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Decomposition of Convex Figures into Similar Pieces

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Abstract. If a convex plane figure P can be decomposed into finitely many nonoverlapping convex figures such that one of these pieces is similar to P, then P is a polygon. Also, if P can be decomposed into infinitely many nonoverlapping sets such that each of the pieces is similar to P, then P is a polygon.

By a *convex figure* we mean a compact convex subset of the plane with nonempty interior. The following statement is mentioned, without proof, on p. 1 of [1].

If a convex figure P is the union of finitely many (*but at least two*) *nonoverlapping and congruent sets similar to P, then P is a polygon.*

In this note we prove the following two generalizations of this result.

Theorem 1. *Suppose that the convex figure P is the union of finitely many* (*but at least two*) *nonoverlapping convex figures such that one of them is similar to P. Then P is a polygon.*

Theorem 2. *Suppose that the convex figure P is the union of infinitely many nonoverlapping sets similar to P. Then P is a polygon.*

We use the following notation. The diameter, interior, closure, boundary, and derived set (set of points of accumulation) of a set A are denoted by diam A, int A, cl A, ∂A, and A'. If A is convex, then we denote the set of extremal points of A by $E(A)$. The isolated points of $E(A)$ are called vertices, and the set of vertices of A is denoted by $V(A)$. It is easy to see that $p \in V(A)$ if and only if p is the common endpoint of two nonparallel segments contained in ∂A. The angle of these segments is denoted by $\alpha_A(p)$. It is clear that $V(A)$ is always countable and for every $\varepsilon > 0$ the set $\{p \in V(A)\colon \alpha_A(p) < \pi - \varepsilon\}$ is finite. Therefore, if $V(A)$ is infinite and $p_1, p_2, \ldots$ is an enumeration of $V(A)$, then $\alpha_A(p_n) \to \pi$.

We denote the set $(E(A))'$ by $K(A)$. Thus $E(A) = V(A) \cup K(A)$. It is easy to see that $p \in \partial A \setminus K(A)$ if and only if there is an open subarc I of ∂A such that $p \in I$ and I consists of at most two line segments. Clearly, $K(A) = \emptyset$ if and only if A is a polygon.

To prove Theorem 1, suppose that P is a convex figure, $P = \bigcup_{i=1}^{n} P_i$, where $P_1, \ldots, P_n$ are nonoverlapping convex figures and P_1 is similar to P. Then

$$\partial P_1 \cap \operatorname{int} P \subset \bigcup_{i=2}^{n} (P_1 \cap P_i). \tag{1}$$

Indeed, every $x \in \partial P_1 \cap \operatorname{int} P$ is a point of accumulation of $\bigcup_{i=2}^{n} P_i$. Thus $x \in P_1 \cap P_i' \subset P_1 \cap P_i$ for at least one $i \geq 2$. Since the sets $P_1 \cap P_i$ are line segments (or points), it follows from (1) that $\partial P_1 \cap \operatorname{int} P$ can be covered by finitely many lines. Therefore the next lemma yields Theorem 1.

Lemma 1. *Let P and $P_1 \subsetneqq P$ be similar convex figures such that $\partial P_1 \cap \operatorname{int} P$ can be covered by finitely many lines. Then P is a polygon.*

Proof. First we note that $E(P_1) \cap \operatorname{int} P$ is finite, as $E(P_1)$ intersects every line in at most two points. Let ϕ denote the similarity transformation mapping P onto P_1. We show that $\phi(K(P)) \subset K(P)$. Let $p \in K(P)$ and $p_1 = \phi(p)$. Obviously, $p_1 \in K(P_1)$ and hence $p_1 \in E(P_1)'$. Since $E(P_1) \cap \operatorname{int} P$ is finite, it follows that $p_1 \in (E(P_1) \cap \partial P)'$. We claim that $p_1 \in K(P)$. If this is not true, then $p_1 \in \partial P \setminus E(P)$ or $p \in V(P)$. In both cases there are two line segments, I and J, in ∂P having p_1 as a common endpoint. Since $p_1 \in (E(P_1) \cap \partial P)'$, one of these line segment, say I, contains at least two distinct elements $q \neq p_1$ and $r \neq p_1$ of $E(P_1) \cap \partial P$. Then p_1, q, and r are collinear points of ∂P_1 and hence there is a line segment $I' \subset I \cap \partial P_1$ containing p_1, q, and r. However, in this case q and r cannot be both extremal points of P_1, a contradiction. Therefore $p_1 \in K(P)$ as we stated.

