Discrete Comput Geom 13:161–176 (1995)

Discrete & Computational
Geometry

On Minimum and Maximum Spanning Trees of Linearly Moving Points

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Abstract. In this paper we investigate the upper bounds on the numbers of transitions of minimum and maximum spanning trees (MinST and MaxST for short) for linearly moving points. Here, a transition means a change on the combinatorial structure of the spanning trees. Suppose that we are given a set of n points in d-dimensional space, $S = \{p_1, p_2, \ldots, p_n\}$, and that all points move along different straight lines at different but fixed speeds, i.e., the position of p_i is a linear function of a real parameter t. We investigate the numbers of transitions of MinST and MaxST when t increases from $-\infty$ to $+\infty$. We assume that the dimension d is a fixed constant. Since there are $O(n^2)$ distances among n points, there are naively $O(n^4)$ transitions of MinST and MaxST. We improve these trivial upper bounds for L_1 and L_∞ distance metrics.

Let $\kappa_p(n)$ (resp. $\mathscr{K}_p(n)$) be the number of maximum possible transitions of MinST (resp. MaxST) in L_p metric for n linearly moving points. We give the following results in this paper: $\kappa_1(n) = O(n^{5/2}\alpha(n))$, $\kappa_\infty(n) = O(n^{5/2}\alpha(n))$, $\mathscr{K}_1(n) = \Theta(n^2)$, and $\mathscr{K}_\infty(n) = \Theta(n^2)$ where $\alpha(n)$ is the inverse Ackermann's function. We also investigate two restricted cases, i.e., the c-oriented case in which there are only c distinct velocity vectors for moving n points, and the case in which only k points move.

1. Introduction

Computational geometry problems for moving objects are theoretically interesting and have important applications in motion planning in robotics. The pioneering

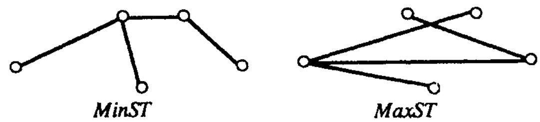

Fig. 1. MinST and MaxST.

work in this field was done by Atallah [4], who gave nontrivial upper bounds on the number of combinatorial transitions of several fundamental geometric structures such as convex hulls for moving points. Voronoi diagrams and Delaunay triangulations for moving points have recently been investigated by Imai and Imai [11], Fu and Lee [7], and Guibas *et al.* [8].

Although the two-dimensional minimum spanning tree (MinST) is a subgraph of the Delaunay triangulation, it is not even clear that the number of transitions of MinST is smaller than that of Delaunay triangulation. Recently, Monma and Suri [15] have investigated the case where only one point is allowed to move in an arbitrary manner, and gave an $O(n^{2d})$ bound (as well as a $\Theta(n^2)$ tight bound in Euclidean two-dimensional space) for transitions of MinST. This bound has been recently improved to $O(n^d \log^{c(d)} n)$ by Aronov *et al.* [3], where a constant $c(d)$ depends on d. However, to the authors' knowledge, no one has ever succeeded in improving naive bounds on the numbers of combinatorial transitions of MinST and the maximum spanning tree (MaxST) (Fig. 1) when all points move linearly.

In this paper we investigate the upper bounds on the numbers of transitions of MinST and MaxST for linearly moving points. Our paper is the first to give nontrivial upper bounds for these numbers.

Let us formulate the problem: Suppose that we are given a set of n points in general d-dimensional space, $S = \{p_1, p_2, \ldots, p_n\}$, and that all points move along different straight lines at different but fixed speeds, i.e., the position of p_i is a linear function of a real parameter t. We investigate the numbers of transitions of MinST and MaxST when t increases from $-\infty$ and $+\infty$. We assume that the dimension d is a fixed constant. Figure 2 illustrates transitions of the MinST of five points when a point (shaded) moves linearly along the dashed line.

When t is fixed, MinST and MaxST are determined only by the relative order of edge lengths. This implies that MinST (resp. MaxST) changes only if the relative order of the lengths for some pair of edges changes.

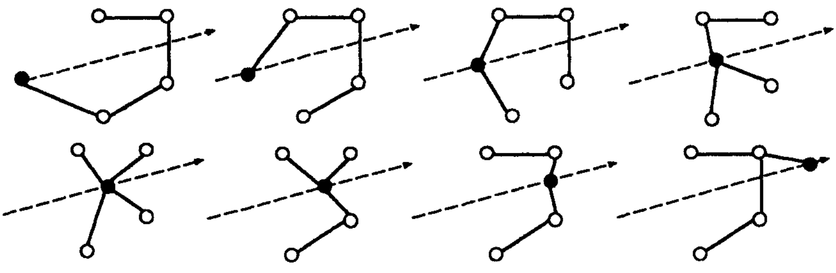

Fig. 2. Transition of MinST.

Since there are $O(n^2)$ distances among n points, there are naively $O(n^4)$ transitions of MinST and MaxST. On the other hand, it is easy to construct an example that requires $\Omega(n^2)$ transitions for each of MinST and MaxST. Therefore, there is a rather big gap between the lower and upper bounds of such transitions. Note that known bounds for the number of transitions of an L_2 planar Delaunay triangulation are $O(n^3)$ and $\Omega(n^2)$ [11], [8]. Very recently, Chew [5] showed an $O(n^2\alpha(n))$ upper bound for a planar Delaunay triangulation in L_1 or L_∞ metric.

