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Oriented Matroids with Few Mutations

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Abstract. This paper defines a "connected sum" operation on oriented matroids of the same rank. This construction is used for three different applications in rank 4. First it provides nonrealizable pseudoplane arrangements with a low number of simplicial regions. This contrasts the case of realizable hyperplane arrangements: by a classical theorem of Shannon every arrangement of n projective planes in $\mathbb{RP}^{d-1}$ contains at least n simplicial regions and every plane is adjacent to at least d simplicial regions [17], [18]. We construct a class of uniform pseudoarrangements of $4n$ pseudoplanes in $\mathbb{RP}^3$ with only $3n + 1$ simplicial regions. Furthermore, we construct an arrangement of 20 pseudoplanes where one plane is not adjacent to any simplicial region.

Finally we disprove the "strong-map conjecture" of Las Vergnas [1]. We describe an arrangement of 12 pseudoplanes containing two points that cannot be simultaneously contained in an extending hyperplane.

0. Introduction

An *arrangement of hyperplanes* of rank d is a collection X of n hyperplanes in $\mathbb{R}^d$ all passing through the origin $\mathbf{0}$ and satisfying $\bigcap X = \mathbf{0}$. Without loss of information we can intersect the arrangement with the unit sphere S^{d-1} and obtain an arrangement of $(d-2)$-spheres embedded in S^{d-1}. An arrangement is called *uniform* if no d hyperplanes meet in a line (equivalently no d spheres meet in an antipodal pair of points.) In a natural way every arrangement of hyperplanes decomposes the S^{d-1} into a $(d-1)$-dimensional cell complex $\mathscr{C}_X$. The cells of maximal dimension in $\mathscr{C}_X$ are called the *regions* or *topes* of X. Topes bounded by exactly d hyperplanes are the *simplicial regions* of X. Throughout this paper we only consider arrangements where no point in S^{d-1} is incident to more than $n-2$ of the hyperplanes. For the rest of this paper we assume that all arrangements are of this type.

Simplicial regions in uniform arrangements correspond to possibilities of local deformations. Intuitively, at every simplicial region one of the involved hyperplanes can be "pushed" over the vertex obtained by intersecting the remaining $d-1$ hyperplanes. Only the orientation of the corresponding antipodal pair of simplices changes, without influencing the structure of the rest of the arrangement. By this "switching of a simplex" a new, in general nonisomorphic, cell complex is obtained. We refer to such an antipodal pair of simplicial regions as a *mutation* of the arrangement [3], [16]. A local perturbation performed at a mutation also makes sense on a purely topological (or combinatorial) level. The "switching-operation" can be completely described on the level of the cell complex. Restricting to the combinatorial data of the face lattice of $\mathscr{C}_X$ we obtain cell complexes corresponding to purely combinatorial descriptions of *arrangements of pseudo-planes* through the origin (see [3] and [8]). When later we consider pseudoplane arrangements we always mean the equivalence classes of all arrangements having isomorphic cell complexes. An arrangement of pseudoplanes is called *realizable* if there is an arrangement of (straight) hyperplanes providing an isomorphic cell complex. It was conjectured by Las Vergnas [12] that every uniform pseudoplane arrangement can be obtained by starting with a uniform arrangement of realizable hyperplanes and performing a finite sequence of local perturbations as described above. Up to now it is not even known whether every arrangement of pseudoplanes possesses at least one mutation.

For realizable arrangements of hyperplanes Shannon [17], [18] proved the following, now classical, result concerning the number of mutations:

Theorem 0.1 (Shannon). *Let X be an arrangement of n hyperplanes of rank d and let H be any hyperplane in X. Then there exist at least d mutations incident to H and at least $n-d$ mutations not incident to H.*

This especially implies the existence of at least n mutations in such arrangements. The situation changes if arrangements of pseudoplanes rather than arrangements of hyperplanes are considered. The following was proved in [16]:

Theorem 0.2 (Roundneff and Sturmfels). *There exists a uniform arrangement $X(8)$ of eight pseudoplanes in rank 4 having exactly seven mutations.*[1]

Bokowski and Richter-Gebert [6] showed that no other uniform arrangement on eight planes shares this property. Iteratively enlarging $X(8)$ by lexicographic extensions Bokowski proved (personal communication):

Theorem 0.3 (Bokowski). *For every $n \geq 8$ there exists a uniform arrangement of n pseudoplanes in rank 4 having exactly $n-1$ mutations.*

[1] A three-dimensional model of this arrangement was built by Bokowski and Richter-Gebert. It can be seen in the entrance-hall of the ZIF-Bielefeld (Germany).

In this paper we provide geometric constructions which improve this upper bound on the minimal number of mutations. We prove:

Theorem 2.2. *For every $n \geq 2$ there exists a uniform rank 4 arrangement of $4n$ pseudoplanes having exactly $3n + 1$ mutations.*

Moreover we prove the following result that contrasts Shannon's theorem of mutations adjacent to a hyperplane:

Theorem 2.3. *There is a rank 4 uniform arrangement $R(20)$ of 20 pseudoplanes containing one plane p not adjacent to any mutation.*

The arrangement $R(20)\backslash p$ was used in [14] as an example where the space of all one-element extensions, considered as a simplicial complex [19], contains an isolated element. The result there depends on the constructions of this paper.

For our purposes it is convenient to use the language of oriented matroids instead of the language of pseudoplane arrangements. For an introduction to oriented matroid theory the reader is referred to [3], [5], and [7]. The topological representation theorem of Folkman and Lawrence states that (oriented) pseudoarrangements with n planes in rank d are in one-to-one correspondence to the rank d oriented matroids on the ground set $E := \{1, \ldots, n\}$ (here we have to allow the pseudoarrangement also to contain repeated copies of the same pseudoplane and a copy of $\mathbb{R}^d$ itself to cover also the cases of *parallel elements* and *loops* in oriented matroids) [3], [8]. In this picture the antipodal pairs of simplicial cells of a uniform pseudoarrangement are mapped to the mutations of the corresponding oriented matroid. If a uniform oriented matroid $\mathcal{M}$ is given by its basis orientations $\chi_{\mathcal{M}}: \Lambda(E, d) \to \{-1, +1\}$, a mutation corresponds to a basis whose orientation can be reversed without violating the oriented matroid axioms. We denote the set of all mutations of $\mathcal{M}$ by $Mut(\mathcal{M}) \in \Lambda(E, d)$. All mutations containing a certain element $e \in E$ are denoted by $Mut_e(\mathcal{M})$.

