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Successive Minima, Intrinsic Volumes, and Lattice Determinants

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Abstract. In Euclidean d-space E^d we prove a lattice-point inequality for arbitrary lattices and for the intrinsic volumes V_i (i.e., normalized quermassintegrals) of convex bodies. The V_i are not equi-affine invariant (except the volume), hence suitable functionals of the lattice have to be introduced. The result generalizes an earlier result of Henk for the integer lattice $\mathbb{Z}^d$.

1. Introduction and Results

In the following let E^d, $d \geq 2$, denote the Euclidean d-space and let $\mathscr{L}^d$ denote the set of lattices $L \subset E^d$ with $\det(L) \neq 0$. Further, let $\mathscr{K}^d$ denote the set of convex bodies $K \subset E^d$ and let $\mathscr{K}_0^d$ denote the subset of 0-symmetric convex bodies. For $K \in \mathscr{K}^d$ let $V_i(K)$, $i = 0, \ldots, d$, be its intrinsic volumes or normalized quermassintegrals (see [7]). In particular, $V_d(K) = V(K)$ is the volume and $V_{d-1}(K) = \frac{1}{2}F(K)$ is the half-surface area of K. For $K \in \mathscr{K}_0^d$ and $L \in \mathscr{L}^d$ let $\lambda_i(K, L)$ denote the ith successive minimum of K with respect to the lattice L. For the special case $K = B^d$ we have the successive minima $\lambda_i(L) = \lambda_i(B^d, L)$ of the lattice L (see [6]).

For centrally symmetric $K \in \mathscr{K}_0^d$ Henk [5] proved

$$\lambda_{i+1}(K, \mathbb{Z}^d) \cdot \cdots \cdot \lambda_d(K, \mathbb{Z}^d) V_d(K) \leq 2^{d-i} V_i(K), \qquad i = 1, \ldots, d-1, \quad (1)$$

which for $i = 0$ is Minkowski's second theorem (see p. 59 of [2]).

Clearly, a generalization of (1) to arbitrary lattices is desirable. The problem is that the proof of (1) uses special properties of $\mathbb{Z}^d$, and that the V_i (except V_d) are not equi-affine invariant. The basic idea to overcome these difficulties is to introduce functionals of L, which correspond to the V_i as, e.g., the minimal determinants (see [14]),

$$D_i(L) = \min\{\det(L_i): L_i \text{ is an } i\text{-dimensional sublattice of } L\}, \qquad i = 1, \ldots, d,$$

and $D_0(L) = 1$. Obviously $D_i(\mathbb{Z}^d) = 1$, $i = 0, \dots, d$, and $D_d(L) = \det(L)$. With D_i and the last successive minimum λ_d of the lattice, generalizations of a lattice-point inequality for convex bodies by Bokowski *et al.* [1] and of an isoperimetric inequality for lattice periodic sets by Hadwiger [4] for the integer lattice $\mathbb{Z}^d$ to arbitrary lattices have been given (see [8]–[11]).

Further, the following generalization of (1) is conjectured.

Conjecture. *Let $K \in \mathscr{K}_0^d$ and $L \in \mathscr{L}^d$. Then*

$$\lambda_{i+1}(K, L) \cdot \cdots \cdot \lambda_d(K, L) \frac{V_d(K)}{D_d(L)} < 2^{d-i} \frac{V_i(K)}{D_i(L)}, \qquad i = 1, \dots, d-1. \quad (2)$$

A first result of this type is given implicitly by Wills [13]. He proved that (2) is true if the factor $i!$ is added to the right-hand side.

In Section 2 we prove that the Conjecture is best possible for each lattice.

A proof of the Conjecture seems to be hard. It is the purpose of this paper to introduce some related lattice functionals instead of D_i and to prove tight inequalities related to (1) for arbitrary $L \in \mathscr{L}^d$. These functionals are

$$C_i(L) = \max_{L_{d-i}} \min_{L_{d-i} \cap L_i = \{0\}} \det(L_i), \qquad i = 1, \dots, d, \quad (3)$$

where L_i and L_{d-i} are i- and $(d-i)$-dimensional sublattices of L, respectively, but not necessarily $L_i + L_{d-i} = L$. Further, let $C_0(L) = 1$. Obviously $C_d(L) = \det(L)$ and $C_i(\mathbb{Z}^d) = 1$, $i = 0, \dots, d$. These and other properties of C_i are collected in the following proposition.

Proposition.

