligned} \tag{2.6}$$

where $C_1 := \max_i \|1 \otimes \alpha_i\| = \max_i \sqrt{\sum_j c_{ij}^2}$, where the c_{ij} are as in (2.4). Now the edges of Q_r all arise from parts of edges of copies of P that lie on the boundary of Q_r, and all these copies touch the sphere of radius r around $\mathbf{0}$, so that all copies lie inside the concentric region

$$\Omega_r := B(r + d_P) - B(r - d_P),$$

where d_P is the diameter of P. Since $\mathrm{Vol}(\Omega_r) \leq C_2 r^2$ as $r \to \infty$, there are at most $C_2(\mathrm{Vol}(P))^{-1} r^2$ such copies of P, so their total edge-length is therefore $\leq C_3 r^2$. Thus (2.6) gives

$$\|\Delta(Q_r)\| \leq C_1 C_3 r^2 \qquad \text{as} \quad r \to \infty.$$

Comparing this with (2.5) gives a contradiction unless $\|\Delta(P)\| = 0$, whence $\Delta(P) = 0$ as required. □

3. Dehn Invariants for Polytopes that Tile $\mathbb{R}^n$

The basic argument of Section 2 carries over to Dehn invariants in $\mathbb{R}^n$. For this we use the framework of Sah [29].

We first define the classical total Euclidean Dehn invariant. Let vol_n denote n-dimensional volume in Euclidean space. Let $\mathscr{P}E$ be the abelian group generated by all polytopes in Euclidean space (of any dimension), with relations $P = Q$ if P and Q are congruent and $P = Q + R$ if P can be dissected into Q and R. For $n \geq 0$, we take the n-sphere, S^n, to be the set of all rays from the origin in $\mathbb{R}^{n+1}$. We also allow $n = -1$, where we define S^{-1} to be $\{0\}$. For $n \geq 0$, we can define convex polytopes in spherical space S^n by taking the convex hull of a finite set of rays in $\mathbb{R}^{n+1}$, and obtain other polytopes by gluing together convex polytopes. The only polytope in S^{-1} is $\{0\}$. We assign volumes[3] to n-dimensional polytopes in S^n by taking the volumes of their intersection with the ball around the origin in $\mathbb{R}^{n+1}$ with volume 2^{n+1}; the volume of $\{0\}$ is 1. Let $\mathscr{P}S$ be defined analogously to $\mathscr{P}E$ but with all polytopes in S^n, for some n. For spherical polytopes Q and R, we define the product $Q * R$ by letting it be the Minkowski sum $Q + R'$, where R' is a spherical polytope congruent to R and such that every ray in R' is orthogonal to every ray in Q. This extends to make $\mathscr{P}S$ a ring. The polytope $\{0\}$ becomes the multiplicative identity in this ring. Let $\mathscr{C}S$ be the ideal in $\mathscr{P}S$ generated by a ray ρ, and let $\mathscr{C}_m S$ be the mth power of this ideal (for $m \in \mathbb{Z}_{\geq 0}$). Call a union of polyhedra P_i *interior disjoint* if the interiors of the P_i's are disjoint.

For P a Euclidean polytope, x a point in P, and $\lambda \in \mathbb{R}_{>0}$, let $X_{P,x}(\lambda)$ be the union of the rays $\mathbb{R}_{\geq 0}(w - x)$, taken over all $w \in P$ at distance less than λ from x. For sufficiently small λ, $X_{P,x}(\lambda)$ is independent of λ, and we then call it $Y_{P,x}$; it is a spherical polytope of dimension $\dim(P) - 1$. Let m_x be maximal such that $Y_{P,x} \in \mathscr{C}_{m_x} S$. We call m_x the *angle dimension* of P at x. Then $Y_{P,x} = \rho^{m_x} * Z_{P,x}$ for some $Z_{P,x} \in \mathscr{P}S$, and the image of $Z_{P,x}$ in $\mathscr{P}S/\mathscr{C}S$ is well defined, up to torsion, according to Sah [29, Chapter 6, Theorem 3.29].