The associated cell complex of the arrangement translates into the set of all covectors $\mathcal{L}(\mathcal{M})$ of the corresponding oriented matroid $\mathcal{M}$ (see [3]). For this, each cell of the complex is represented by a sign-vector $C \in \{-, 0, +\}^E$ describing its relative position with respect to every hyperplane. The face lattice of the cell complex is the poset of all covectors where the order relation is induced by the relations "$0 < +$" and "$0 < -$." An oriented matroid is completely described by its covectors. For an element $g \in E$ the *affine oriented matroid* $(\mathcal{M}, g) := \{C \in \mathcal{L}(\mathcal{M}) \mid C_g = +\}$ can be considered as an affine arrangement of pseudoplanes in $\mathbb{R}^{d-1}$ where g plays the role of a plane at infinity (see [3], [9], [13], and [14]).

For an element $e \in E$ the *contraction* $\mathcal{M}/e$ is the oriented matroid on $E\backslash\{e\}$ obtained by taking all covectors $\{C \in \mathcal{L}(\mathcal{M}) \mid C_e = 0\}$ and deleting the eth component. In the corresponding pseudoarrangement a contraction $\mathcal{M}/e$ corresponds to the arrangement induced on the pseudoplane e by intersecting with all other pseudoplanes.

Finally, we disprove a conjecture of Las Vergnas concerning strong maps. A *strong map* is a pair $(\mathcal{M}_1, \mathcal{M}_2)$ of oriented matroids such that every covector of $\mathcal{M}_2$ is also a covector of $\mathcal{M}_1$, that is, $\mathcal{L}(\mathcal{M}_2) \subseteq \mathcal{L}(\mathcal{M}_1)$ (see [1] and [3]). If $(\mathcal{M}_1, \mathcal{M}_2)$ is a strong map we write $\mathcal{M}_1 \to \mathcal{M}_2$. Clearly, in this case the rank d_2 of $\mathcal{M}_2$ cannot exceed the rank d_1 of $\mathcal{M}_1$. Geometrically a strong map corresponds to an embedding of a d_2-dimensional pseudosubspace $\mathcal{M}_2$ through the origin in the d_1-dimensional pseudoarrangement of $\mathcal{M}_1$. If, for instance, we consider a concrete hyperplane arrangement X_1 with oriented matroid $\mathcal{M}_1$ and a new linear d_2-dimensional subspace X_2, intersecting X_1 arbitrarily, every covector on X_2 is also a covector of X_1. Therefore $\mathcal{M}_1 \to \mathcal{M}_2$, where $\mathcal{M}_2$ is the induced oriented matroid on X_2, forms a strong map. Las Vergnas conjectured in 1975 (see [1]) that for every strong map $\mathcal{M}_1 \to \mathcal{M}_2$ the oriented matroid $\mathcal{M}_2$ can be obtained from $\mathcal{M}_1$ by taking a suitable extension followed by a contraction. This is especially true for the realizable case and the cases $d_2 = d_1 - 1$, $d_2 = 1$ [15]. We prove:

Corollary 3.5. *There is a strong map $\mathcal{M}_1 \to \mathcal{M}_2$ from a rank 4 oriented matroid $\mathcal{M}_1$ on 12 elements to a rank 2 oriented matroid $\mathcal{M}_2$ such that no one-element extension of $\mathcal{M}_1$ meets all covectors of $\mathcal{M}_2$.*

This especially implies that the strong map $\mathcal{M}_1 \to \mathcal{M}_2$ does not factor in an extension followed by a contraction. We even prove the stronger result that $\mathcal{M}_1$ contains *two* points that cannot simultaneously lie in an extending plane. This improves a result of Goodman and Pollack. They proved that there is an oriented matroid on eight points such that the corresponding pseudoarrangement contains *three* points that cannot lie simultaneously in an extension plane [10].

1. Composition of Oriented Matroids

The constructions of oriented matroids presented in this paper are based on a new rank-preserving "composition" operation for oriented matroids. In this section we discuss the general aspects of rank-preserving compositions of oriented matroids. In Section 2 we specialize these techniques to a concrete construction in rank 4.

For two given rank d oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on disjoint ground sets E_1 and E_2 we define a composition operation—the *connected sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$—yielding an oriented matroid $\tilde{\mathcal{M}}$ containing both of them. Geometrically, such a composition can easily be obtained by interlocking the corresponding pseudoarrangements in a suitable way. Clearly, there are many ways to do this. Moreover, in general it is also difficult to make sure that all pseudoplanes intersect properly. We overcome these difficulties by specifying the particular geometric situation by assigning an *oriented flag* to each of the summands, which specify the position where the other summand has to be glued in. For a rank d oriented matroid $\mathcal{M}$ on E we denote by $\chi_{\mathcal{M}}\colon \Lambda(E, d) \to \{-1, 0, 1\}$ its corresponding chirotope. Since both $\chi_{\mathcal{M}}$ and $-\chi_{\mathcal{M}}$ are chirotopes of $\mathcal{M}$ we assume, furthermore, that the lexicographically fist basis λ of $\mathscr{M}$ satisfies $\chi_{\mathscr{M}}(\lambda) = +1$. Conversely, for a chirotope χ the corresponding oriented matroid is denoted by $\mathscr{M}_\chi$. In what follows we always assume that $E := \{1, \ldots, n\}$ together with certain extended elements $a_1, \ldots, a_d$ are linearly ordered by $1 < \cdots < n < a_1 < \cdots < a_d$.

Definition 1.1. Let $\mathscr{M}$ be an oriented matroid of rank d on the set $E \uplus \{a_1, \ldots, a_d\}$. The sequence $\mathbf{a} := (a_1, \ldots, a_d)$ is called an *oriented flag* of $\mathscr{M}$ if $\mathbf{a}$ is a basis of $\mathscr{M}$ and for all $k \leq d$ and $\lambda \in \Lambda(E, k)$ we have $\chi_{\mathscr{M}}(\lambda, \tau_1) = \chi_{\mathscr{M}}(\lambda, \tau_2)$ for all

$$\tau_1, \tau_2 \in \Lambda(\{a_1, \ldots, a_d\}, d - k).$$

The pair $(\mathscr{M}, \mathbf{a})$ is called a *flagged* oriented matroid.