(a) *The C_i are invariant under rigid motion and homogeneous of degree i.*

(b) *Let $k_1 \le \cdots \le k_d$ and $i = 1, \dots, d$. Then*

$$C_i(\operatorname{diag}(k_1, \dots, k_d)\mathbb{Z}^d) = k_{d-i+1} \cdot \cdots \cdot k_d.$$

(c) $C_d(L) = D_d(L) = \det(L)$.

(d) *The C_i exist and* $C_i(L) \le \lambda_{d-i+1}(L) \cdot \cdots \cdot \lambda_d(L)$, $i = 1, \dots, d$.

(e) $C_1(L) = \lambda_d(L)$.

(f) $C_i(L) \ge D_i(L)$, $i = 0, \dots, d$.

(g) $D_{i+j}(L) \le D_i(L) C_j(L)$, $i = 0, \dots, d$; $j = 0, \dots, d-i$.

(h) *For each inequality there is an L with strict inequality.*

Our main result is now:

Theorem 1. *Let $K \in \mathscr{K}_0^d$ and $L \in \mathscr{L}^d$. Then*

$$\lambda_{i+1}(K, L) \cdot \cdots \cdot \lambda_d(K, L) V_d(K) \le 2^{d-i} C_{d-i}(L) V_i(K), \qquad i = 0, \dots, d-1.$$

This inequality is tight, i.e., $C_i(L)$ cannot be replaced by $C_i(L) - \varepsilon$. From (b) in the Proposition it follows that Theorem 1 generalizes (1) and from (g) and (h) in the

Proposition it follows that Theorem 1 is weaker than the Conjecture. From (d) and (h) of the Proposition it follows that Theorem 1 is an improvement of (see [11])

$$\lambda_{i+1}(K,L)\cdot \cdots \cdot\lambda_d(K,L)V_d(K)$$
$$\le 2^{d-i}\lambda_{i+1}(L)\cdot \cdots \cdot\lambda_d(L)V_i(K), \qquad i=0,\ldots,d-1.$$

Further, we can give the following geometric interpretation of the relation between λ_i and V_i:

Corollary 1. *Let $K\in\mathscr{K}_0^d$ and $L\in\mathscr{L}^d$. Then (with $\lambda_i=\lambda_i(K,L)$)*

$$V_d(\tfrac{1}{2}\lambda_{i+1}K)\le C_{d-i}(L)V_i(\tfrac{1}{2}\lambda_{i+1}K), \qquad i=0,\ldots,d-1.$$

For $i=0$ Corollary 1 is Minkowski's main theorem in *Geometry of Numbers*.

In Section 3 we give some basic properties of C_i. In particular we prove the Proposition and the following theorem.

Theorem 2. *Let $L\in\mathscr{L}^d$. Then*

$$C_i(L)=C_d(L)C_{d-i}(L^*), \qquad i=0,\ldots,d,$$

where L^* is the dual lattice of L.

2. Tightness of the Conjecture

Now we give a sequence of convex bodies such that the defect in the Conjecture tends to zero. For $i=0$ the Conjecture is the second theorem of Minkowski, which is tight for each lattice (e.g., for the DV- or Voronoi-cell (see, e.g., [2]) of the lattice, equality holds). Here we consider the case $1\le i\le d-1$. Let L_i be an i-dimensional sublattice of L with $\det(L_i)=D_i(L)$ and let $E_i=\operatorname{lin}(L_i)$. Further, let $K_0\subset E_i$ temporarily be an arbitrary convex body and let Z be the DV-cell of the $(d-i)$-dimensional lattice $L/E_i^\perp$ (where / denotes the orthogonal projection), then it follows from Lemma 1 in [8] that

$$V_{d-i}(Z)=\det(L/E_i^\perp)=\frac{\det(L)}{\det(L_i)}=\frac{D_d(L)}{D_i(L)},$$

where V_j denotes the j-dimensional volume. In the following let $K:=K_0+Z$. We have

$$\lambda_{i+j}(K,L)\ge\lambda_j(K/E_i^\perp,L/E_i^\perp)=\lambda_j(Z,L/E_i^\perp)=2.$$

Further,

$$V_d(K)=V_i(K_0)V_{d-i}(Z)=\frac{V_i(K_0)D_d(L)}{D_i(L)}.$$

Hence it follows that

$$\frac{2^{d-i}V_i(K)D_d(L)}{D_i(L)\lambda_{i+1}(K,L)\cdot\ \cdots\ \cdot\lambda_d(K,L)V_d(K)} \le \frac{V_i(K)D_d(L)}{D_i(L)V_d(K)} = \frac{V_i(K_0+Z)}{V_i(K_0)}.$$

So it suffices to give a sequence of K_0 such that $V_i(K_0+Z)/V_i(K_0)$ tends to one.

