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The Mayer–Vietoris and *IC* Equations for Convex Polytopes

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Abstract. In this paper we use geometric dissection to obtain linear equations on the flag vectors on convex polytopes. These results provide new proofs and expressions of the complete system of such equations originally discovered by Bayer and Billera. The Mayer–Vietoris equation applies to a situation where two convex polytopes overlap to produce union and intersection, both convex polytopes. The operators I and C applied to a polytope produce the cylinder (or prism) and cone (or pyramid), respectively, with the given polytopes as base. The IC equation relates the flag vectors of the polytopes obtained in this way. As a consequence, it becomes easier to define linear functions of the flag vector, via initial data and their law of transformation under the operators I and C.

1. Introduction

In this paper we use geometric dissection to obtain linear equations on the flag vectors of convex polytopes. A flag δ on a d-dimensional convex polytope Δ is a sequence

$$\delta = (\delta_1 \subset \delta_2 \subset \dots \subset \delta_r \subset \Delta)$$

of proper faces of Δ, each strictly contained in the next. Its *type* or *dimension* is the ascending sequence

$$d_1 < d_2 < \dots < d_r < d,$$

where $d_i = \dim \delta_i$ is the dimension of the ith *term* of δ. It is convenient to consider the type of a flag to be a subset S of $\{0, 1, \dots, d-1\}$.

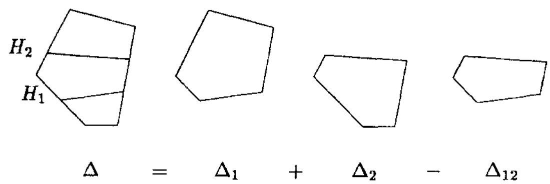


Fig. 1

For each such S the quantity f_S is the number of flags on Δ with type S. There are 2^d subsets of $\{0, 1, \ldots, d-1\}$. The *flag vector* $f = f\Delta$ of Δ has for components the 2^d quantities f_S. It counts not only vertices, edges, faces, etc., but also incidence relations.

According to Bayer and Billera [1], the flag vectors of d-dimensional polytopes span a space whose dimension is the $(d+1)$st Fibonacci number. Moreover, they also show that this space has as a basis the flag vectors of the polytopes formed by successively applying the operators C and IC to the point polytope.

Here C is the operator that takes a polytope Δ to the cone or pyramid $C\Delta$ with base Δ, while $I\Delta$ is the product of Δ with an interval. (In fact, because they prefer simplicial polytopes to simple polytopes, Bayer and Billera use the bipyramid operator rather than the prism or cylinder operator I. That C and IC produce a basis corresponds under the duality induced by polarization to the basis result proved in their paper.)

In algebraic topology the Mayer–Vietoris equation relates the homology of a union $A \cup B$ to that of A, of B, and of the intersection $A \cap B$. Here, the Mayer–Vietoris equation is an inclusion–exclusion result which applies when $\Delta = \Delta_1 \cup \Delta_2$ expresses the convex polytope Δ as a suitable overlapping union of polytopes Δ_1 and δ_2 (see Fig. 1).

Of the 2^d polytopes of dimension d that can be formed by applying I and C to the point polytope, only a Fibonacci number are linearly independent. (The easy part of [1] is to show that at most a Fibonacci number are independent. That this bound is sharp is the hard part.) The other purpose of this paper is to make clear the relationship between the operators I and C. In particular, an equation between I and C is presented here. As a consequence, it becomes easier to define linear functions of the flag vector, via their law of transformation under I and C. This will be exploited elsewhere.

2. A Proof of a Result of Bayer and Billera

Here another proof is given of their bound on the dimension of span of the flag vectors of convex polytopes, a proof which is related to (the polarization of) McMullen's celebrated proof [4] by shelling of the Dehn–Sommerville equations for simple polytopes. In this proof the appearance of the Fibonacci numbers is perhaps more natural. This proof leads us to the Mayer–Vietoris theorem.