If we consider the (oriented) pseudoarrangement corresponding to $\mathscr{M}$, the hyperplanes in $\mathbf{a}$ are in general position. Moreover, all subspaces of dimension k obtained by intersecting elements in $\mathbf{a}$ have the same relative position in the pseudoarrangement corresponding to $\mathscr{M} \backslash \mathbf{a}$. We can view the flag $\mathbf{a}$ in $\mathscr{M}$ as a tower of (oriented) pseudosubspaces

$$a_1 \supset (a_1 \cup a_2) \supset (a_1 \cup a_2 \cup a_3) \supset \cdots \supset (a_1 \cup a_2 \cup \cdots \cup a_d) = \varnothing$$

embedded in the pseudoarrangement corresponding to $\mathscr{M}$.

Equivalently, we can characterize an oriented flag by its corresponding sequence of oriented subspaces in the pseudoarrangement.

Definition 1.2. Let $(\mathscr{M}, \mathbf{a})$ be a flagged oriented matroid of rank d. The *extension sequence* of $(\mathscr{M}, \mathbf{a})$ is the sequence $\hat{\mathscr{M}}^0, \ldots, \hat{\mathscr{M}}^{d-1}$ of oriented matroids given by

$$\hat{\mathscr{M}}^i := (\mathscr{M}/\{a_1, \ldots, a_{i-1}\}) \backslash \{a_{i+1}, \ldots, a_d\}.$$

Accordingly $\hat{\mathscr{M}}^i$ has rank $d - i + 1$ for $i = 1, \ldots, d - 1$. Furthermore, the oriented matroid $\hat{\mathscr{M}}^{i+1}$ is a single-element extension of $\mathscr{M}^i/a_i$ for $i \in \{0, \ldots, d - 1\}$. Notice that a flagged oriented matroid $(\mathscr{M}, \mathbf{a})$ is uniquely determined by its extension sequence.

We now define the *connected sum* $\tilde{\mathscr{M}} := \mathscr{M}_1 \oplus_{\mathbf{a}}^{\mathbf{b}} \mathscr{M}_2$ of two uniform flagged oriented matroids $(\mathscr{M}_1, \mathbf{a})$ and $(\mathscr{M}_2, \mathbf{b})$. Intuitively, $\tilde{\mathscr{M}}$ consists of the oriented matroid $\mathscr{M}_1$ where the elements $\mathbf{a}$ are replaced by the configuration $\mathscr{M}_2 \backslash \mathbf{b}$. Since the definition is completely symmetric in both summands, we can also consider $\tilde{\mathscr{M}}$ as generated by taking $\mathscr{M}_2$ and replacing $\mathbf{b}$ by $\mathscr{M}_1 \backslash \mathbf{a}$.

Definition and Theorem 1.3. *Let $(\mathscr{M}_1, \mathbf{a})$, $(\mathscr{M}_2, \mathbf{b})$ be two flagged rank d oriented matroids on disjoint ground sets $E \uplus \{a_1, \ldots, a_d\}$ and $F \uplus \{b_1, \ldots, b_d\}$, respectively. The alternating map $\chi_{\tilde{\mathscr{M}}}: \lambda(E \uplus F, d) \to \{-1, +1\}$ given by*

$$\chi_{\tilde{\mathscr{M}}}(\lambda, \tau) := \chi_{\mathscr{M}_1}(\lambda, a_1, \ldots, a_{d_2}) \cdot \chi_{\mathscr{M}_2}(b_{d_1}, \ldots, b_1, \tau)$$

for $\lambda \in \Lambda(E, d_1)$ *and* $\tau \in \Lambda(F, d_2)$ *with* $d_1 + d_2 = d$, *again defines an oriented matroid* $\tilde{\mathcal{M}} := \mathcal{M}_1 \oplus_{\mathbf{a}}^{\mathbf{b}} \mathcal{M}_2$, *the* connected sum *of* $(\mathcal{M}_1, \mathbf{a})$ *and* $(\mathcal{M}_2, \mathbf{b})$.

Proof. We only have to prove that all Grassmann–Plücker relations for the map $\chi_{\tilde{\mathcal{M}}}$ as defined above are fulfilled. If so $\chi_{\tilde{\mathcal{M}}}$ defines an oriented matroid. For convenience we set $\chi_1 := \chi_{\mathcal{M}_1}$, $\chi_2 := \chi_{\mathcal{M}_2}$, and $\tilde{\chi} := \chi_{\tilde{\mathcal{M}}}$. Following [7] we consider Grassmann–Plücker relations as polynomial relations over the fuzzy ring of GF_3. Let $\lambda \in \Lambda(E, d_1)$ and $\tau \in \Lambda(F, d_2)$ with $d_1 + d_2 = d - 2$. Furthermore, let $e_1, \ldots, e_4 \in E$ and $f_1, \ldots, f_4 \in F$. We abbreviate the three summand Grassmann–Plücker polynomials by (see [7])

$$\{\lambda, \tau | a, b, c, d\}_\chi := \chi(\lambda, \tau, a, b) \cdot \chi(\lambda, \tau, c, d) - \chi(\lambda, \tau, a, c) \cdot \chi(\lambda, \tau, b, d) + \chi(\lambda, \tau, a, d) \cdot \chi(\lambda, \tau, b, c).$$

If we obtain, over the fuzzy ring of GF_3,

$$\{\lambda, \tau | a, b, c, d\}_\chi \in \{0, *\}$$

for all $(\lambda, \tau) \in \Lambda(E \uplus F, d)$ and $a, b, c, d \in E \uplus F$, then $\tilde{\chi}$ defines an oriented matroid. Applying Definition 1.3 we obtain

$$\{\lambda, \tau | e_1, e_2, e_3, e_4\}_{\tilde{\chi}} = \sigma \cdot \{\lambda, a_1, \ldots, a_{d_2} | e_1, e_2, e_3, e_4\}_{\chi_1} \cdot (\chi_2(b_{d_1}, \ldots, b_1, \tau))^2,$$

where $\sigma \in \{-1, +1\}$ has to be chosen in a suitable way. Therefore the Grassmann–Plücker relation is fulfilled since the corresponding Grassmann–Plücker relation was satisfied for χ_1. The same argument proves the correctness for the Grassmann–Plücker relation $\{\lambda, \tau | f_1, f_2, f_3, f_4\}_{\tilde{\chi}}$.