If we write the formula for quermassintegrals of a sum of convex bodies lying in complementary subspaces (see p. 215 of [3]) in terms of the intrinsic volumes and apply it to $K_0 + Z$ we obtain

$$V_i(K_0+Z) = \sum_{\nu=0}^{d-i} V_{i-\nu}(K_0)V_\nu(Z) = V_i(K_0) + \sum_{\nu=1}^{d-i} V_{i-\nu}(K_0)V_\nu(Z).$$

With $R := \max_{1\le\nu\le d-i} V_\nu(Z)$ it follows that $V_i(K_0+Z) \le V_i(K_0) + R\sum_{\nu=0}^{i-1} V_\nu(K_0)$.

Now let $K_0 := rB^i$ be the ball with radius $r > 0$, then it follows, with $V_i(B^d) = \binom{d}{i}\kappa_d/\kappa_{d-i}$ (where κ_j denotes the volume of the j-dimensional unit ball), that

$$\frac{V_i(K_0+Z)}{V_i(K_0)} \le 1 + R\sum_{\nu=0}^{i-1}\frac{V_\nu(K_0)}{V_i(K_0)} \le 1 + R\sum_{\nu=0}^{i-1}\frac{r^\nu\binom{i}{\nu}\frac{\kappa_i}{\kappa_{i-\nu}}}{r^i\binom{i}{i}\frac{\kappa_i}{\kappa_0}}$$

$$= 1 + R\sum_{\nu=0}^{i-1} r^{\nu-i}\frac{\binom{i}{\nu}}{\kappa_{i-\nu}} \to 1 \qquad (r\to\infty).$$

3. Properties of C_i

To give a slightly different definition of C_i we need the following lemma, which is a straightforward application of the dimension formula for submodules (see p. 120 of [12]) and for linear subspaces, respectively.

Lemma 1. *Let L_i and L_{d-i} be i- and $(d-i)$-dimensional sublattices of a lattice $L \in \mathscr{L}^d$, respectively. Then*

$$L_{d-i}\cap L_i = \{0\} \quad\Leftrightarrow\quad \mathrm{lin}(L_{d-i})\cap\mathrm{lin}(L_i) = \{0\},$$

where $\mathrm{lin}(M)$ *is the linear hull of the set M.*

The following lemma shows that we can maximize over arbitrary subspaces instead of sublattices in the definition of C_i.

Lemma 2. *Let $L \in \mathscr{L}^d$. Then*

$$C_i(L) = \max_{E_{d-i}}\ \min_{E_{d-i}\cap L_i=\{0\}} \det(L_i),$$

where E_{d-i} is an arbitrary $(d-i)$-dimensional subspace of E^d.

Proof. Let E_{d-i} be an arbitrary $(d-i)$-dimensional subspace of E^d and let $L_r := L \cap E_{d-i}$ be an r-dimensional sublattice of L $(0 \leq r \leq d-i)$. Then we can complete L_r to a $(d-i)$-dimensional sublattice L_{d-i} of L, such that $L_r \subset L_{d-i}$. If $L_{d-i} \cap L_i = \{0\}$, then $L_r \cap L_i = \{0\}$ and so $E_{d-i} \cap L_i = E_{d-i} \cap L \cap L_i = L_r \cap L_i = \{0\}$. Consequently, we have

$$\min_{L_{d-i} \cap L_i = \{0\}} \det(L_i) \geq \min_{E_{d-i} \cap L_i = \{0\}} \det(L_i),$$

which yields "$\geq$." The reverse inequality follows from Lemma 1. □

Proof of the Proposition. (a), (b), and (c) are clear, (f) follows from (3).