Let $\kappa_p(n)$ (resp. $\mathscr{K}_p(n)$) be the number of maximum possible transitions of MinST (resp. MaxST) in L_p metric for n linearly moving points. In this paper we restrict ourselves to the cases of $p = 1$ and ∞ (except in Section 3), and give improved bounds for them as follows:

$$\kappa_1(n) = O(n^{5/2}\alpha(n)), \qquad \kappa_\infty(n) = O(n^{5/2}\alpha(n)),$$

$$\mathscr{K}_1(n) = \Theta(n^2), \qquad \mathscr{K}_\infty(n) = \Theta(n^2),$$

where $\alpha(n)$ is the inverse Ackermann's function and is very slowly growing [1], [16]. In particular, a $\Theta(n^2)$ tight bound for MaxST is attained.

We then consider two restricted cases. The first is the c-oriented case in which there are only c distinct velocity vectors for moving n points. The second is the case in which only k points move, while the other points remain in their original positions. We improve the above upper bounds for these cases.

L_1 and L_∞ metrics are referred to as linear metrics in the subsequent discussion. The common technique we use to derive our upper bounds is the generalization of the combinatorial results obtained by Gusfield [9] and Katoh and Ibaraki [12] for the number of transitions of the minimum (or maximum) weight base in a matroid in which the weights of all elements are linear functions of a single parameter t. Note that the minimum (or maximum) weight base in a matroid is an abstract notion of MinST and MaxST for general graphs.

The distance between two points is a piecewise-linear convex function in t for linear metrics. If $p(t) = (p^{(1)}(t), \ldots, p^{(d)}(t))$ and $q^{(t)} = (q^{(1)}(t), \ldots, q^{(d)}(t))$, the distance function $d_1(p(t), q(t)) = \sum_{i=1}^{d} |p^{(i)}(t) - q^{(i)}(t)|$ is a sum of d piecewise-linear convex functions, and, hence, is a piecewise-linear convex function itself. For the L_∞ metric, the distance function is written as a function whose value is the maximum of d piecewise-linear convex functions and, hence, is piecewise-linear convex itself.

Therefore, we must generalize the result of [9] and [12] to the piecewise-linear convex case. For this purpose, we introduce a minimum (resp. maximum) weight-base problem for matroids appropriately defined on certain multigraphs such that the weights of all elements are linear in t, and the transition of the minimum (resp. maximum) weight base occurs if the transition of MinST (resp. MaxST) for the original graphs occurs.

From this, we obtain $O(m^{3/2})$ and $O(m\sqrt{n})$ nontrivial upper bounds on the numbers of transitions of MinST and MaxST, respectively, for general graphs with n vertices and m edges in which each edge length is piecewise-linear convex in a single parameter t with a constant number of breakpoints. As a direct consequence of these results, we have $\kappa_1(n) = O(n^3)$ and $\mathscr{K}_1(n) = O(n^{5/2})$. These bounds are further improved to those stated above through geometric insights into the structures of the problems. In particular, we use Yao's lemma [20], with which [20] developed efficient algorithms for the Euclidean MinST.

Finally we study the problem of finding the value of t at which the total length of MaxST is minimized. For linear and L_2 metrics, we give nearly linear-time algorithms for moving points in a plane, based on the parametric search technique developed by Megiddo [13].

2. Linear Metric Spanning Trees of Moving Points

We derive the upper bounds on the number of transitions of MinST and MaxST in L_1 and L_∞ metrics. Since the results we obtain and the techniques we use are the same for both metrics and for any d-dimensional space, we concentrate only on the L_1 metric case and on $d = 2$. Let $p_i(t) = (x_i(t), y_i(t))$ denote the position of point p_i at t, where $x_i(t)$ and $y_i(t)$ are linear functions of t. The L_1 distance between two points in the plane is a piecewise-linear convex function in t with at most two breakpoints. Here t' is said to be a breakpoint of a piecewise-linear function if the slope of the function changes at t'. The L_p distance between points p_i and p_j is denoted by $d_p(p_i, p_j)$. Since $d_p(p_i, p_j)$ is a function of t, it should be written as $d_p(p_i(t), p_j(t))$, but for convenience we omit the argument t unless there is a possibility of confusion.

2.1. *Number of Minimum Weight Bases of a Linearly Weighted Matroid*

First, we introduce a theorem on the minimum weight base of a linearly weighted matroid, previously presented by Gusfield [9] and Katoh and Ibaraki [12]. Let E be a finite set and let $\mathscr{B}$ be a family of subsets of E. The pair $(E, \mathscr{B})$ is called a *matroid* $M(E, \mathscr{B})$, and the elements of $\mathscr{B}$ are the bases of $M(E, \mathscr{B})$, if the following two axioms hold [18]:

(A1) For any $B, C \subset E$ with $B \neq C$, if $B \in \mathscr{B}$ and $C \subset B$, $C \notin \mathscr{B}$.
(A2) For any $B, B' \subset \mathscr{B}$ with $B \neq B'$ and for any $e \in B - B'$, there is $e' \in B' - B$ such that $(B - \{e\}) \cup \{e'\} \in \mathscr{B}$.