Likewise, if we consider the Grassmann–Plücker relation $\{\lambda, \tau | e_1, e_2, e_3, f_1\}_{\tilde{\chi}}$ (or symmetrically $\{\lambda, \tau | e_1, f_1, f_2, f_3\}_{\tilde{\chi}}$) we observe

$$\{\lambda, \tau | e_1, e_2, e_3, f_1\}_{\tilde{\chi}} = \sigma \cdot \{\lambda, a_1, \ldots, a_{d_2} | e_1, e_2, e_3, e_4\}_{\chi_1} \cdot \chi_2(b_{d_1}, \ldots, b_1, \tau) \cdot \chi_2(b_{d_1 - 1}, \ldots, b_1, f_1, \tau),$$

which proves the correctness of those Grassmann–Plücker relations. It remains to prove the relation on the form $\{\lambda, \tau | e_1, e_2, f_1, f_2\}_{\tilde{\chi}}$. In this case after expanding we observe that the Grassmann–Plücker relation has the form

$$\{\lambda, \tau | e_1, e_2, f_1, f_2\}_{\tilde{\chi}} = A + B - B,$$

where A and B are suitable products. This implies that this Grassmann–Plücker relation is also satisfied. □

2. A Geometric Construction in Rank 4

In this section we describe a special composition construction in rank 4. We especially point out the geometric content of the construction and take care of the mutation structure of the resulting oriented matroid.

For a given oriented matroid $\mathcal{M}$ on n elements having k mutations we describe how to obtain an extension $\mathcal{M}'$ by four new elements that has exactly $k + 3$ mutations. Inductive application of this construction leads to an infinite family of oriented matroids on $4n$ elements having only $3n + 1$ mutations.

We therefore first analyze the geometric structure of the unique uniform rank 4 oriented matroid $X(8)$ on eight elements with only seven mutations. This oriented matroid is a basic building block of our constructions. $X(8)$ was first presented in [16]. It was proved in [6] that $X(8)$ is the smallest example of a uniform oriented matroid possessing less mutations than elements. The infinite class of oriented matroids we want to describe can be obtained by iteratively taking the connected sum of several copies of $X(8)$.

The easiest way to describe $X(8)$ geometrically is by describing a geometric situation that can be perturbed to $X(8)$. Let $\mathcal{M}$ be the oriented matroid on the ground set $E := \{1, \ldots, 8\}$ consisting of eight planes with normal vectors given by the following homogeneous coordinates:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

If we add a plane at infinity $g = (1, 1, 1, 1)^T$ and consider the affine oriented matroid $(\mathcal{M} \cup g, g)$, these hyperplanes correspond to two tetrahedra $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$ with same center and parallel faces. For each pair

$$(i, j) \in \Lambda(\{1, 3, 5, 7\}, 2)$$

the planes $i, i + 1, j, j + 1$ meet in a point at infinity. These are the only degeneracies in $\mathcal{M}$. We now define $X(8)$ by its basis orientations:

$$\chi_{X(8)}(\lambda) := \begin{cases} \chi_{\mathcal{M}}(\lambda) & \text{if } \chi_{\mathcal{M}}(\lambda) \neq 0, \\ +1 & \text{if } \chi_{\mathcal{M}}(\lambda) = 0, \end{cases}$$

for $\lambda \in \Lambda(E, 4)$. Observe that in $X(8)$ the pairs $(i, i + 1)$ are inseparable for $i = 1, 3, 5, 7$. The seven mutations of $X(8)$ are given by

$$Mut(X(8)) := \{(1, 2, 3, 4), (1, 2, 5, 6), (1, 2, 7, 8), (3, 4, 5, 6), (3, 4, 7, 8), (5, 6, 7, 8), (1, 3, 5, 7)\}.$$

Consider the affine arrangement $(X(8), 8)$. Figure 2.1(a) illustrates the geometric situation of the hyperplanes $1, \ldots, 6$. The lines of intersection of two hyperplanes are labeled by the corresponding two planes. A description of the complete arrangement is obtained if the intersections of the lines with planes 7 and 8 are specified. Each line should intersect plane 8 at the end marked by a point and intersect plane 7 at the opposite end.

Observe that the region given by the positive side of 7 and the negative side of 8 contains exactly four mutations. Figure 2.1(b) describes the situation in the contraction at element 8. We later replace a substructure of a hyperplane arrangement by the arrangement given in Fig. 2.1(a). To get the labeling we need in this application we define an oriented matroid $\tilde{X}(8)$ on the ground set $\{1, 1_1, 1_2, 2_1, 2_2, 2, 3, 4\}$ isomorphic to $X(8)$ by the following relabeling and reorientation:

$$1 \to 1, \quad 6 \to -1_1, \quad 5 \to 1_2, \quad 4 \to -2_1, \quad 3 \to -2_2, \quad 2 \to -2, \quad 7 \to 3, \quad 8 \to 4.$$

The corresponding situation in the contraction on element 3 of $\tilde{X}(8)$ can be found in Fig. 2.2(c). $\tilde{X}(8)$ possesses the mutations:

$$(1, 2, 1_1, 1_2), (1, 2, 2_1, 2_2), (1_1, 1_2, 2_1, 2_1), (1, 2, 3, 4),$$
$$(1_1, 1_2, 3, 4), (2_1, 2_2, 3, 4), (1, 1_2, 2_1, 3).$$

We now describe a special extension construction for oriented matroids of rank 4 that decreases the ratio of mutations and elements. We frequently identify the cells in pseudoarrangements with the corresponding covectors. In the affine oriented matroid $(\mathcal{M}, g)$ we also identify the elements in $E \backslash g$ with the corresponding pseudoplanes in $(\mathcal{M}, g)$. For $i, j \in E$ we abbreviate the affine line in the intersection of i and j by $l_{i,j} := i \cap j$.

Let $\mathcal{M}$ be a uniform oriented rank 4 matroid on $E := \{1, \ldots, n\}$ with $n \geq 5$ elements. If $\mathcal{M}$ contains a mutation $(1, 2, 3, 4) \in Mut(\mathcal{M})$ where the pair $(1, 2)$ is an inseparable pair [3] of hyperplanes, we construct an extension $\mathcal{M}_{(1,2,3,4)}$ by four new hyperplanes such that $\mathcal{M}_{(1,2,3,4)}$ possesses exactly $|Mut(\mathcal{M})| + 3$ mutations. Moreover, $\mathcal{M}_{(1,2,3,4)}$ will satisfy the relation $|Mut_4(\mathcal{M}_{(1,2,3,4)})| = |Mut_4(\mathcal{M})| - 1$.