To prove (d) let $u_j \in \lambda_j(B^d, L)B^d \cap L$, $j = 1, \dots, d$, be d linearly independent lattice points. Then $|u_j| = \lambda_j(L)$ and $U = \{u_1, \dots, u_d\}$ forms a basis of E^d. Let L_{d-i} be a $(d-i)$-dimensional sublattice of L and let $A = \{a_1, \dots, a_{d-i}\}$ be a basis of L_{d-i}. Let $U_1 \subset U$ be a maximal subset of U with $A \cup U_1$ linearly independent. Then $u \in \text{lin}(A \cup U_1)$ for all $u \in U$ and so $A \cup U_1$ forms a basis of E^d. Hence we have $|U_1| = i$ and the i-dimensional lattice L_i spanned by U_1 has the property $\dim(L_{d-i} + L_i) = d$ and so we can conclude, as in the proof of Lemma 1, that $L_{d-i} \cap L_i = \{0\}$. Now (d) follows from

$$\det(L_i) \leq \prod_{u \in U_1} |u| \leq \lambda_{d-i+1}(L) \cdot \cdots \cdot \lambda_d(L). \tag{4}$$

To prove (e) it suffices to prove $C_1 \geq \lambda_d$ (we omit the L). If $\lambda_d > C_1$, then there is an $r \geq 1$ such that $\lambda_{r+1} > C_1 \geq \lambda_r$, since $C_1 \geq D_1 = \lambda_1$. Let $u_1, \dots, u_r \in L$ with $|u_i| = \lambda_i$, $i = 1, \dots, r$, and let L_r be the r-dimensional sublattice spanned by $u_1, \dots, u_r$. Then we can complete L_r to a $(d-1)$-dimensional sublattice L_{d-1} and by the definition of C_1 a lattice vector $u_{r+1} \notin L_{d-1}$ with $|u_{r+1}| \leq C_1 < \lambda_{r+1}$ exists. Since $u_1, \dots, u_{r+1}$ are linearly independent, this is a contradiction to the definition of λ_{r+1}.

In (g) the cases $i = 0$, $i = d$, and $j = 0$ are clear. For $1 \leq i \leq d-1$ and $1 \leq j \leq d-i$ let L_i be an i-dimensional sublattice of L with $\det(L_i) = D_i(L)$. We can complete L_i to a $(d-j)$-dimensional sublattice L_{d-j} with $E_{d-j} := \text{lin}(L_{d-j})$. Consider the lattice L_j with $L_j \cap E_{d-j} = \{0\}$ and minimal determinant, then

$$C_j(L) \geq \det(L_j). \tag{5}$$

Let P_1 and P_2 be the fundamental epipeds to L_i and L_j, respectively. Then $P = P_1 + P_2$ is a fundamental epiped to the $(i+j)$-dimensional sublattice $L_{i+j} = L_i + L_j$. With the principle of Cavalieri and (5) it follows that

$$\begin{aligned} V_{i+j}(P) &= V_i(P_1)V_j\left(P_2/E_{d-j}^{\perp}\right) \leq V_i(P_1)V_j(P_2) \\ &= \det(L_i)\det(L_j) \leq D_i(L)C_j(L). \end{aligned} \tag{6}$$

Now (g) follows from $D_{i+j}(L) \leq \det(L_{i+j}) = V_{i+j}(P)$.

Now we prove (h). For (f) choose the lattice with basis $\{e_1, 2e_2, \ldots, de_d\}$. Then $D_i = i!$ and $C_i = d!/(d-i)!$ and so $C_i/D_i = \binom{d}{i} > 1$, for $i = 1, \ldots, d-1$. For (d) and (g) we choose a lattice L, such that $v \cdot w \neq 0$, for all $v, w \in L \setminus \{0\}$ (e.g., the lattice with basis $(1, 0, \ldots, 0)^t$, $(\pi, \pi, 0, \ldots, 0)^t$, $(\pi^2, \pi^2, \pi^2, 0, \ldots, 0)^t$, $\ldots, (\pi^{d-1}, \ldots, \pi^{d-1})^t$ has this property since π is transcendental). Then we have, in (4) and (6) (in (4) only for $i > 1$) in the proofs of (d) and (g), strict inequality. □

Proof of Theorem 2. For $i = 0$ and $i = d$ the assertion follows from $C_0(L) = 1$ and $C_d(L) = \det(L)$. For $1 \leq i \leq d-1$ it suffices to prove $C_i(L) \leq C_{d-i}(L^*)\det(L)$, because we can apply this to L^* and $d-i$ instead of i and obtain the reverse inequality.