For instance, let $\mathscr{T}$ be a set of spanning trees in an undirected connected graph $G = (V, E)$; then $(E, \mathscr{T})$ forms a matroid and $\mathscr{T}$ is a set of bases [18].

The number $|B|$ of elements of a base $B \in \mathscr{B}$ is independent of the choice of B [18], and is denoted by p. Let $m = |E|$, and assume the elements of E to be indexed from 1 through m. We assume that each element i has a real-valued weight $w_i(t) = a_i t + b_i$ that is linear in the parameter t. The minimum (resp. maximum) weight base is the one in which the sum of weights of elements is minimum (resp. maximum). It is known [18] that the minimum (resp. maximum) weight base changes only if the relative order of weights of some two elements i and j changes.

Since the weight functions of two elements have at most one intersection, we have an $O(m^2)$ trivial upper bound on the number of transitions of the minimum (resp. maximum) weight base of $M(E, \mathscr{B})$. This was improved by [9] and [12], as is shown in the following theorem. The outline of its proof is given because the technique used is useful for deriving the results in the restricted cases discussed in Section 2.5.

We remark that if we consider the trivial matroid where $\mathscr{B}$ consists of all subsets of cardinality p of E, Theorem 2.1 implies the well-known fact (see [6]) that the complexity of the pth level of an arrangement of m lines is $O(m \min\{\sqrt{p}, \sqrt{m-p}\})$.

Theorem 2.1 [9], [12]. *When all $w_i(t)$ are linear in t, the number of transitions is*

$$O\left(m \min\left\{\sqrt{p}, \sqrt{m-p}\right\}\right). \tag{1}$$

Proof. We briefly show the proof of Theorem 2.1 which was originally given by [9] and [12]. We consider only the case of the minimum weight base (the case of the maximum weight base can be treated in the same manner). For ease of exposition, we assume that all a_i's are distinct, where a_i is the coefficient of t in $w_i(t)$. We rearrange the indices 1 through m in decreasing order of a_i. Let

$$t_{ij} = \frac{b_j - b_i}{a_i - a_j},$$

which is the value of t at which the relative order of the weights of two elements i and j changes.

We further assume for simplicity that all t_{ij}'s are distinct. This assumption implies that when the minimum weight base changes at t from B to B', B is transformed into B' by a single exchange, i.e., $B' = B \cup \{i\} - \{j\}$. We represent a base B by an m-dimensional 0–1 vector, such that the ith element in the vector, denoted x_i is equal to 1 if $i \in B$, and is equal to 0 otherwise. Let us define the potential of B as

$$\pi(B) = \sum_{i=1}^{m} i x_i. \tag{2}$$

By definition,

$$\frac{p(p+1)}{2} = \sum_{i=1}^{p} i \leq \pi(B) \leq \sum_{i=m-p+1}^{m} i = \frac{p(m+1)}{2}. \tag{3}$$

Let $B_1, B_2, \ldots, B_N$ denote the minimum weight bases generated by increasing t from $-\infty$ to $+\infty$. When the minimum weight base changes from B_k to B_{k+1} at t_k, it follows from the above assumption that $i, j \in E$ exist such that $i \in B_k - B_{k+1}$, $j \in B_{k+1} - B_k$, and $B_{k+1} = (B_k \cup \{j\}) - \{i\}$. In this case we say that B_k is transformed into B_{k+1} by an exchange $\langle i, j\rangle$. It is clear that $t_k = t_{ij}$ and $a_i > a_j$ hold, and that the change from B_k to B_{k+1} increases the potential by $j - i$. The *distance* of exchange $\langle i, j\rangle$ is defined as $j - i$.

For any i and j with $i < j$ at most one k exists such that B_k is transformed into B_{k+1} by an exchange $\langle i, j\rangle$, because t_k must equal t_{ij}. Therefore, at most $m - l$ k's exist such that the change from B_k to B_{k+1} increases the potential by l.

From inequality (3), we have the following inequality:

$$\pi(B_N) - \pi(B_1) \leq \frac{p(m-p)}{2}. \tag{4}$$

Therefore, the case, where N becomes as large as possible, happens when the potential increase due to each change from B_k to B_{k+1} is as small as possible. More precisely, this case happens when $m-1$ exchanges with distance one exist, and $m-2$ exchanges with distance two exist, and so on. From (4), the upper bound of N is obtained by first (i) computing the minimum $q = q^*$ that satisfies

$$\sum_{l=1}^{q} l(m-1) \geq \frac{p(m-p)}{2}, \tag{5}$$

and then (ii) computing

$$\sum_{l=1}^{q^*} (m-1). \tag{6}$$

It is easy to see that $q^* = O(\min\{\sqrt{p}, \sqrt{m-p}\})$. Therefore, the term in (6) becomes $O(m \min\{\sqrt{p}, \sqrt{m-p}\})$. □

2.2. *Number of Distinct MinSTs and MaxSTs with Piecewise-Linear Convex Weight Functions*

Next, let us apply Theorem 2.1 to analyze the number of transitions of MinST (and MaxST) of a graph with piecewise-linear convex weights.