To simplify our considerations we assume that a hyperplane $g \in E \backslash \{1, \ldots, 4\}$ plays the role of a hyperplane at infinity. Without loss of generality we assume that the affine pseudoarrangement $(\mathcal{M}, g)$ is given such that the hyperplanes 1, 2, 3, and 4 are realized as flat hyperplanes. By $\mathcal{A} := \{C \in (\mathcal{M}, g) | C_i \neq 0 \text{ for all } i \in E\}$ we denote the set of all full-dimensional regions of $(\mathcal{M}, g)$. The mutations of $\mathcal{M}$ are in one-to-one correspondence to the tetrahedra in $\mathcal{A}$. Finally, we assume without loss of generality that after a suitable reorientation the tetrahedral region corresponding to the mutation $(1, 2, 3, 4)$ lies on the positive side of every pseudoplane in $E \backslash 2$ and on the negative side of plane 2. By this choice $(1, 2)$ becomes a covariant pair in the terminology of [3]. By $s_{3,4}$ we denote the directed line-segment on $l_{3,4}$ lying in the boundary of the tetrahedron $(1, 2, 3, 4)$ pointing from on the hyperplane 1 to the hyperplane 2 (Fig. 2.2(a)).

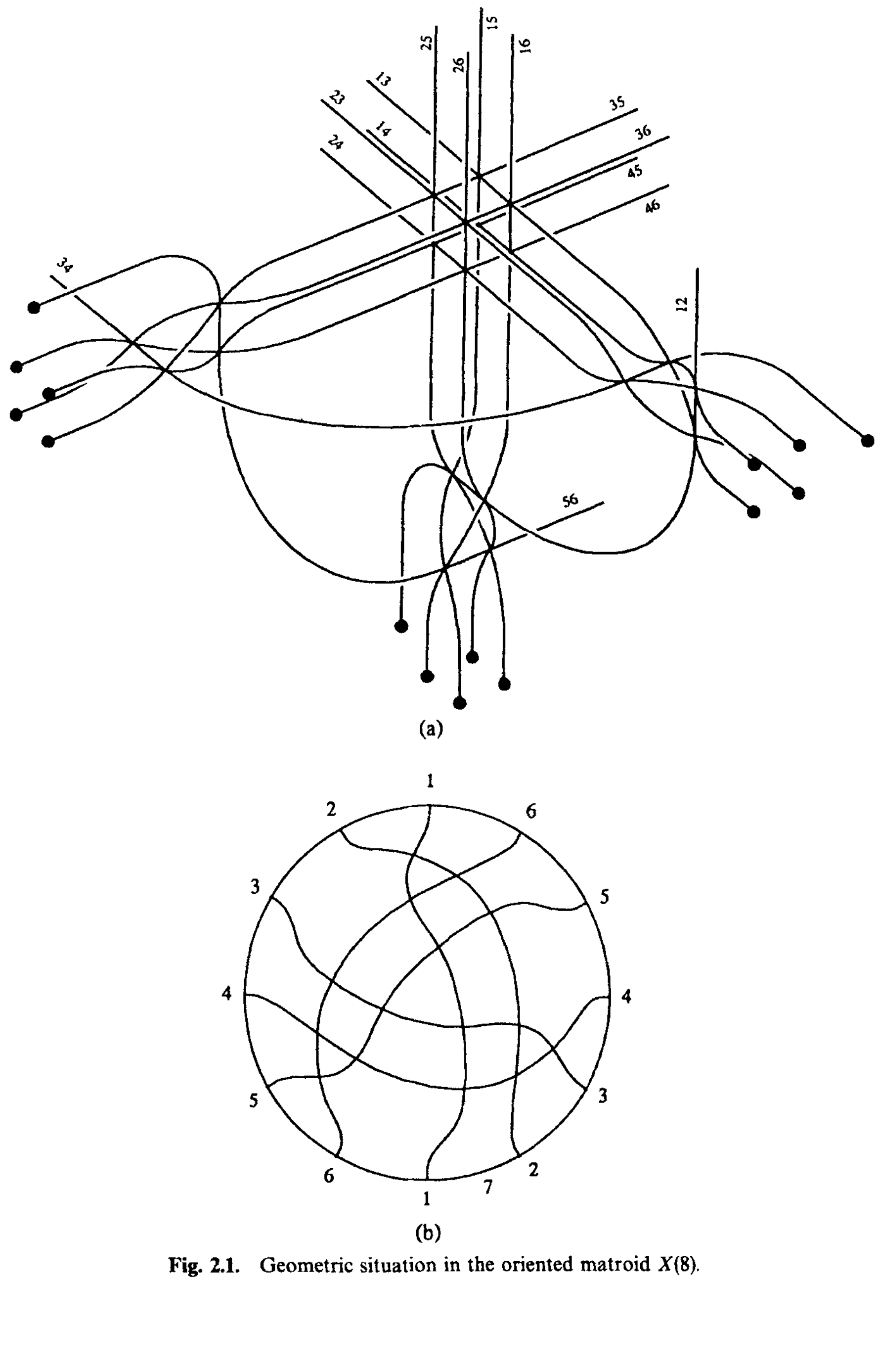

Fig. 2.1. Geometric situation in the oriented matroid $X(8)$.