Let L_{d-i} be a $(d-i)$-dimensional sublattice, such that

$$C_i(L) = \min_{L_{d-i} \cap \Lambda_i = \{0\}} \det(\Lambda_i)$$

and let $E_{d-i} = \operatorname{lin}(L_{d-i})$, then $E_{d-i}^{\perp}$ is an i-dimensional subspace, which is spanned by a sublattice of L^* (see [8]). Let $\tilde{L}_{d-i}$ be a $(d-i)$-dimensional sublattice of L^* with $\tilde{L}_{d-i} \cap E_{d-i}^{\perp} = \{0\}$ and $\det(\tilde{L}_{d-i})$ minimal, then

$$C_{d-i}(L^*) \geq \det\left(\tilde{L}_{d-i}\right). \tag{7}$$

We can assume that $\tilde{L}_{d-i}$ is primitive in L^*, since from Lemma 1 it follows that $\operatorname{lin}(\tilde{L}_{d-i}) \cap E_{d-i}^{\perp} = \{0\}$, and otherwise the lattice $L^* \cap \operatorname{lin}(\tilde{L}_{d-i})$ would be a "better" lattice. Now let $L_i := L \cap (\operatorname{lin}(\tilde{L}_{d-i}))^{\perp}$, then L_i is an i-dimensional sublattice of L with $L_i \cap E_{d-i} = \{0\}$.

Let $x \in L_i \cap E_{d-i}$. As in the proof of Lemma 1, we can show that $\operatorname{lin}(\tilde{L}_{d-i}) + E_{d-i}^{\perp} = E^d$, i.e., we can represent each $y \in E^d$ as $y = u + v$, where $u \in \operatorname{lin}(\tilde{L}_{d-i})$ and $v \in E_{d-i}^{\perp}$. Then it follows that $x \cdot y = x \cdot u + x \cdot v = 0 + 0 = 0$, since $x \in L_i \subset (\operatorname{lin}(\tilde{L}_{d-i}))^{\perp}$ and $x \in E_{d-i} = (E_{d-i}^{\perp})^{\perp}$. Consequently, $x = 0$.

Hence we have

$$\det(L_i) \geq \min_{L_{d-i} \cap \Lambda_i = \{0\}} \det(\Lambda_i) = C_i(L). \tag{8}$$

It further follows from [8], Theorem 1, and (8) that

$$\det\left(\tilde{L}_{d-i}\right) = \det(L^*)\det(L_i) = \frac{\det(L_i)}{\det(L)} \geq \frac{C_i(L)}{\det(L)}. \tag{9}$$

Finally it follows from (7) and (9) that

$$C_{d-i}(L^*) \geq \det\left(\tilde{L}_{d-i}\right) \geq \frac{C_i(L)}{\det(L)},$$

and Theorem 2 is proved. □

4. Proof of Theorem 1

For $i = 0$ Theorem 1 is the second theorem of Minkowski, so it suffices to consider the case $1 \leq i \leq d-1$. For $j = 1, \ldots, i$ let $y_j \in \lambda_j(K, L)K \cap L$ be i linearly independent lattice points. Let $E_i = \text{lin}\{y_1, \ldots, y_i\}$, then it follows by the definition of $C_{d-i}(L)$, that a $(d-i)$-dimensional sublattice L_{d-i} of L exists such that:

(1) $L_{d-i} \cap E_i = \{0\}$ (and, with Lemma 1 in Section 3, $\text{lin}(L_{d-i}) \cap E_i = \{0\}$).
(2) $\det(L_{d-i}) \le C_{d-i}(L)$.

Let $K_{d-i} := K \cap \text{lin}(L_{d-i})$, then it follows, with the second fundamental theorem of Minkowski for K_{d-i} and L_{d-i}, that

$$\lambda_1(K_{d-i}, L_{d-i}) \cdot \cdots \cdot \lambda_{d-i}(K_{d-i}, L_{d-i}) V_{d-i}(K_{d-i}) \le 2^{d-i} \det(L_{d-i}) \le 2^{d-i} C_{d-i}(L).$$

By the choice of L_{d-i} (see (1)), $\lambda_j(K_{d-i}, L_{d-i})K$ contains $i+j$ linear-independent lattice points of L, such that

$$\lambda_{i+j}(K, L) \le \lambda_j(K_{d-i}, L_{d-i}), \qquad j = 1, \ldots, d-i,$$

and so

$$\lambda_{i+1}(K, L) \cdot \cdots \cdot \lambda_d(K, L) V_{d-i}(K_{d-i}) \le 2^{d-i} C_{d-i}(L). \tag{10}$$

Now Theorem 1 follows from (10) and $V_d(K) < V_i(K) V_{d-i}(K_{d-i})$ (see [5]). □

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