Let α be a linear function that takes a different value on every vertex of Δ. It will be thought of as a height function. Now let δ be a flag on Δ. Let δ^α be the highest point on the first term δ_1 of δ, and let δ_α be the lowest. By hypothesis on α, these points exist, and are vertices of Δ.

Now let v be a vertex of Δ. Let $f^\alpha v$ be a local flag vector, that counts all flags on Δ with $\delta^\alpha = v$. Thus, $f\Delta$ is the sum of $f^\alpha v$ over all vertices of Δ. Similarly, let $f_\alpha v$ count flags on Δ with $\delta_\alpha = v$. The result of Bayer and Billera would follow if the self-dual sum

$$f^\alpha v + f_\alpha v$$

for any vertex v on a d-polytope were known to lie in a fixed space of dimension at most the $(d+1)$st Fibonacci number.

Around v the polytope Δ looks like a cone on something. That something is the link $L = L_v$ which is constructed as follows. Let $\{\beta = 0\}$ be a hyperplane that supports Δ at v. In other words, β is zero at v and strictly positive on the rest of Δ. The *link* $L = L_v$ is the intersection $\{\beta = \varepsilon\} \cap \Delta$ of Δ with a parallel displacement of $\{\beta = 0\}$, where ε is a suitably small positive number.

The hyperplane $\{\alpha = \alpha v\}$ will in general meet the link L_v. (It will not if v is the highest or lowest vertex of Δ). It will divide it into two. The *upper* and *lower* links are

$$L_v^+ = L_v \cap \{\alpha \geq \alpha v\},$$

$$L_v^- = L_v \cap \{\alpha \leq \alpha v\},$$

respectively, while

$$L_v^0 = L_v \cap \{\alpha = \alpha v\}$$

is the *level* link, and L_v itself is the *total* link (see Fig. 2).

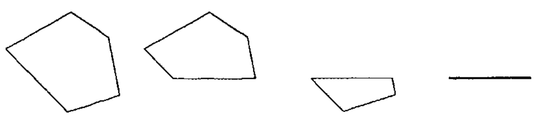
Note that the upper, lower, and total links have dimension $(d-1)$, while the level link has dimension $(d-2)$. The next step is to relate $f^\alpha v + f_\alpha v$ to the upper, lower, level, and total links. In fact:

Proposition. *There are linear functions A and B such that the equation*

$$f^\alpha v + f_\alpha v = A(fL^+v + fL^-v) - B(fL^0 v)$$

holds. Note that A and B depend on the dimension d.

This equation entails the Bayer–Billera bound on the span of flag vectors. To begin with, for 0-polytopes and 1-polytopes also the span of the flag vectors has dimension 1. The above equation, together with $2f\Delta$ being the sum of $f^\alpha v + f_\alpha v$



L_v L_v^+ L_v^- L_v^0

Fig. 2

over all vertices of Δ, now suffices to support an inductive proof of the Bayer–Billera bound.

The ease of proof of the proposition depends very much on the degree to which the reader is accustomed to linear functions of the flag vector. For now, note that flags δ with $\delta^{\alpha} = v$ or $\delta_{\alpha} = v$ can be constructed from $L^{-}v$ and from $L^{+}v$, respectively. This construction will not apply to all flags on $L^{-}v$ or $L^{+}v$. Such inadmissible flags can be constructed—and thus counted—by $L^{0}v$. More details are given in Section 4.

3. The Mayer–Vietoris Equation

The configuration of upper, lower, level, and total links is not so different from that of the Mayer–Vietoris theorem, which is now stated.