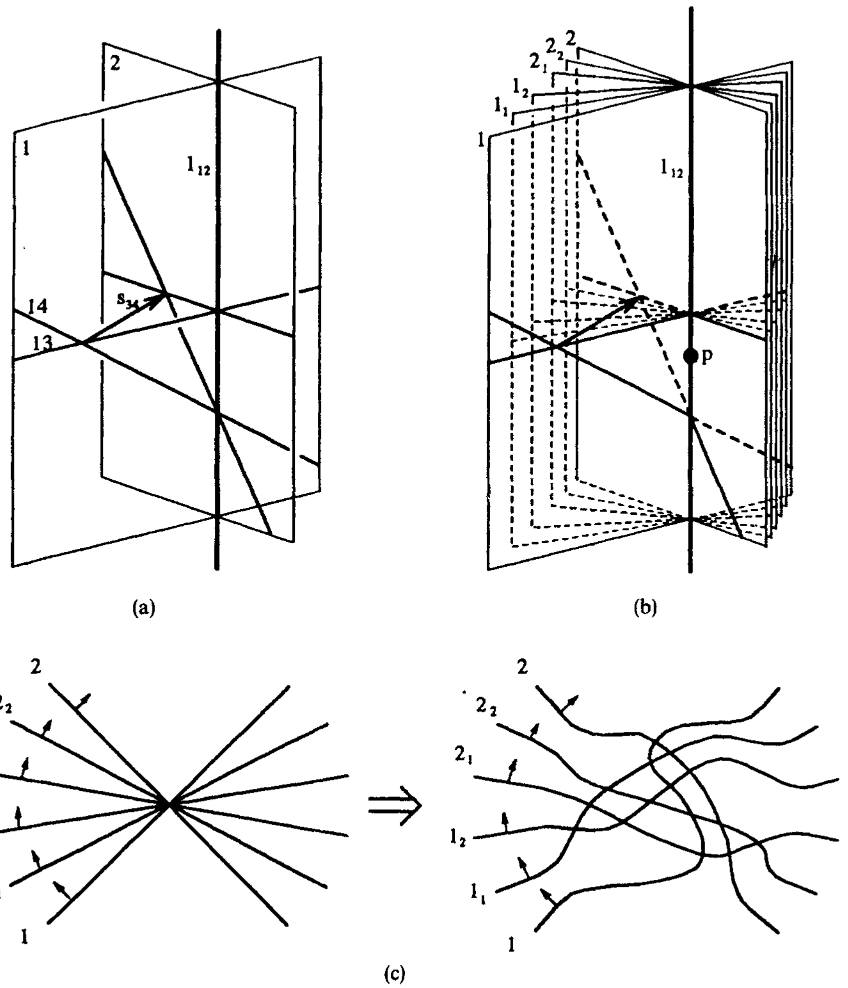

Fig. 2.2. Construction of $\mathcal{M}'$ from $\mathcal{M}$.

We now replace the pair of planes (1, 2) by six hyperplanes

$$F := \{1, 1_1, 1_2, 2_1, 2_2, 2\}$$

all incident with the line $l_{1,2}$, intersecting $s_{3,4}$ in the order 1, 1_1, 1_2, 2_1, 2_2, 2 (Fig. 2.2(b)). We assume that the orientations are chosen in such a way that after removing any four hyperplanes in $\{1, 1_1, 1_2, 2_1, 2_2, 2\}$ we obtain our original arrangement $(\mathcal{M}, g)$. This is obtained if all orientations point to the direction indicated by the directed edge $s_{3,4}$. The resulting oriented matroid is called $\mathcal{M}'$. Notice that $\mathcal{M}'$ is no longer uniform. We now describe how to perturb the hyperplanes in F in order to obtain $\mathcal{M}_{(1,2,3,4)}$ with the desired properties. Consider a point p on the boundary of the tetrahedron $(1, 2, 3, 4)$ lying on line $l_{1,2}$. We rotate each of the planes in F slightly around point p in a way that the situation in the intersection of the planes in F with each of the planes $j \in E\backslash\{1, 2\}$ is deformed as described in Fig. 2.2(c). (This rotation can be done keeping all planes of F flat. For our purposes a topological deformation in the described way serves as well.) If we make the deformation sufficiently small we can still keep the property that after removing four hyperplanes from $\{1, 1_1, 1_2, 2_1, 2_2, 2\}$ we obtain our original arrangement $(\mathcal{M}, g)$. This way we obtain an oriented matroid $\mathcal{M}''$ where all hyperplanes of F pass through one point p in the boundary of the tetrahedron $(1, 2, 3, 4)$. If we now consider the restriction of $\mathcal{M}''$ to the elements

$$\tilde{E} := \{1, 1_1, 1_2, 2_1, 2_2, 2, 3, 4\}$$

we observe that we have exactly the situation of $\tilde{\mathcal{X}}(8)$ with all vertices not lying on 3 or 4 collapsed to one single point p. Therefore we can perturb the oriented matroid $\mathcal{M}''$ in a small neighborhood of p such that its restriction to $\tilde{E}$ equals our oriented matroid $\tilde{\mathcal{X}}(8)$. The resulting configuration is the oriented matroid $\mathcal{M}_{(1,2,3,4)}$ with the desired properties. Notice that by our construction the pairs $(1_1, 1_2)$ and $(2_1, 2_2)$ are inseparable in $\mathcal{M}_{(1,2,3,4)}$. Furthermore, any inseparable pair of $\mathcal{M}$ with elements in $E - \{1, 2, 3, 4\}$ will be also inseparable in $\mathcal{M}_{(1,2,3,4)}$.

Before analyzing the structure of the mutations in $\mathcal{M}_{(1,2,3,4)}$ we give a more formal definition of $\mathcal{M}_{(1,2,3,4)}$ by its basis orientations. We assume that $F \uplus E\backslash\{1, 2\}$ is equipped with the linear order

$$\underbrace{1 < 1_1 < 1_2 < 2_1 < 2_2 < 2}_{F} < \underbrace{3 < 4 < \cdots < n}_{E\backslash\{1,2\}}.$$

We also assume that the planes are oriented as described above. If we set $\chi_1 := \chi_{\mathcal{M}}$ and $\chi_2 := \chi_{\tilde{\mathcal{X}}(8)}$ we obtain the bases orientation of $\chi := \chi_{\mathcal{M}_{(1,2,3,4)}}$ as follows. Let $\tau \in \Lambda(F, d_1)$ and $\lambda \in \Lambda(E\backslash\{1, 2\}, d_2)$ with $d_1 + d_2 = d$. We have

$$\chi(\tau, \lambda) := \begin{cases} \chi_1(\lambda) & \text{if } |\tau| = 0, \\ \chi_1(1, \lambda) & \text{if } |\tau| = 1, \\ \chi_1(1, 2, \lambda) & \text{if } |\tau| = 2, \\ \chi_2(\tau, 3) & \text{if } |\tau| = 3, \\ \chi_2(\tau) & \text{if } |\tau| = 4. \end{cases}$$

The different cardinalities of τ correspond to different steps in the construction described above. The whole construction corresponds to a suitable connected sum as described in the last section. To simplify the construction and point out the geometric situation we just encoded the flags directly in the chosen labeling and orientation of the planes.