Consequently, $K(P)$ is a compact subset of ∂P that is mapped into itself by the contraction ϕ. If P is not a polygon, then $K(P) \neq \emptyset$ and it follows that ϕ has a fixed point $p_0 \in K(P)$. Let S denote the angular domain between the half-tangents of P at p_0 and containing P (S is a half-plane if P has a tangent at p_0). Since $\phi(P) = P_1 \subset P$, it is easy to see that $\phi(S) = S$. This implies that either ϕ is a homothetic transformation or it is a homothetic transformation followed by a reflection. In both cases, q, $\phi^2(q)$, and p_0 are collinear for every point q.

Since $p_0 \in K(P)$, we have $p_0 = \phi(p_0) \in K(P_1) = E(P_1)'$. Since $E(P_1) \cap \operatorname{int} P$ is finite, it follows that $p_0 \in (E(P_1) \cap \partial P)'$. This implies, in particular, that $E(P_1) \cap \partial P$ is infinite. Let $q_1 \in E(P_1) \cap \partial P$ be arbitrary, and put $q = \phi^{-1}(q_1)$ and $q_2 = \phi(q_1)$. Then $q \in E(P) \subset \partial P$ and $q_2 \in \partial P_1$, since $q_1 \in \partial P$. As we remarked above, the points q, q_2, and p_0 are collinear. Now we distinguish between two cases. If $q_2 \in \partial P$, then q, q_2, and p_0 are collinear points of ∂P and thus the segment I with endpoints q and p_0 belongs to ∂P. In this case $q_1 = \phi^{-1}(q_2) \in \phi^{-1}(I)$. Clearly, there are at most two lines that intersect ∂P in a segment with endpoint p_0, and hence those points q_1 of $E(P_1) \cap \partial P$ for which $q_2 \in \partial P$, can be covered by at most two lines.

If, on the other hand, $q_2 \in \text{int } P$, then $q_1 = \phi^{-1}(q_2) \in \phi^{-1}(\partial P_1 \cap \text{int } P)$. Since, by assumption, $\partial P_1 \cap \text{int } P$ can be covered by finitely many lines, we obtain that those points q_1 of $E(P_1) \cap \partial P$ for which $q_2 \notin \partial P$, can be covered by finitely many lines. Summing up, $E(P_1) \cap \partial P$ can be covered by finitely many lines. However, $E(P_1)$ contains at most two elements of every line and hence $E(P_1) \cap \partial P$ must be finite. This contradicts our previous statement that $E(P_1) \cap \partial P$ is infinite. Therefore $K(P) = \emptyset$; that is, P is a polygon. □

Now we turn to the proof of Theorem 2, and suppose that $P = \bigcup_{i=1}^{\infty} P_i$, where $P_1, P_2, \ldots$ are nonoverlapping sets similar to P. We say that a point p is *critical*, if every neighborhood of p intersects infinitely many of the sets P_i.

Lemma 2. *Let $G \subset P$ be open and suppose that G contains no critical points. Then $E(P_i) \cap G$ is finite for every i.*

Proof. If $p \in (\partial P_i) \cap G$, then there is a neighborhood U of p that intersects only a finite number of the sets P_j. This easily implies that every point of $(\partial P_i) \cap U$ is contained in a P_j with some $j \neq i$ (see the proof of (1)), and hence $(\partial P_i) \cap U$ is covered by finitely many sets of the form $P_i \cap P_j$ $(i \neq j)$. Since these sets are line segments (or points), it follows that $E(P_i) \cap U$ is finite, and hence $E(P_i) \cap G = V(P_i) \cap G$ for every i.

Suppose that $E(P_i) \cap G = V(P_i) \cap G$ is infinite, and let $p_1, p_2, \ldots$ be an enumeration of $V(P)$. As we remarked earlier, we have $\alpha_P(p_n) \to \pi$ $(n \to \infty)$, where $\alpha_P(p_n)$ denotes the angle of the two line segments of ∂P with endpoint p_n. If $p \in V(P_i) \cap G$, then there is a neighborhood of p that intersects only a finite number of the sets P_j such that p is a common vertex of these P_j's. If the angles at these vertices are $\alpha_P(p_{n_i})$ $(i = 1, \ldots, k)$, then $\alpha_P(p_{n_1}) + \cdots + \alpha_P(p_{n_k}) = v\pi$, where $v = 1$ or 2 (we have $v = 1$ if $p \in \partial P_j \setminus E(P_j)$ for some j). Since $\alpha_P(p_n) > 0$ and $\alpha_P(p_n) \to \pi$, it is easy to see that the number of sets of indices $\{n_1, \ldots, n_k\}$ satisfying these equations is finite. However, if $V(P_i) \cap G$ is infinite, then infinitely many different angles must occur among these indices (namely, the angles of P_i at the points of $V(P_i) \cap G$), which is a contradiction. □