The weight $w_i(t)$ of an edge i of a graph $G = (V, E)$ is a piecewise-linear convex function of a parameter t. Let $|V| = n$ and $|E| = m$. Let l_i denote the number of breakpoints of $w_i(t)$, and let

$$M = \sum_{i \in E} (l_i + 1). \tag{7}$$

When t increases from $-\infty$ to $+\infty$, we want to estimate the numbers $N_{\min}$ and $N_{\max}$ of transitions of MinST and MaxST of G. Notice that MinST (resp. MaxST) changes only if the relative order between the weights for a pair of edges changes. For each pair of edges i and j, the functions $w_i(t)$ and $w_j(t)$ have at most $l_i + l_j + 1$ intersections. Therefore, the trivial upper bound for both of $N_{\max}$ and $N_{\min}$ is $O(\sum_{i \neq j} l_i + l_j + 1) = O(Mm)$.

In order to improve this bound, we construct a multigraph $G' = (V, E')$ from the original graph $G = (V, E)$ in such a way that the vertex set of G' is V, the weight of each edge of G' is linear in t, and the minimum (or maximum) weight base of an appropriate matroid defined on G' changes if (not necessarily only if) the topology of MinST (or MaxST) changes. Thus, the number of transitions of the matroid is at least the number of transitions of MinST (or MaxST).

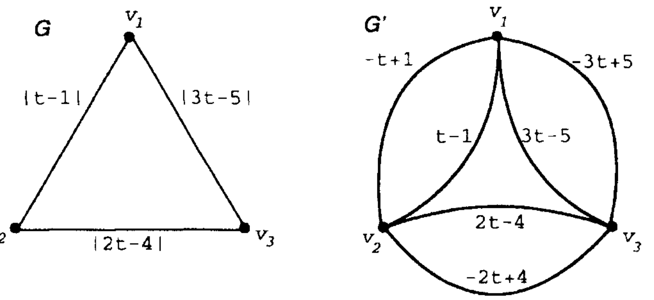

Fig. 3. An example of graphs G and G^d.

The convex function $w_i(t)$ can be thought of as the upper envelope of $l_i + 1$ linear functions. Let such $l_i + 1$ linear functions be

$$z_i^k(t) = a_i^k t + b_i^k, \qquad k = 1, \ldots, l_i + 1. \tag{8}$$

The edge set E' consists of $l_i + 1$ multiple edges $e_i^1, e_i^2, \ldots, e_i^{l_i+1}$ connecting two endpoints for each edge i of G. The edge e_i^k has the linear weight $z_i^k(t)$ defined by (8). $|E'| = M$ holds from the definition of M.

Figure 3 illustrates an example of graph G and its corresponding multigraph G'. As illustrated on the left-hand side of the figure, G has three vertices and three edges, and the weights of the edges are piecewise-linear convex functions $|t-1|$, $|2t-4|$, and $|3t-5|$. The corresponding multigraph G' has six edges with linear weights as illustrated in the figure.

Lemma 2.2.

(i) *Let C be a subset of E' such that at most one edge among $\{e_i^1, e_i^2, \ldots, e_i^{l_i+1}\}$ does not belong to C for each i, and the set*

$$\{i \in E \,|\, \textit{all edges } e_i^1, e_i^2, \ldots, e_i^{l_i+1} \textit{belong to } C\} \tag{9}$$

is a spanning tree in G. Let $\mathscr{C}$ be the set of all such C's. Then $(E', \mathscr{C})$ is a matroid.

(ii) *Let $\mathscr{T}$ be the set of spanning trees of the multigraph G'. Then $(E', \mathscr{T})$ is a matroid.*

Proof. Since (ii) is obvious, we prove only (i). Since any $C \in \mathscr{C}$ has $M - (m - n + 1)$ edges, axiom (A1) holds. For axiom (A2), let us consider, $C' \in \mathscr{C}$ with $C \neq C'$. Choose an arbitrary $e_i^k \in C - C'$. Let T and T' be two spanning trees defined by (9) for C and C', respectively. The following two cases are possible:

Case 1: $i \notin T$. Some $e_i^{k'} \in C' - C$ exists because there are exactly l_i edges with subscript i in C, as $i \notin T$, and at least l_i edges with subscript i in C', by definition of $\mathscr{C}$. Thus $(C \cup e_i^{k'}) - e_i^k$ again belongs to $\mathscr{C}$.

Case 2: $i \in T$. Since $e_i^k \notin C'$, $i \notin T'$ follows. Thus, a unique path in T' connecting both endponts of i exists. Choose an edge j on the path such that $j \notin T$ (such an edge always exists). Then $(T \cup \{j\}) - \{i\}$ is again a spanning tree. Since a unique edge $e_j^{k'}$ exists such that $e_j^{k'} \in C' - C$, $(C \cup e_j^{k'}) - e_i^k$ again belongs to $\mathscr{C}$. □

Theorem 2.3. *for an undirected graph $G = (V, E)$ in which the edge weights are piecewise-linear convex functions of a single parameter t:*

(i) *$O(M\sqrt{m})$ transitions of MinST exist.*
(ii) *$O(M\sqrt{n})$ transitions of MaxST exist.*

Proof. (i) consider the matroid in Lemma 2.2(i). Given a MinST T at a certain value t, the set C defined by

$$C = E' - \left\{ e_i^{k'} | i \notin T,\, z_i^{k'}(t) = \max_{1 \le k \le l_i + 1} z_i^k(t) \right\}$$

is a minimum weight base in the matroid as can be easily shown. Conversely, for a minimum weight base in the matroid at a certain t, the corresponding spanning tree defined by (9) is MinST for the same t. Therefore, if MinST changes, the corresponding minimum weight base in the matroid always changes. Thus, since every base in the matroid has $M - (m - n + 1)$ elements, the theorem follows from Theorem 2.1.