Theorem (Mayer–Vietoris or Inclusion–Exclusion). *Let Δ be a convex polytope. Let H_1 and H_2 be two hyperplanes, cutting Δ as in Fig. 1. Neither H_1 nor H_2 are to pass through any vertex of Δ, nor are they to meet on Δ. Choose a height function α so that $H_2 \cap \Delta$ lies entirely above $H_1 \cap \Delta$.*

Then the four polytopes

$$\begin{aligned} \Delta &= \text{the original polytope}, \\ \Delta_1 &= \text{all points of } \Delta \text{ on or above } H_1, \\ \Delta_2 &= \text{all points of } \Delta \text{ on or below } H_2, \\ \Delta_{12} &= \Delta_1 \cap \Delta_2 \end{aligned}$$

satisfy the equation

$$\Delta = \Delta_1 + \Delta_2 - \Delta_{12}$$

by which is meant that the equation $f\Delta = f\Delta_1 + f\Delta_2 - f\Delta_{12}$ is satisfied by their flag vectors.

Corollary. *Consider Fig. 2. Then*

$$L_v = L_v^{+} + L_v^{-} - IL_v^{0}$$

holds as an equation between flag vectors, where I is the operator that takes a polytope Δ to the product $[0, 1] \times \Delta$ of Δ with an interval.

Proof. The level link L_v^0 can be replaced by a thin slice of L_v, which has the combinatorial type of IL_v^0, by cutting L_v with a slight translate of $\{\alpha = \alpha v\}$ as well as $\{\alpha = \alpha v\}$ itself. This displacement will not change the combinatorial structure of the polytopes L_v^{+} and L_v^{-}. The corollary now follows from Mayer–Vietoris. □

Corollary. *The flag vector fL_v is a linear function of $fL_v^+ + fL_v^-$ and fL_v^0. (Notation is as before.)*

Proof. The flag vector of $I\Delta$ is a linear function of that of Δ. This is left to the reader. □

The rest of the section is devoted to the proof of the Mayer–Vietoris theorem. It has two phases. The first is the matching up of the vertices, the second the matching up of flags.

A bijection is required between the vertices of Δ_1 and Δ_2 on the one side, and Δ and Δ_{12} on the other. (The reader is advised to attempt this before proceeding.) Certain vertices of Δ_1 are also vertices of Δ. Those that are not arise from $\Delta \cap H_1$ and so lie on Δ_{12}. Similarly, certain vertices of Δ_2 lie on Δ_{12} and those that do not lie on Δ.

This map is a bijection. This can be made clearer by introducing a function first(v, X, Y) where v is a vertex and X and Y are polytopes. It is defined by

v as a vertex of X if v is a vertex of X,

else v as a vertex of Y if v is a vertex of Y,

else "an error has occurred"

and using it the map defined above is

$$\Delta_1 \ni v \mapsto \text{first}(v, \Delta, \Delta_{12}),$$

$$\Delta_2 \ni v \mapsto \text{first}(v, \Delta_{12}, \Delta),$$

while the rules

$$\Delta \ni v \mapsto \text{first}(v, \Delta_1, \Delta_2),$$

$$\Delta_{12} \ni v \mapsto \text{first}(v, \Delta_2, \Delta_1),$$

provide the inverse.

To prove that these two maps are inverse to each other requires chasing the vertices through the rules. The facts

a vertex of Δ_1 not of Δ is not of Δ_2,

a vertex of Δ_2 not of Δ_{12} is not of Δ_1,

a vertex of Δ not of Δ_1 is not of Δ_{12},

a vertex of Δ_{12} not of Δ_2 is not of Δ

will prove useful. The rest is left to the reader.

This bijection on vertices can now be extended to flags. First, note that if v is a vertex of two polytopes X and Y chosen from $\{\Delta, \Delta_1, \Delta_2, \Delta_{12}\}$, then X and Y are isomorphic in a neighbourhood of v and so any flag δ on X whose first term δ_1 contains v determines a unique flag on Y, and vice versa.

Now choose a height function α such that every vertex of the four Δ polytopes has a different height, and such that $\Delta \cap H_2$ lies above $\Delta \cap H_1$. Each flag δ will have a unique highest vertex $v = \delta_1^\alpha$ of the first term. This vertex is subject to a bijection, which the flag then follows. This concludes the proof.