For an n-simplex P, it is clear that the set of points in P with angle dimension $d \leq n$ is, up to a set of exceptions of dimension less than d, a polytope or union of polytopes of dimension d, and that, in fact, we can write this polytope or union of polytopes as an interior disjoint union of polytopes of dimension d, such that $Y_{P,x}$ is constant on all of each of these polytopes, except a set of dimension less than d. If the two preceding facts are true for polytopes P_i, they remain true for the interior disjoint union of P_i's. This is a consequence of the fact that if P is the interior disjoint union of P_i's, then $Y_{P,x}$ is the interior disjoint union of $Y_{P_i,x}$. It follows that they hold for all polytopes. Similarly, for simplexes $P, Y_{P,x}$ takes on only a finite number of different values, so this is also true for polytopes.

Now, for a given polytope P in $\mathbb{R}^n$, approximate the set of all $x \in P$ with angle dimension $d \leq n$ by an interior disjoint union $\bigcup_{i=1}^{n_d} P_{di}$ of polytopes as above, such that $Y_{P,x}$ is constant almost everywhere on each polytope. For each P_{di}, pick some point x_{di}, not in the exceptional set, and let $\bar{Y}_{di} = Z_{P,x_{di}}$. Then we define the *classical total Euclidean Dehn invariant* $E\Phi(P)$ to be

$$E\Phi(P) = \sum_{d=0}^{n} \Delta_{n,d}(P)T^d := \sum_{d=0}^{n} \sum_{i=1}^{n_d} \operatorname{vol}_d(P_{di})\left(\frac{T}{2}\right)^d \otimes \bar{Y}_{di} \in \mathbb{R}[T] \otimes_{\mathbb{Z}} (\mathscr{P}S/\mathscr{C}S). \tag{3.1}$$

[3] We use a ball of volume 2^{n+1} for consistency with Sah [29, p. 101]. This convention produces the factor of 2 appearing in $(T/2)^d$ in formula (3.1) below, to match Sah [29, pp. 135, 137].

This is well defined since, as we have already noted, the image of $\bar{Y}_{di}$ in $\mathscr{PS}/\mathscr{CS}$ is well defined, up to torsion, and any torsion vanishes when we tensor by the divisible abelian group $\mathbb{R}[T]$. The map $E\Phi$: $\mathscr{PE} \to \mathbb{R}[T] \otimes_{\mathbb{Z}} (\mathscr{PS}/\mathscr{CS})$ is then an additive homomorphism.

In this definition, we have $\Delta_{n,n}(P) = \mathrm{vol}_n(P)$, and $\Delta_{n,n-j}(P) = 0$ whenever j is odd or $j = n$. Furthermore, $\Delta_{3,1}(P)$ agrees with $\Delta(P)$ in Section 2, under a natural isomorphism.[4] Finally we note that an n-cube Q has $\Delta_{n,j}(Q) = 0$ for $0 \leq j \leq n-1$. (In fact this follows from Theorem 2, since an n-cube certainly tiles $\mathbb{R}^n$.)

Proof of Theorem 2. Let P have dimension n, and fill $\mathbb{R}^n$ with congruent copies. For $r \in \mathbb{R}_{\geq 0}$ large, let Q_r be the polytope that is the union of all copies of P in our tiling that intersect $B(r)$, the open ball of radius r around $\mathbf{0}$. Let P have diameter d_P. Evidently $B(r) \subseteq Q_r \subseteq B(r + d_P)$. Let S_r be the polytope consisting of the union of all copies of P in Q_r that intersect ∂Q_r. Then $S_r \cap B(r - d_P) = \varnothing$. Hence $S_r \subseteq B(r + d_P) - B(r - d_P)$, so the volume of S_r is $O(r^{n-1})$. It follows that S_r is the union of $O(r^{n-1})$ copies of P. Also, since $B(r) \subseteq Q_r$, the polytope Q_r is the union of $\Omega(r^n)$ congruent copies of P.