(ii) Consider the matroid defined in Lemma 2.2(ii). It is clear that if MaxST changes, the corresponding maximum weight base in the matroid always changes. Thus, since every base in the matroid has $n - 1$ elements, the theorem follows from Theorem 2.1. □

2.3. *Number of Transitions of MinST*

Based on Theorem 2.3(i), the following theorem is immediate.

Theorem 2.4. $\kappa_1(n) = O(n^3)$.

Proof. Consider the complete graph $G = (S, S \times S)$, where S is the set of n points in the plane, and the length of an edge between two points is measured in the L_1 metric. Since the L_1 distance between two points is a piecewise-linear convex function in t with at most two breakpoints, we have $M = 3n(n - 1)/2$ from (7). Thus, the theorem follows from Theorem 2.3(i). □

This bound is further improved by using the technique developed by Yao [20]. We first define an L_1-version of Yao's graph (sometimes called the *local neighborhood graph*) introduced by Yao [20]. For a given t and a given point p_i, we divide the plane into eight regions relative to p_i. The regions are formed by four lines passing

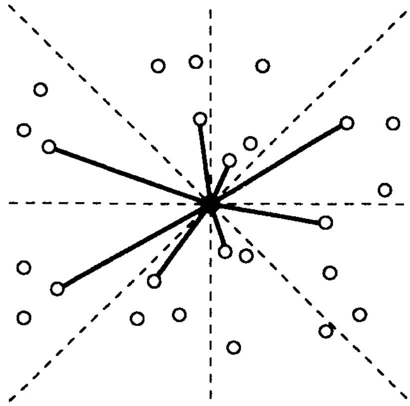

Fig. 4. Local nearest neighbors.

through p_i and forming angles of 0°, 45°, 90°, and 135°, respectively, with the x-axis. We number the regions counterclockwise, and use $R_l(p_i)$ to denote the set of points in the lth region (including the boundary), for $1 \leq l \leq 8$. We then have the following lemma:

Lemma 2.5 [20]. *If p_j and p_k are points in $R_l(p_i)$ for some l, then $d_1(p_j, p_k) \leq \max\{d_1(p_i, p_j), d_1(p_i, p_k)\}$.*

For each $R_l(p_i)$, let p_k be the one such that

$$d_1(p_i, p_k) = \min\{d_1(p_i, p_j) | j \neq i, p_j \in R_l(p_i)\}.$$

The point p_k is called the *local nearest neighbor to p_i in $R_l(p_i)$* (Fig. 4).

An L_1-version of Yao's graph, $G = (S, E)$, is the one such that S is the set of n points in the plane, and $(p_i, p_j) \in E$ if and only if p_j is the nearest neighbor to p_i in $R_l(p_i)$ for some l with $1 \leq l \leq 8$. $G = (S, E)$ contains at most $8n$ edges.

Lemma 2.6 [20]. *The edge set E of $G = (S, E)$ contains a MinST in L_1 metric.*

Figure 5 illustrates the containment of a MinST (bold edges) in the associated Yao's graph.

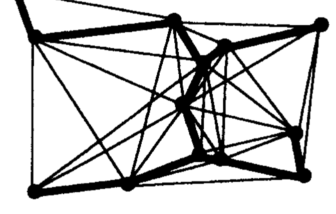

Fig. 5. Yao's graph and MinST.

Since $G = (S, E)$ depends on the parameter t, we write it as $G(t) = (V, E(t))$. How many times does $E(t)$ change as t increases from $-\infty$ to $+\infty$? The lemma below follows from the theory of upper envelopes of line segments [10], [19].

Lemma 2.7. *For each p_i and each l with $1 \leq l \leq 8$, the nearest neighbor to p_i in $R_l(p_i)$ changes $O(n\alpha(n))$ times when t moves from $-\infty$ to $+\infty$, where the function $\alpha(n)$ is the inverse Ackermann's function.*

Proof. Suppose that a point $p_j(t)$ enters into and goes out of $R_l(p_i(t))$ at $t = t'$ and $t = t''$, respectively. During the time interval $[t', t'']$, the L_1 distance $d_1(p_i(t), p_j(t))$ is a linear function in t as is easily shown. Consider the two-dimensional space (t, z). Then $z = d_1(p_i(t), p_j(t))$ can be thought of as a line segment connecting (t', z') and (t'', z''), where z' and z'' are values of $d_1(p_i(t), p_j(t))$ at t' and t'', respectively. Since at most n points enter into and go out of $R_l(p_i(t))$ over the entire range of t, we can get $O(n)$ such line segments. The graph of the distance between $p_i(t)$ and the nearest neighbor to $p_i(t)$ in $R_l(p_i(t))$ is then the lower envelope of such $O(n)$ line segments. Thus, the lemma is immediate from the results of Hart and Sharir [10]. □

Thus, we have the following lemma:

Lemma 2.8. *The edge set $E(t)$ of $G(t) = (V, E(t))$ changes $O(n^2\alpha(n))$ times.*