4. Remarks on the Mayer–Vietoris Equation

It is now possible to be more explicit about the calculation of $f^\alpha + f_\alpha$ from the link. Flags with first terms v can readily be calculated from L_v $(= L_v^+ + L_v^- - IL_v^0)$. Other flags have a first term δ_1 which meets either L_v^+ or L_v^-, but not L_v^0. The inadmissible flags fall into two types. Those for which δ_1 lies on L_v^0 can readily be counted. Those for which δ_1 only intersects L_v^0 can be counted in a manner similar to the counting of IL_v^0.

An important application of Mayer–Vietoris is to the situation where two parallel hyperplanes are used to truncate Δ along a face δ. Suppose, for definiteness, that δ lies below H_1.

From a rational polytope Δ a torus embedding $\mathbf{P}_\Delta$ can be constructed [3]. Under this correspondence the truncation of Δ along δ becomes a monoidal transformation of $\mathbf{P}_\Delta$ along $D = \mathbf{P}_\delta$. This geometry supports a conjecture concerning the middle perversity intersection homology Betti numbers of algebraic varieties, which will be described elsewhere.

To conclude, here is an observation that arose in conversation with McMullen. According to Mayer–Vietoris the equation

$$h\Delta + h\Delta_{12} = h\Delta_1 + h\Delta_2$$

holds for h any linear function of the flag vector. If a suitable homology theory $H(\cdots)$ for convex polytopes (and torus enbeddings) can be defined, then an isomorphism

$$H(\Delta) \oplus H(\Delta_{12}) \simeq H(\Delta_1) \oplus H(\Delta_2)$$

should be expected, and conversely this isomorphism could be used to help construct $H(\cdots)$ and derive its properties. This conjectured isomorphism helps justify the use of the name Mayer–Vietoris.

5. The Polytopes $CC\Delta$ and $IC\Delta$ Compared

Suppose that Δ is a convex polytope. The cone $C\Delta$ has Δ for its base, and around Δ the combinatorial structure of $C\Delta$ is the same as that of $I\Delta$ around one of its end facets, say $\{0\} \times \Delta$. However, at its apex a the cone $C\Delta$ is very different, for it there has Δ as the link. For example, if Δ is simple than $C\Delta$ is simple away from its apex, where it is nonsimple unless the base Δ is in fact a simplex.

The cone $CC\Delta$ on $C\Delta$ has its own apex a' as well as the apex a of $C\Delta$. The preceding description might lead one to expect a' to be a most distinguished point of

$CC\Delta$, and indeed the author was so deceived for some time. However, this is not so.

The cone $C\Delta$ is no more than the free join of Δ with the line segment aa'. In other words, in some suitable affine space containing (a copy of) Δ, choose a line segment aa' that is skew to the subspace spanned by Δ, and take the convex hull of Δ with aa'. This is $CC\Delta$.

Along aa' the combinatorial structure of $CC\Delta$ does not change. The structure is locally similar to that of a product, for example, that of $IC\Delta$ along $I\{a\}$, where a is still the apex of $C\Delta$.

Proposition. *The polytopes $CC\Delta$ and $IC\Delta$ have the same local combinatorial structure along the edges aa' and $I\{a\}$, respectively.*

Proof. It is enough to show that at the four vertices a and a' of $CC\Delta$ and $\{0\} \times \{a\}$ and $\{1\} \times \{a\}$ of $IC\Delta$, respectively, the combinatorial structure is the same. Firstly, $CC\Delta$ is the free join of aa' to $C\Delta$ and so the structure at a is the same as at a'. Secondly, $C\Delta$ is the base of $CC\Delta$ and so the structure at a on $CC\Delta$ is the same as that of $\{0\} \times \{a\}$ on $IC\Delta$. Finally, $IC\Delta$ is a product, so $\{0\} \times \{a\}$ and $\{1\} \times \{a\}$ have the same structure. □

6. The Polytopes $IC\Delta$ and $CC\Delta$ Truncated

The similarity between $IC\Delta$ and $CC\Delta$ has just been seen. The difference will become more apparent when that which they have in common is removed. This is done by the process of truncation.