Lemma 3. *Suppose that P is not a polygon. Let $G \subset P$ be an open set containing a critical point and let n be a positive integer. Then there is a closed disk $D \subset G \setminus P_n$ such that* int D *also contains a critical point.*

Proof. First we assume that $G \setminus P_n$ does not contain critical points. This implies, by Lemma 2, that $E(P_i) \cap (G \setminus P_n)$ is finite for every i. If $i \neq n$, then $E(P_i) \cap P_n$ is also finite, since P_i and P_n are nonoverlapping convex sets. That is, $E(P_i) \cap G$ is finite for every $i \neq n$.

Since the sets P_i are similar and are contained in P, it follows that diam $P_i \to 0$. As every neighborhood of a critical point $p \in G$ intersects infinitely many P_i's, it follows that G contains infinitely many of the sets P_i. If $P_i \subset G$ and $i \neq n$, then $E(P_i)$ is finite; that is, P_i is a polygon. Then so is P, contradicting our assumption.

Therefore $G \setminus P_n$ must contain at least one critical point p. If D is a small closed disk around p such that $D \cap P_n = \emptyset$, then D satisfies the requirements of the lemma. □

Now we prove Theorem 2 assuming that int P contains a critical point. If P is not a polygon, then, by Lemma 3, there is a closed disk $D_1 \subset \operatorname{int} P$ such that $D_1 \cap P_1 = \emptyset$ and int D_1 contains a critical point. Applying Lemma 3 again, we can choose a closed disk $D_2 \subset \operatorname{int} D_1$ such that $D_2 \cap P_2 = \emptyset$ and int D_2 contains a critical point. Continuing this process, we can define the nested sequence of closed disks $D_1 \supset D_2 \supset \cdots$. If $p \in \bigcap_{n=1}^{\infty} D_n$, then $p \in P$ and $p \notin \bigcup_{n=1}^{\infty} P_n$, a contradiction.

Therefore, in order to prove Theorem 2, we may assume that int P contains no critical points. In this case, by Lemma 2, the sets $V_i = E(P_i) \cap \operatorname{int} P$ are finite for every i.

Our next aim is to show that, for at least one i, $\partial P_i \cap \operatorname{int} P$ can be covered by finitely many lines. For a given i, consider the components of the set $\partial P_i \setminus (\partial P \cup V_i)$. These components are open subarcs of ∂P_i, and, as they do not contain extremal points of P_i, they are line segments. These line segments are of two kinds: either their endpoints are in ∂P; that is, they are chords of P, or at least one of their endpoints is in int P. In the latter case these endpoints belong to V_i and hence the number of these line segments is finite (at most twice the cardinality of V_i). That is, if $\partial P_i \cap \operatorname{int} P$ cannot be covered by finitely many lines, then ∂P_i contains infinitely many chords of P. It is easy to see that in this case $\partial P \setminus P_i$ has infinitely many components.

Suppose this happens for every i. Then $\partial P \setminus P_1$ has infinitely many components; let I_1 be a component with diam $I_1 < 1$. Clearly, I_1 is an open subarc of ∂P. Let $P_{i_2} \cap I_1 \neq \emptyset$; then, as P_1 and P_{i_2} are nonoverlapping convex sets, it follows that P_{i_2} is contained in the convex hull of I_1. This implies that each component of $\partial P \setminus P_{i_2}$, with at most one exception, is also a component of $I_1 \setminus P_{i_2}$. Let I_2 be one of these components with cl $I_2 \subset I_1$ and diam $I_2 < 1/2$. Let $P_{i_3} \cap I_2 \neq \emptyset$; then P_{i_3} is contained in the convex hull of I_2. Thus every component of $\partial P \setminus P_{i_3}$, except one, is also a component of $I_2 \setminus P_{i_3}$; let I_3 be one of these components with cl $I_3 \subset I_2$ and diam $I_3 < 1/3$. In this way we can define the nested sequence of subarcs $I_1 \supset I_2 \supset \cdots$. If $p \in \bigcap_{n=1}^{\infty}$ cl I_n, then p cannot be covered by any of the sets P_i, which is impossible.