Letting $t_1, t_2, \ldots, t_r$ with $t_1 < t_2 < \cdots < t_r$ be the sequence of t's at which $E(t)$ changes, $[-\infty, +\infty]$ is divided into $O(n\alpha(n))$ disjoint intervals $I_1, I_2, \ldots, I_{n\alpha(n)}$ so that each interval contains $O(n)$ t_k's. Now let us consider the interval I_k and define

$$E_k = \{(p_i, p_j) | (p_i, p_j) \in E(t) \text{ for some } t \in I_k\}. \tag{10}$$

Then $|E_k| = O(n)$ follows. Consider the graph $G_k = (S, E_k)$ in which the weight of each edge $p_i, p_j)$ is equal to $d_1(p_i(t), p_j(t))$. Note that MinST of $G_k = (S, E_k)$ changes at some $t \in I_k$ if and only if MinST for the same point set in the plane changes. From $|E_k| = O(n)$ and Theorem 2.3(i), the number of transitions of MinST of G_k over the interval I_k is $O(n^{3/2})$. Therefore, we have the following theorem:

Theorem 2.9. $\kappa_1(n) = O(n^{5/2}\alpha(n))$.

The Euclidean (i.e., L_2) MinST is also contained in the edge set of Yao's graph (of L_2 norm). We can easily show that the number of transitions of Yao's graph is $n\lambda_4(n) = \Theta(n^2 2^{\alpha(n)})$ for the L_2 norm by using the result of [1], where $\lambda_4(n)$ is the maximum length of a Davenport–Schinzel sequence of order 4. Thus, by applying an argument similar to the one given after Lemma 2.8, we immediately obtain the following:

Proposition 2.10. $\kappa_2(n) = O(n^3 2^{\alpha(n)})$.

The main reason we have not succeeded in obtaining a subcubic upper bound complexity for the L_2 case is the lack of a good counterpart of Theorem 2.1. Instead of a family of linear functions, we should consider a family of quadratic functions. As we remarked before, Theorem 2.1 is a generalization of an upper-bound theorem of the complexity of a level of an arrangement of lines. Thus, a straightforward counterpart of Theorem 2.1 for the L_2 case should involve a nontrivial upper-bound result on the complexity of a level of an arrangement of quadratic curves, which seems to be a hard open problem in computational geometry.

2.4. *Number of Transitions of MaxST*

As a direct consequence of Theorem 2.3(ii), we get $\mathscr{K}_1(n) = O(n^{5/2})$ by the same argument as in the proof of Theorem 2.4. This upper bound is further improved to $O(n^2)$. We also prove that $\mathscr{K}_1(n) = \Omega(n^2)$. Thus we establish the tight bound $\mathscr{K}_1(n) = \Theta(n^2)$.

Theorem 2.11. $\mathscr{K}_1(n) = \Theta(n^2)$.

Proof. First, we give the lower bound. In the one-dimensional case the MaxST coincides with the furthest neighbor graph, which is a graph obtaining by connecting each point to the point (one of the points) furthest from it. Let $S(t) = S_0(t) \cup S_+(t) \cup S_-(t)$ be a set of $3n$ moving points on the real number line. $S_0(t) = \{0, 1/n^3, 2/n^3, \ldots, (n-1)/n^3\}$ is a set of n stable points. $S_+(t) = \{p_1(t), \ldots, p_n(t)\}$, where $p_i(t) = 2i - 1 + (1 + (2i-1)^2/n^2)t$. $S_-(t) = \{q_1(t), \ldots, q_n(t)\}$, where $q_i(t) = -2i - (1 - (2i)^2/n^2)t$. The point $p_i(t)$ is the furthest neighbor of the point j/n^3 of S_0 if and only if $(n^3 - 2j)/(4i-3)n \le t \le (n^3 + 2j)/(4i-4)n$. Hence, transitions of the MaxST of $S(t)$ occur at no less than n^2 t's.

Next, we show the $O(n^2)$ upper bound for the planar case. It is straightforward to generalize it for any fixed-dimensional case. As shown in [14], MaxST contains the furthest neighbor graph (FNG). The L_1 hull of S is the set of points which maximizes one of the linear form $x+y$, $x-y$, $-x+y$, and $-x-y$. From the definition, the L_1 furthest neighbor of a point of S is located on the L_1 hull. It is easy to see that the number of transitions of the L_1 hull is $O(n)$. We can assume that there are only a constant number of points on the L_1 hull at an arbitrary t (we can apply the perturbation method otherwise), and hence the number of transitions of the FNG is $O(n^2)$.

The FNG contains at most two (in the higher-dimensional case, 2^{d-1}) connected components. Let l be the longest distance between the connected components. Then l is the distance between a point in the L_1 hull of one component and a point in the L_1 hull of the other. MaxST of S is the union of the FNG and l. The number of transitions of the L_1 hull of each connected component is $O(n^2)$. Since at most four points are located on the L_1 hull if the points are in general position, the edge l is changed $O(1)$ times for a fixed topology of the L_1 hulls of components. Thus, we obtain the $O(n^2)$ upper bound. □

2.5. *Restricted Cases*

We consider in this section two restricted cases: the c-oriented case and the case where only k points move. We are interested in the case where c and k are small compared with n. In order to deal with the c-oriented case, we first give the following lemma, which is a counterpart of Theorem 2.1 for the c-oriented case.