Now if x has angle dimension $d \leq n-1$ or less in Q_r, it must be in ∂Q_r and thus have angle dimension d or less in S_r. It must then come from some point of some copy of P with angle dimension d or less. (This follows from the additivity of $Y_{R,x}$, as before.) Also, $Y_{Q_r,x} = Y_{S_r,x}$ must be an interior disjoint union of $Y_{P_i,x}$'s, where each P_i is a copy of P. However, there are only a finite number of possible distinct $Y_{P,w}$'s; hence there is a minimal volume $Y_{P,w}$ can have, which bounds the size of our interior disjoint union. These facts imply that the $Y_{S_r,x}$'s fall into only a finite number of classes in $\mathscr{PS}$, independent of r. We can then write the term $\Delta_{n,d}(Q_r)$ as $\sum_{j=1}^{k} V_j \otimes w_j$, for some j and w_j's independent of r, and some scalar V_j's with $\sum_{j=1}^{k} V_j = O(r^{n-1})$, since the total set of points in our $O(r^{n-1})$ copies of P with angle dimension d or less must have d-dimensional volume $O(r^{n-1})$. However,
we also know that $\Delta_{n,d}(Q_r)$ has the form $l\Delta$, where the scalar $l = \Omega(r^n)$ and $\Delta := \Delta_{n,d}(P)$ is an element of $\mathbb{R} \otimes_{\mathbb{Z}} (\mathscr{PS}/\mathscr{CS})$. We want to prove that $\Delta = 0$. Now $\mathbb{R} \otimes_{\mathbb{Z}} (\mathscr{PS}/\mathscr{CS})$ is an $\mathbb{R}$-vector space; hence if $\Delta \neq 0$, we can pick an $\mathbb{R}$-homomorphism f: $\mathbb{R} \otimes_{\mathbb{Z}} (\mathscr{PS}/\mathscr{CS}) \to \mathbb{R}$ such that $f(\Delta) \neq 0$. Then

$$lf(\Delta) = \sum_{j=1}^{k} V_j f(1 \otimes w_j).$$

The right-hand side is $O(r^{n-1})$ and the left-hand side is $\Omega(r^n)$, a contradiction. Hence Δ must be 0. This is the desired result. □

[4] More precisely, we can identify the dimension 1 part of $\mathscr{PS}/\mathscr{CS}$ with $\mathbb{R}/(\pi/2)\mathbb{Z}$ by taking the length of arcs in the dimension 1 piece of $\mathscr{PS}$. We take these lengths in the usual way, so that the semicircle has length π. Now $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/(\pi/2)\mathbb{Z} \cong \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$, and one isomorphism sends $a \otimes (b + (\pi/2)\mathbb{Z})$ to $a \otimes (b + \pi\mathbb{Z})$, for all a and b. Under this isomorphism, $\Delta_{3,1}(P)$ agrees with $\Delta(P)$.

4. Generalizations

The idea behind the proofs of Theorems 1 and 2 applies also to scissors congruence for some other subgroups of the group of Euclidean motions. The most well studied of these is translation scissors congruence, which uses the group T_n, of all translations of $\mathbb{R}^n$. A complete set of translation scissors congruence invariants, called *Hadwiger invariants*, is known in all dimensions.

Theorem 3. *Any (convex or nonconvex) polytope P that fills $\mathbb{R}^n$ by translations is translation scissors congruent to an n-cube.*

Proof. There is a Hadwiger invariant $\Omega^n(\mathrm{v}_1, \dots, \mathrm{v}_j)(P)$ for each ordered sequence[5] $(\mathrm{v}_1, \dots, \mathrm{v}_j)$ of orthonormal vectors in $\mathbb{R}^n$, where $0 \le j \le n-1$. The invariant $\Omega^n(\mathrm{v}_1, \dots, \mathrm{v}_j)$ adds up, with appropriate signs, the volumes of codimension j faces of P normal to all of $\{\mathrm{v}_1, \dots, \mathrm{v}_j\}$ which are contained in codimension $j-1$ faces of P normal to $\{\mathrm{v}_1, \dots, \mathrm{v}_{j-1}\}$, which must be contained in codimension $j-2$ faces of P normal to $\{\mathrm{v}_1, \dots, \mathrm{v}_{j-2}\}$, and so on. See Section 2.6 of [29] for a precise definition. Such invariants were originally introduced by Hadwiger, [13], [14], and were later proved complete by Jessen and Thorup [21] and independently by Sah [29, Chapter 4]. (These invariants are not independent, and have nontrivial relations, which were determined by Dupont [8].)