Let $\{\alpha = 0\}$ be an affine hyperplane that supports $IC\Delta$ along $I\{a\}$. In other words, α is zero on $I\{a\}$ and strictly positive on the remainder of $IC\Delta$. The intersection

$$\{\alpha \geq \varepsilon\} \cap IC\Delta,$$

where ε is a suitably small positive number, this is the truncation of $IC\Delta$ along $I\{a\}$. The truncation of $C\Delta$ along a is equivalent combinatorially to $I\Delta$, and so $IC\Delta$ truncated along $I\{a\}$ is equivalent to $II\Delta$.

The geometry of $CC\Delta$ truncated along aa' is more subtle (Fig. 3). Each vertex v of Δ gives rise to three vertices on the truncation—on the base Δ the vertex v itself, and the intersection of the truncating hyperplane with the edges va and va'. In fact, the truncation of $CC\Delta$ along aa' is the convex hull of three copies of Δ, of which two have been translated and shrunk in size. It should not become clear that, just as $C\Delta$ truncated at a is equivalent to $I\Delta$, so $CC\Delta$ truncated along aa' is equivalent to $S_2 \times \Delta$, where S_2 is a two-dimensional simplex. (This product is the convex hull of Δ and two parallel translates of Δ. Shrinking the two translates does not change the combinatorial structure.) The thick horizontal lines on Fig. 3 represent Δ and its translates. The thickened part of the other edges belong to $\Delta \times S_2$, which should be seen to be $CC\Delta$ truncated.

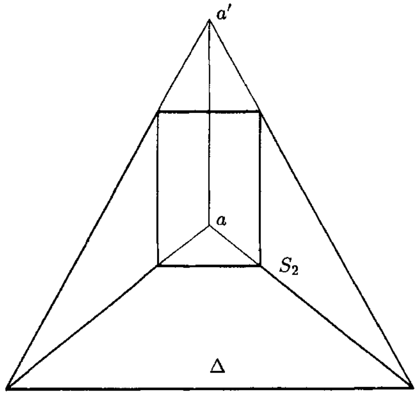


Fig. 3

7. The *IC* Equation

Our goal now is to reveal the relationship between the operators I and C. Truncation of $IC\Delta$ along $I\{a\}$ and of $CC\Delta$ along aa' will produce the same change on flag vectors, for the combinatorial structure is the same along the two edges. (This is also a consequence of the Mayer–Vietoris equation for convex polytopes.) Thus the equation

$$(I - C)CX = IIX - S_2 \times X$$

holds for any polytope X, in the following sense. The flag vector of ICX less that of CCX is equal to that of IIX less that of $S_2 \times X$.

This equation applied to Δ yields

$$(I - C)C\Delta = II\Delta - S_2 \times \Delta, \qquad (*)$$

while applied to $I\Delta$ the equation

$$(I - C)CI\Delta = III\Delta - S_2 \times I\Delta$$

is produced. By virtue of the trivial equation

$$I(S_2 \times X) = S_2 \times IX$$

and left multiplication of $(*)$ by I, the equation

$$I(I - C)C\Delta = (I - C)CI\Delta \qquad (**)$$

follows.

According to Bayer and Billera the polytopes formed by successively applying IC and C to a point provide a basis for the span of the flag vectors of all polytopes. The equation $(**)$ is used to help understand this fact.

Theorem (The IC Equation). *The operators I and C satisfy the equation*

$$IIC + CCI = ICC + ICI$$

or more memorably

$$I\ \ (I - C)C \quad = \quad (I - C)C\ \ I$$

in the sense that an equation holds between the flag vectors of the polytopes formed by applying IIC, etc., to a polytope Δ.