Lemma 2.12. *Let $M(E, \mathscr{B})$ be a matroid with m elements in which the weight of each element is linear in t, and suppose that there are only c' distinct slopes among all weight functions. Then the number of changes of the minimum (resp. maximum) weight base is $O((c'mp)^{1/2} + c'p)$, where p denotes the number of elements in a base.*

Proof. We consider only the case of the minimum weight base. Let $1, 2, \ldots, m$ be the indices of elements of a matroid rearranged in nonincreasing order of the slopes of the weight functions $w_i(t) = a_i(t) + b_i$. To be precise, if $i < j$, then either $a_i > a_j$, or $a_i = a_j$ and $b_i < b_j$. Since there are c' distinct slopes, the sequence of $1, 2, \ldots, m$ is divided into c' clusters, $C_1, C_2, \ldots, C_{c'}$, each of which contains elements with weights of the same slope.

An element is called *active* at t_0 if it is in the current matroid base when $t = t_0$. Otherwise, it is called *inactive* (at t_0). We make each cluster size at least $\lfloor m/c' \rfloor$, by adding (at most m) dummy elements which never become active. The total number m' of matroid elements after the above modification is at most $2m$.

We use similar notation to that used in the proof of Theorem 2.1. Particularly, the same potential is used. The total change of the potential is at most $p(m' - p)/2$. In the analysis we use the fact that, when B_k is transformed into B_{k+1} by an exchange $\langle i, j \rangle$, i and j belong to different clusters.

As in the proof of Theorem 2.1, we consider the case where the number of N of changes becomes as large as possible. This occurs when the potential increase due to each change of the minimum weight base is as small as possible. Let s_q be the size of the cluster C_q. The ith element of C_q is denoted by $f(i, q)$ and the $(s_q - j)$th element of C_q is denoted by $l(j, q)$.

If $h < m/2c'$, an exchange that increases the potential by h is the form $\langle l(j, q), f(h - j + 1, q + 1) \rangle$, and j must be smaller than h.

Let us assume $j < m/2c'$ from now on. The $h' > h$, suppose that $\langle l(j, q), f(h - j + 1, q + 1) \rangle$ and $\langle l(j, q), f(h' - j + 1, q + 1) \rangle$ both appear in the sequence. The exchange $\langle l(j, q), f(h - j + 1, q + 1) \rangle$ makes $l(j, q)$ inactive. Before $\langle l(j, q), f(h' - j + 1, q + 1) \rangle$ takes place, $l(j, q)$ must be activated again. In order to activate $l(j, q)$, there must be an exchange $\langle f(i, q'), l(j, q) \rangle$ for some q' with $q' \le q - 1$ and a suitable i. This exchange increases the potential by at least $s_q - j > m/2c'$.

We charge m/rc' of the cost of $\langle f(i, q'), l(j, q) \rangle$ to $\langle l(j, q), f(h' - j + 1, q + 1) \rangle$. We call this new cost the *amortized cost*. That the same $\langle f(i, q'), l(j, q) \rangle$ is charged more than once does not occur; thus the amortized cost of $\langle f(i, q'), l(j, q) \rangle$ is at least $m/4c'$.

Then, for each index $l(j, q)$ $(q = 1, 2, \ldots, c' - 1, j + 0, 1, \ldots, s_q - 1)$, only one (if any) index x exists such that the exchange $\langle l(i, q), x \rangle$ increases the potential less than $m/4c'$ (with respect to the amortized cost). Moreover, the increase of the potential is at least j for the exchange.

Under the above condition, N is maximized when there are exactly $c' - 1$ exchanges which increase the amortized potential j for each $j = 1, 2, \ldots, \min\{m/4c', \sqrt{p(m' - p)/(c' - 1)}\}$, and all other exchanges increase the amortized potential at least $m/4c'$. Since the total increases of the potential is at most $p(m' - p)/2 < pm$, there are less than $4c'p$ exchanges which increase the potential more than or equal to $m/4c'$. Thus, we have $N = O((c'mp)^{1/2} + c'p)$. □

Now the number of transitions of MinST in the c-oriented case, where only c distinct velocity vectors exists, can be analyzed in the same fashion as in Section 2.2. It is easy to show that the number of transitions of Yao's graph is $O(\min\{\alpha(n), c\}n^2)$ for the c-oriented case. There are at most c^2 distinct slopes among the weight functions. Thus we establish the following theorem from Lemma 2.12:

Theorem 2.13. *$\kappa_1(n) = O(\min\{c, \alpha(n)\}c^2n^2)$ holds in the c-oriented case.*

Proof. We consider the time interval in which $O(n)$ change occurs on Yao's graph. In this interval, $m = O(n)$, $p = m - (n - 1) = O(n)$, and $c' = c^2$. Thus, the number of transitions in the interval is $O((c^2n^2)^{1/2} + c^2n) = O(c^2n)$. Since there are $O(\min\{c, \alpha(n)\}n)$ such intervals, the total number of transitions is $O(\min\{c, \alpha(n)\}c^2n^2)$. □

The above bound is tight for fixed c, since it is easy to show the $\Omega(n^2)$ lower bound for the 2-oriented case.