Therefore at least one of the sets $\partial P_i \cap \operatorname{int} P$ must be contained in a finite system of lines. Then an application of Lemma 1 completes the proof of Theorem 2. □

Problems and Remarks. 1. The analogue of Theorem 1 in higher dimension is not true: every cone C can be decomposed into two nonoverlapping convex sets such that one of them is similar to C. In connection with this example, T. Zamfirescu asked the following (personal communication, June 1994):

Let P be a convex figure in $\mathbf{R}^3$ and suppose that P is the union of finitely many nonoverlapping convex figures such that *two* of the pieces are similar to P. Does this imply that P is a polyhedron?

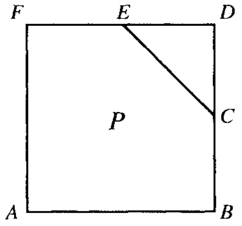

Fig. 1

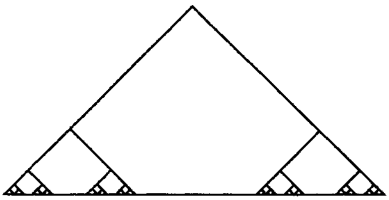

Fig. 2

2. We do not know whether or not the higher dimension analogue of Theorem 2 is true.

3. Not every convex polygon P can be decomposed into infinitely many nonoverlapping sets similar to P. A necessary condition for the existence of such a decomposition is that 2π is a linear combination of the angles of P with nonnegative integer coefficients. Indeed, suppose that this condition is not satisfied, and, still, there is a decomposition $P = \bigcup_{n=1}^{\infty} P_n$ with the given properties. This easily implies that whenever $p \in \operatorname{int} P$ is the vertex of any of the polygons P_n, then each neighborhood of p contains infinitely many polygons P_k. Then we can find a sequence of closed disks D_n such that $\operatorname{int} P \supset D_1 \supset D_2 \supset \dots$, the center of each D_n is a vertex of one of the polygons P_k, and $D_n \cap P_n = \emptyset$ for every n. If $p \in \bigcap_{n=1}^{\infty} D_n$, then $p \in P \setminus \bigcup_{n=1}^{\infty} P_n$ which is impossible.

This necessary condition is not sufficient. For example, it can be shown that the regular hexagon cannot be decomposed into infinitely many nonoverlapping regular hexagons.

4. It is easy to prove that if a polygon P can be decomposed into finitely many nonoverlapping sets similar to P, then it can also be decomposed into infinitely

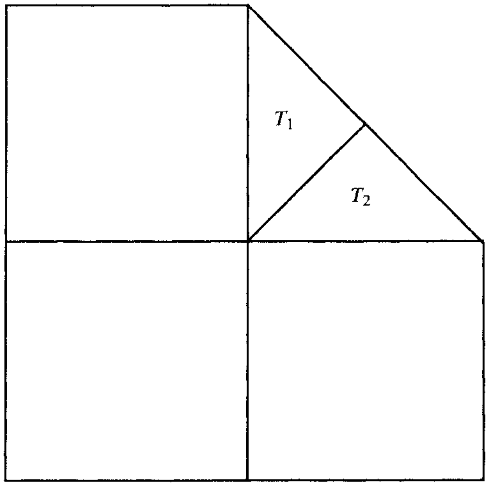

Fig. 3

many such sets. The converse, however, is not true, as the following simple example shows.

Let P denote the pentagon with vertices $ABCEF$ on Fig. 1, where $ABDF$ is a square and C and E are the middle points of the sides BD and DF. It is clear that P cannot be decomposed into finitely many nonoverlapping sets similar to P. For the sake of brevity we say that a set S has an infinite P-tiling if S can be decomposed into infinitely many nonoverlapping sets similar to P. We want to show that P has an infinite P-tiling.

(i) As Fig. 2 shows, removing a Cantor-set from the hypotenuse of an equilateral right triangle, the remaining set has an infinite P-tiling.

(ii) Then it follows that every square has an infinite P-tiling (see Fig. 1).

(iii) Finally, Fig. 3 shows an infinite P-tiling of P itself; the sets T_1 and T_2 have infinite P-tilings by (i), and the remaining three squares have infinite P-tilings by (ii).

Reference

1. G. Valette and T. Zamfirescu, Les partages d'un polygone convexe en 4 polygones semblambes au premier, *J. Combin. Theory Ser. B* **16** (1974), 1–16.

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