8. Reduction to the IC and C Basis

The IC equation allows expressions in I and C to be rewritten, replacing IIC with $ICC + ICI - CCI$. By itself this is not enough to reduce, say, $IIC\cdot$ (where $\cdot$ is the point polytope) to a combination of polytopes in the Bayer–Billera IC and C basis. To deal with the terms $ICI\cdot$ and $CCI\cdot$ an additional rule is required—that $I\cdot$ is equal to $C\cdot$.

Proposition (Reduction).

(i) *Let W be a word in I and C. Successive application of the rules*

$$IIC = ICC + ICI - CCI,$$
$$I\cdot = C\cdot$$

to $W\cdot$ *will eventually result in a formal sum to which neither of these rules can be applied.*

(ii) *If* $W\cdot$ *is a word in I and C, terminated by* $\cdot$, *to which neither of the rules applies, then W is a succession of the operators IC and C.*

Proof. Part (ii) is immediate. The problem is to prove termination. The first rule replaces IIC by ICC and CCI, each of which has only one I, and by ICI, which has two. The second rule similarly replaces an I by a C.

Define a partial order on words in I and C, where the fewer I's a word has the simpler (smaller) it is. Thus, the second rule replaces a word by a simpler word, and the first replaces a word by two simpler words, and one of the same simplicity.

It is necessary to refine the order. The replacement of IIC by ICI results in a C moving left (or an I moving right). This can be measured, for it will shorten the length of a block of I's in the word, at the cost of lengthening the next block.

It is now possible to define an order that will prove termination. Of two words, the smaller is the one with the fewer I's, or if equal, the one whose first block of I's is shorter, or if equal, the one whose second block of I's is shorter, and so forth. According to this order, both of the rules result in a word being replaced by simpler word(s) and, as positive integers cannot indefinitely be decreased, the process terminates. □

Corollary (Truncation Proof of Bayer–Billera Equations). *The span of the flag vectors of d-polytopes has dimension at most the* $(d+1)$*st Fibonacci number.*

Proof. Take a d-dimensional polytope and truncate it at the vertices to obtain Δ_1. According to the Mayer–Vietoris equation the flag vector of Δ is expressible in terms of that of Δ_1, corrected by that of I and C applied to some $(d-1)$ polytopes. Let Δ_2 be the truncation of Δ_1 along (the residue of) the edges of Δ. Again, Δ_2 differs from Δ_1 by II and and IC applied to $(d-2)$ polytopes.

This truncation process finishes at Δ_{d-2} which is a simple polytope. It is part of Dehn–Sommerville for simple polytopes that the polytopes

$$I^i C^j \cdot , \qquad \text{where} \quad i+j=d \quad \text{and} \quad i \le j,$$

form a basis for the space of face (and flag) vectors of simple polytopes.

The result now follows by induction on the dimension d and application of the reduction proposition. (It is trivial to show that there are a Fibonacci number of words in IC and C of length d.) □

9. Linear Functions of the Flag Vector

Suppose that h is a linear function on the flag vector of polytopes, with the property that $hI\Delta$ and $hC\Delta$ can be calculated from $h\Delta$. For the moment let $\hat{I}$ and $\hat{C}$ denote the linear functions that calculate $hI\Delta$ and $hC\Delta$ from $h\Delta$. By virtue of the IC equation the functions $\hat{I}$ and $\hat{C}$ must satisfy

$$\hat{I} \ (\hat{I}-\hat{C})\hat{C} \ = \ (\hat{I}-\hat{C})\hat{C} \ \hat{I}$$

and the equation

$$\hat{I}h(\cdot) = \hat{C}h(\cdot)$$

must also hold, by virtue of $I\cdot = C\cdot$.