If we define S^{n-1} as in Section 3, we can look at the group $\mathscr{G}_n$ generated by the $(n-1)$-dimensional spherical polytopes in S^{n-1}, with relations $Q = R + T$ if Q can be dissected into R and T, and no congruence relations at all. Define the homomorphism $\mathrm{vol}_{n-1}\colon \mathscr{G}_n \to \mathbb{R}$ by extending it from vol_{n-1} on spherical polytopes in S^n. For each unit vector v in $\mathbb{R}^n$, let U be the subspace of $\mathbb{R}^n$ normal to v. We then view S^{n-2} as the rays in U. For convex polytopes $Q \in \mathscr{G}_n$, we define $\phi_{\mathrm{v}}(Q) \in \mathscr{G}_{n-1}$ as being F if Q has a face F contained in U and Q is on the same side of U as v, as being $-F$ if Q has a face F contained in U and Q is on the opposite side of U to v, and as 0 otherwise. We define $\phi_{\mathrm{v}}\colon \mathscr{G}_n \to \mathscr{G}_{n-1}$ by extending this linearly.

If we have an n-dimensional Euclidean polytope P, and $x \in \mathbb{R}^n$, if we take $Y_{P,x}$ as in Section 3, and if we have orthonormal vectors $\mathrm{v}_1, \dots, \mathrm{v}_j \in \mathbb{R}^n$, then $\phi_{\mathrm{v}_1, \dots, \mathrm{v}_j}(Y_{P,x})$, by which we mean $(\phi_{\mathrm{v}_j} \circ \cdots \circ \phi_{\mathrm{v}_1})(Y_{P,x})$, will be 0 except when x is in a codimension j subset of $\mathbb{R}^n$. As in Section 3, we approximate this set by an interior disjoint union $\bigcup_{i=1}^{m} P_i$ of $(n-j)$-dimensional polytopes, with $Y_{P,x}$ being constant almost everywhere on each polytope. For each P_i, pick x_i not in the exceptional set. Then we can write

$$\Omega^n(\mathrm{v}_1, \dots, \mathrm{v}_j)(P) = c_j \sum_{i=1}^{m} \mathrm{vol}_{n-j}(P_i)\mathrm{vol}_{n-j-1}\big(\phi_{\mathrm{v}_1, \dots, \mathrm{v}_j}(Y_{P,x_i})\big),$$

for some nonzero constant $c_j \in \mathbb{R}$. We apply the volume-versus-surface area argu-

[5] This data actually determines an oriented flag of subspaces $V_1, \dots, V_r$ of $\mathbb{R}^n$ with V_j having the oriented basis $[\mathrm{v}_1, \dots, \mathrm{v}_j]$.

ment of Theorem 2 to show that any P that fills $\mathbb{R}^n$ by translation has $\Omega^n(\mathrm{v}_1, \ldots, \mathrm{v}_j)(P) = 0$ for $j \geq 1$. □

Venkov [39] proved that any convex body that tiles $\mathbb{R}^n$ by translations has a lattice tiling, see also [1]. This result was rediscovered by McMullen [26]. On the other hand Stein [34] and Szabó [38] found nonconvex polytopes in $\mathbb{R}^n$ for various $n \geq 3$ which fill $\mathbb{R}^n$ by translations but which do not possess any lattice tilings. These provide nontrivial examples to which Theorem 3 applies. In passing we remark that Kuperberg [25] has constructed polyhedral tiles that are knotted, which nevertheless tile $\mathbb{R}^3$ with a lattice tiling having no linking of the polyhedra.