We now analyze the structure of the mutations of $\mathcal{M}_{(1,2,3,4)}$, using the fact that the mutations are in one-to one correspondence to the simplicial regions of arrangements. For an oriented matroid $\mathcal{M}$ on $A \cup B$ and two integers $i, j \geq 0$ with $i + j = 4$ we denote by

$$Mut^{(i,j)}_{(A,B)}(\mathcal{M}) := \{\lambda \in Mut(\mathcal{M}) \,|\, |\lambda \cap A| = i \text{ and } |\lambda \cap B| = j\}$$

the set of all mutations containing i elements of A and j elements of B. Furthermore, for a set of d-tuples M we denote by $M[i \leftrightarrow j]$ the set of d-tuples where all occurrences of index i are replaced by index j.

Theorem 2.1. *Let $\mathcal{M}$ be a uniform rank 4 oriented matroid on E where*

$$(1, 2, 3, 4) \in Mut(\mathcal{M})$$

and $(1, 2)$ *is an inseparable pair. Then the oriented matroid* $\mathcal{M}_{(1,2,3,4)}$ *on* $E \uplus \{1_1, 1_2, 2_1, 2_2\}$ *has the following properties*:

(i) $|Mut(\mathcal{M}_{(1,2,3,4)})| = |Mut(\mathcal{M})| + 3$.
(ii) $Mut_4(\mathcal{M}_{(1,2,3,4)}) = Mut_4(\mathcal{M}) - \{(1, 2, 3, 4)\}$.

Proof. We prove this result by carefully analyzing the geometric structure of $\mathcal{M}_{(1,2,3,4)}$ and keeping track of the simplicial cells in the pseudoarrangement. We first observe that according to our construction we have

$$Mut^{(4,0)}_{(E-\{1,2\},F)}(\mathcal{M}_{(1,2,3,4)}) = Mut^{(4,0)}_{(E-\{1,2\},\{1,2\})}(\mathcal{M}).$$

This states simply that the regions not incident with 1 or 2 are not influenced by our construction. Conversely, the mutations having no elements in $E \backslash \{1, 2\}$ are those that are induced by the inner structure of $\tilde{X}(8)$:

$$Mut^{(0,4)}_{(E-\{1,2\},F)}(\mathcal{M}_{(1,2,3,4)}) = Mut^{(0,4)}_{(\{3,4\},F)}(\tilde{X}(8)).$$

These are exactly the three mutations $(1_1, 1_2, 2_1, 2_2)$, $(1, 2, 1_1, 1_2)$, $(1, 2, 2_1, 2_2)$ of the oriented matroid $\tilde{X}(8)$.

We now have to study the mutations of $\mathcal{M}_{(1,2,3,4)}$ containing elements in $E \backslash \{1, 2\}$ as well as in F. Therefore recall that we have replaced our original line $l_{1,2}$ in the pseudoarrangement of $\mathcal{M}$ everywhere except at the tetrahedron $(1, 2, 3, 4)$ by a cylinder over the contraction given in Fig. 2.2(c). Therefore all mutations in this region of $\mathcal{M}_{(1,2,3,4)}$ have to be incident to the regions in Fig. 2.2(c) which correspond to unbounded wedges. Mutations occur exactly at the places where these regions meet former mutations of $\mathcal{M}$ incident to $l_{1,2}$.

Since $(1, 2)$ is a covariant pair all mutations of $\mathcal{M}$ containing the pair $(1, 2)$ lie in $\mathcal{A}^- := \{C \in \mathcal{A} \,|\, C_1 \cdot C_2 = -\}$. These are exactly the mutations adjacent to the line $l_{1,2}$. We get

$$\begin{aligned} &Mut^{(2,2)}_{(E-\{1,2\},F)}(\mathcal{M}_{(1,2,3,4)})[1_1 \leftrightarrow 1][1_2 \leftrightarrow 2][2_1 \leftrightarrow 1][2_2 \leftrightarrow 2] \\ &\quad = Mut^{(2,2)}_{(E-\{1,2\},\{1,2\})}(\mathcal{M}). \end{aligned}$$

Mutations of $\mathscr{M}_{(1,2,3,4)}$ having exactly one element in F correspond to mutations of $\mathscr{M}$ having exactly one element in $\{1, 2\}$. We obtain

$$Mut^{(3,1)}_{(E-\{1,2\},F)}(\mathscr{M}_{(1,2,3,4)}) = Mut^{(3,1)}_{(E-\{1,2\},\{1,2\})}(\mathscr{M}).$$

Similary we obtain exactly the mutations of $\mathscr{M}_{(1,2,3,4)}$ having three elements in F and one in $E - \{1, 2\}$:

$$Mut^{(1,3)}_{(E-\{1,2\},F)}(\mathscr{M}_{(1,2,3,4)}) = Mut^{(1,3)}_{(\{3,4\},F)}(\tilde{X}(8)).$$

This is exactly the mutation $(1, 1_2, 2_1, 3)$ of the oriented matroid $\tilde{X}(8)$.

Collecting these observations and defining $M^{(i,j)} := |Mut^{(i,j)}_{(E-\{1,2\},F)}(\mathscr{M}_{(1,2,3,4)})|$ we obtain

$$|Mut(\mathscr{M}_{(1,2,3,4)})| := \underbrace{M^{(4,0)} + M^{(3,1)} + M^{(2,2)}}_{=|Mut(\mathscr{M})|-1} + \underbrace{M^{(1,3)}}_{=1} + \underbrace{M^{(0,4)}}_{=3}.$$

This proves statement (i). A similar count proves that the mutations containing element 4 in $\mathscr{M}_{(1,2,3,4)}$ are exactly those of $\mathscr{M}$ after $(1, 2, 3, 4)$ has been deleted. □

We now immediately obtain:

Theorem 2.2. *For every $n = 4 \cdot k \geq 8$ there is a uniform rank 4 oriented matroid $\mathscr{M}^n$ having exactly $3n/4 + 1$ mutations.*

Proof. To prove the result inductively we prove the stronger fact that $\mathscr{M}^n$ has $3n/4 + 1$ mutations and moreover contains an inseparable pair of elements $(1^n, 2^n)$. For $n = 8$ this statement is fulfilled by $X(8)$ and the pair $(1, 2)$. Now assume that there exists an oriented matroid $\mathscr{M}^n$ with the desired properties. Since $(1^n, 2^n)$ is inseparable in $\mathscr{M}$ there is also a mutation $(1^n, 2^n, 3^n, 4^n)$ containing 1^n and 2^n (see Lemma 2.6 of [14]). Therefore $\mathscr{M}^{n+4} := \mathscr{M}^n_{(1^n,2^n,3^n,4^n)}$ has, by Theorem 3.1(i), exactly $3(n + 4)/4 + 1$ mutations. Moreover, $\mathscr{M}^{n+4}$ possesses an inseparable pair $(1^n_1, 1^n_2)$. This proves the claim. □

Finally we give a construction of an oriented matroid $R(20)$ on 20 elements where one element is not contained in any mutation. We start with the oriented matroid $X(8)$ and use our construction to destroy successively all mutations containing element 8. We define

$$R(20) := ((X(8)_{(1,2,7,8)})_{(3,4,7,8)})_{(5,6,7,8)}.$$

Theorem 2.3. $Mut_8(R(20)) = \varnothing$.