For example, the number of vertices f_0 is such a linear function of the flag vector. The formulae

$$\hat{I}n = 2n, \qquad \hat{C}n = n+1$$

calculate $f_0 I\Delta$ and $f_0 C\Delta$ from $f_0\Delta$. As $f_0(\cdot) = 1$ the equations

$$\hat{I}f_0(\cdot) = 2 \times 1 = 2, \qquad \hat{C}f_0(\cdot) = 1 + 1 = 2$$

hold.

The linear functions $\hat{I}$ and $\hat{C}$ defined above should satisfy the IC equation. In fact

$$\hat{I}(\hat{I}-\hat{C})\hat{C}n = 2\{2(n+1)-(n+2)\} = 2n,$$

while

$$(\hat{I} - \hat{C})\hat{C}\hat{I}n = 2(2n + 1) - (2n + 2) = 2n$$

and so $\hat{I}$ and $\hat{C}$ do indeed meet the IC equation.

This process can be reversed. Suitable $\hat{I}$ and $\hat{C}$ together with a boundary condition will define a linear function on flag vectors.

Corollary (Definition via I and C). *Suppose that linear functions $\hat{I}$ and $\hat{C}$ are given, which satisfy the IC equation. Suppose also that an initial value $h\,\cdot$ is given, such that $\hat{I}(\cdot) = \hat{C}(\cdot)$. Then there is a unique linear function h defined on all convex polytopes, which satisfies*

$$hI\Delta = \hat{I}h\Delta, \qquad hC\Delta = \hat{C}h\Delta$$

which also has prescribed initial value $h(\cdot)$.

Proof. The flag vector of any polytope can be expressed as a linear combination of the polytopes formed by successive application of I and C to the point polytope. From such a representation, $h\Delta$ can readily be calculated by the formulae for $\hat{I}$, $\hat{C}$, and $h(\cdot)$.

It will follow immediately that $hI\Delta = \hat{I}h\Delta$ and $hC\Delta = \hat{C}h\Delta$. The problem is that the representation as a linear combination of I and C polytopes is not unique.

However, successive application of the rewrite rules of the reduction proposition will transform an arbitrary combination into one that involves only the polytopes in C and IC. Because $\hat{I}$ and $\hat{C}$ satisfy the IC equation, and $\hat{I}\cdot = \hat{C}h\cdot$, each rewrite leaves the calculated value of $h\Delta$ unchanged. Because the Bayer–Billera basis is a basis, the final expression is completely determined by Δ, and so the value of $h\Delta$ obtained does not depend on the representation chosen for the flag vector in terms of I and C. □

10. Remarks on the IC Equation

Sommerville's original proof [6] of the Dehn–Sommerville equations applied Möbius inversion to Euler's equation, and the proof by Bayer and Billera [1] of the generalization used the same technique. This approach casts the Euler equation as a basic result in polytopes, to be manipulated by combinatorial technique. The proof by truncation presented here treats the Dehn–Sommerville equations as basic, together with the Mayer–Vietoris and IC equations. Geometry has replaced combinatorial technique.

In fact, the Euler equation can be proved quite easily from the Dehn–Sommerville equations, by successive truncation of a general polytope to a simple polytope. Moreover, McMullen in 1971 [4] provided a direct proof of the Dehn–Sommerville equations. Thus, convex polytope theory can be started with simple polytopes satisfying Dehn–Sommerville, or with general polytopes satisfying Euler. The author prefers the first approach.

The IC equation was discovered by the author in 1985 as a consequence of some unpublished calculations involving the *cd*-index (see [2], [5], and [7]). However, its geometric significance remained persistently misunderstood and undiscovered until 1991. The IC equation provides an elegant method for defining linear functions of the flag vector of convex polytopes. Its meaning was found during research on generalized Betti numbers for convex polytopes, which will be presented elsewhere. The geometric proof presented here was found quite easily, once the IC equation had been properly understood.

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