In principle some information about Dehn invariants can also be obtained for finite sets of polytopes $\{P_1, \ldots, P_l\}$ that can fill space $\mathbb{R}^n$ using Euclidean motions. Finite sets of polyhedra in $\mathbb{R}^3$ that fill space aperiodically have been recently studied as tiling models for quasicrystals.[6] For example, Kramer [24] uses seven polyhedra to produce a tiling with icosahedral symmetry around one point. Katz [22] finds 22 "decorated" rhombohedra, and Danzer [4] finds 4 "decorated" tetrahedra all of which give only nonperiodic "allowed" tilings of $\mathbb{R}^3$. Here the "decorations" are matching rules on how edges and faces may meet in "allowed" tilings. Matching rules can be eliminated by encoding them geometrically with (polyhedral) bumps and dents in the polyhedra, resulting in a new set of nonconvex polyhedra in which arbitrary tilings are allowed. Given a tiling of $\mathbb{R}^3$ by such $\{P_1, \ldots, P_l\}$, as in Section 2 we extract from it a sequence of larger and larger polyhedra $\{Q_{r_j}: j = 1, 2, \ldots\}$ covered by the tiles, where Q_{r_j} contains n_{jk} tiles of type P_k for $1 \leq k \leq l$. We associate to Q_{r_j} the tiling frequencies

$$\lambda_{jk} := \frac{n_{jk}}{\sum_{i=1}^{l} n_{ji}}, \qquad 1 \leq k \leq l. \tag{4.1}$$

If $(\lambda_1, \ldots, \lambda_l)$ is any limit point as $j \to \infty$ of these tiling frequencies, then the argument of Section 2 shows that the Dehn invariants $\Delta(P_k)$ must obey the linear relation

$$\sum_{k=1}^{l} \lambda_k \Delta(P_k) = 0. \tag{4.2}$$

We may sometimes obtain several independent linear relations in this way. For tilings obtained by "inflation" rules, the tilings are completely known so that such limiting frequencies $(\lambda_1, \ldots, \lambda_l)$ can be computed explicitly.

The usefulness of (4.2) does not necessarily lie in getting information about the Dehn invariants $\Delta(P_i)$, because typically the tiles $\{P_1, \ldots, P_l\}$ are known in advance, so that the $\Delta(P_k)$ are already directly computable. Rather it may possibly be useful in reverse, as a criterion for proving nonperiodicity of tilings. Suppose that we are

[6] For information and further references on quasicrystals consult [18] and [31].

given a set of polyhedra $\{P_1, \ldots, P_l\}$ that tile $\mathbb{R}^3$, satisfying the hypothesis that all linear relations (4.2) satisfied by their Dehn invariants $\Delta(P_k)$ have $(\lambda_1, \ldots, \lambda_l) \notin \mathbb{Q}^n$. Then we could conclude: *no tiling of* $\mathbb{R}^3$ *using these tiles is periodic.*[7] For a periodic tiling implies that a relation (4.2) exists having all λ_i rational. We do not know of any example where this hypothesis is satisfied, however, and it would be interesting to find one or to show that none exists. The simplest case would be a set of two polyhedra $\{P_1, P_2\}$ which can be used to fill $\mathbb{R}^3$ and which have $\Delta(P_2) = \lambda\Delta(P_1) \neq 0$, with λ irrational.

Finally we address the analogous question for hyperbolic polytopes that tile hyperbolic space $\mathbb{H}^n$ may be considered. Dehn invariants for the group of hyperbolic isometries exist, see Chapter 8 of [29]. These are known to be complete scissors congruence invariants in dimensions 1 and 2; the problem of completeness remains open for all larger dimensions. Are the analogues of Theorems 1 and 2 true in the hyperbolic case? We do not know. The arguments of Sections 2 and 3 fail to apply in the hyperbolic case, because hyperbolic polyhedra may have surface area proportional to their volume.

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[7] A tiling in $\mathbb{R}^n$ is *periodic* if it is invariant under n linearly independent translations. A tiling is *aperiodic* if it is not invariant under only nonzero translation. The argument here cannot prove aperiodicity of all tilings.

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