Now let us analyze the number of transitions of MaxST in the c-oriented case. Consider the complete graph $G = (S, S \times S)$ defined in the proof of Theorem 2.4, and the corresponding multigraph B' introduced in Section 2.1. It is easy to see that there are $O(c^2)$ distinct slopes among $O(n^2)$ edge weights. Thus, from Lemma 2.12, we have the following theorem:

Theorem 2.14. *$\mathscr{K}_1(n) = O(cn^{3/2})$ holds in the c-oriented case.*

Proof. In the corresponding matroid, $m = O(n^2)$, $p = n - 1$, and $c' = c^2$. Thus, the number of transitions is $O((c^2n^2n)^{1/2} + c^2n) = O(cn^{3/2} + c^2n)$. On the other hand, an $O((n^2)$ bound has been already given (Theorem 2.11), and $\min\{n^2, cn^{3/2} + c^2n\} = O(cn^{3/2})$. □

We now consider the case where there are only k moving points. Other points are called *fixed*. Let S' and S'' be the sets of k moving points and $n - k$ fixed points, respectively. Let MaxST(S'') (resp. MinST(S'')) be the MaxST (resp. MinST) for the point set S''. This does not change with time, since the points in S'' are fixed. MaxST (resp. MinST) for any t is contained in the set of the union of MaxST(S'') (resp. MinST(S'')) and edges connecting the points in S' and S ($= S' \cup S''$). There are $O(kn)$ edges in this set. Furthermore, since the situation can be regarded as the

$(k+1)$-oriented case. We have the following theorems:

Theorem 2.15. *$\mathscr{K}_1(n) = O(k^2 n)$ holds when only k points move linearly.*

Proof. In the corresponding matroid, $m = O(kn)$, $p = O(n)$, and $c' = k^2$. We apply Lemma 2.12, and obtain $\mathscr{K}_1(n) = O(\sqrt{k^3 n^2} + k^2 n) = O(k^2 n)$. □

Remark. The above upper bound has been improved to $O(kn + k^2\sqrt{n})$ recently [17].

Theorem 2.16. *$\kappa_1(n) = O(k^3 n)$ holds when only k points move linearly.*

Proof. In the corresponding matroid, $m = O(kn)$, $p = O(kn)$, and $c' = k^2$. Thus, we obtain the theorem. □

3. Finding the Smallest MaxST

It is an interesting problem to find the value of t when the MaxST of linearly moving points satisfies some minimality condition. In this section we give efficient algorithms for finding the value of t when the total edge length of planar MaxST becomes minimum.

Theorem 3.1. *We can find the value of t when the total edge length of MaxST in two-dimensional space becomes minimum in $O(n \log^2 n)$ time and $O(n \log^4 n)$ time for the L_1 and L_2 metrics, respectively.*

Proof. First, note that the length of a given spanning tree is a convex function in t. Thus, the total length of MaxST is also convex in t, since it is an upper envelope of convex functions each of which corresponds to the total length of a spanning tree. Therefore, the optimal value t^* can be found as the supremum of t such that the slope of the function representing the total length of MaxST at t is negative. Thus, for a given t, we can tell whether $t^* < t$, $t^* > t$, or $t^* = t$ in $O(n \log n)$ time by computing the MaxST at t by using the algorithm given by [14] (for the L_2 metric). The time complexity can be reduced to $O(n)$ for the L_1 case.

Next, by directly parallelizing the algorithm of [14], we obtain an $O(\log n)$-time and an $O(n)$-processor algorithm to compute the L_1 MaxST, and an $O(\log^2 n)$-time and an $O(n)$-processor algorithm to compute the L_2 MaxST (for computing the L_2 FNG we use an $O(\log^2 n)$-time and $O(n)$-processor three-dimensional convex hull algorithm [2]).

From the above observations, it is now an easy exercise to apply Megiddo's parametric search [13] in order to obtain the results. □

Notice that these results are valid only for $d = 2$, since no parallel algorithm with the above running time is known for the general d-dimensional case.

4. Concluding Remarks

We have investigated the upper bounds on the number of transitions of dynamic MinST and MaxST of points moving linearly in a fixed-dimensional space.

For linear metrics, we have obtained a tight bound $\Theta(n^2)$ for the MaxST case. On the other hand, for the MinST case, there is still a gap of $\sqrt{n}\alpha(n)$ factor between the lower and upper bounds. We conjecture that the bound for the MinST is also $\Theta(n^2)$. These results can be extended to any convex polyhedral metric, provided that the distance function, which is a piecewise-linear convex function, has a constant number of break points.

It is important to investigate the problem for the Euclidean L_2 metric. So far, we have only been able to show an $O(n^3 2^{\alpha(n)})$ bound for MinST, and a trivial $O(n^4)$ bound for MaxST. However, we believe that these bounds will be significantly improved in future.

We also investigated the problem of finding the minimum length of MaxST for moving points, and proposed an efficient algorithm with $O(n \log^4 n)$ running time for the L_2 metric. The MinST version of this problem is quite important in practical applications, since a point set whose MST is small is usually well-clustered. Thus, it may be applied to, for example, elimination of linear noise from two-dimensional data (such as an automatic-focus system), and estimation of time of explosion or other critical events in chemical or physical experiments from observed current movement of data. However, the total length of MinST is neither concave nor convex in t, and it is left for future research to design subquadratic algorithms for finding the value of t minimizing the total edge length of the dynamic MinST.

Acknowledgments

The authors would like to thank Prof. Hiroshi Imai for his helpful comments. They also thank anonymous referees for many valuable suggestions.

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Received October 1, 1993, and in revised form July 20, 1994.