Proof. We first have to prove that the above construction of $R(20)$ is well defined. For the oriented matroid $X(8)$ we have

$$Mut_8(X(8)) := \{(1, 2, 7, 8), (3, 4, 7, 8), (5, 6, 7, 8)\}.$$

Furthermore, the pairs $(i, i+1)$ are inseparable for $i = 1, 3, 5$. By Theorem 4.1(ii) we obtain $Mut_8(X(8)_{(1,2,7,8)}) = \{(3, 4, 7, 8), (5, 6, 7, 8)\}$. The pairs $(i, i+1)$ are still inseparable for $i = 3, 5$. Therefore we can again apply Theorem 3.1(ii) and obtain a set of mutations $Mut_8((X(8)_{(1,2,7,8)})_{(3,4,7,8)}) = \{(5, 6, 7, 8)\}$, where $(5, 6)$ is still inseparable. A final application of Theorem 3.1(ii) proves the claim. □

The oriented matroid $R(20)$ is used in [14] to prove that there are oriented matroids where the space of all one-point extensions is disconnected. Namely, it can be proved that $R(20)$ is an isolated element in the extension space of $R(20)\backslash 8$. A geometric picture of $R(20)$ in terms of its contractions can also be found in [14].

3. A Nonfactorizable Strong Map

This section is devoted to a rank 4 arrangement of pseudoplanes $R(12) := X(8)_{(7,8,1,2)}$ containing two topes T_1, T_2 such that no extension plane of $R(12)$ intersects T_1 and T_2 simultaneously. This is a consequence of the fact that $X(8)$ is a noneuclidean oriented matroid. For an introduction to the theory of *euclideanness* in oriented matroids and the related theory of *oriented matroid programs* the reader is referred to [3], [9], and [13]. All necessary results needed here can also be found in [14] and [19].

If not explicitly stated otherwise we assume that every oriented matroid in this section is of rank 4. For a set of covectors $A \subset \{-, 0, +\}^E$ and an element $f \in E$ we define $A_f^+ := \{V \in A \mid V_f = +\}$, $A_f^0 := \{V \in A \mid V_f = 0\}$, etc. We say that f *hits* a set of covectors $A \subset \{-, 0, +\}^E$ if $A_f^+ \neq A$ and $A_f^- \neq A$. If in a pseudoarrangement f hits a set of vectors (=faces) A, then not all faces corresponding to elements in A lie in the same open half-space with respect to pseudoplane f. Clearly, if f hits A, then f also hits $-A := \{-V \mid V \in A\}$. If f hits a one-element set $\{V\}$, then $V_f = 0$.

A *cocircuit* of an oriented matroid $\mathcal{M}$ is a covector corresponding to a vertex in the pseudoarrangement of $\mathcal{M}$ (see [3]). We abbreviate the set of cocircuits of $\mathcal{M}$ by $\mathcal{O}^*(\mathcal{M})$. For a given oriented matroid $\mathcal{M}$ on E and $i, j, k \in E$ we also define $Y_i = \mathcal{L}(\mathcal{M})_i^0$, $Y_{i,j} := (Y_i)_j^0$, and $Y_{i,j,k} := (Y_{i,j})_k^0$. Observe that in uniform arrangements the set $Y_{i,j,k}$ forms a pair $\pm Y$ of cocircuits corresponding to the intersection of i, j, and k.

According to Fukuda and Mandel [9], [13] every noneuclidean oriented matroid program $(\mathcal{M}, g, f)$ contains at least one *cyclic component* $\mathbf{c}$ in the edge-graph $G_{(\mathcal{M},g,f)}$ (see [19]). In the terminology of Sturmfels and Ziegler $\mathbf{c}$ is called a *very strong component* of $(\mathcal{M}, g, f)$ and consists of all cocircuits contained in the cyclic component. The vertices at infinity of $(\mathcal{M}, g, f)$ on the positive side of f is abbreviated by $\mathbf{I}^{f,g} := (\mathcal{O}^*(\mathcal{M})_g^0)_f^+$.

The crucial observation about the noneuclidean oriented matroids can now be formulated as a property about hitting the sets $\mathbf{I}^{f,g}$ and $\mathbf{c}$ by an extension plane.

Lemma 3.1. *Let $(\mathcal{M}, g, f)$ be a noneuclidean oriented matroid program with very strong component* $\mathbf{c}$ *and infinite vertices $\mathbf{I}^{f,g}$. Let $\hat{\mathcal{M}} := (\mathcal{M}\backslash f) \cup \hat{f}$ be a one-element extension of $\mathcal{M}\backslash f$. If $\hat{f}$ hits* $\mathbf{c}$ *in $\hat{\mathcal{M}}$, then $\hat{f}$ also hits $\mathbf{I}^{f,g}$.*

Proof. This lemma is a simple reformulation of Lemma 3.6(i) of [19] that the localization of every extension of an oriented matroid program $(\mathcal{M}, g, f)$ has the same value on all elements of a very strong component. The proof goes back to Mandel [13]. □

The oriented matroid program $(X(8), 8, 7)$ has a unique very strong component $\mathbf{c}$. In this case especially we get

$$\mathbf{I}^{8,7} := \{Y_{8,j,k} \mid j, k \in E\backslash\{7, 8\}\}_7^+$$

and

$$\mathbf{c} := \{Y_{1,2,3}, Y_{1,2,4}, Y_{1,3,4}, Y_{2,3,4}, Y_{1,2,5}, Y_{1,2,6}, Y_{1,5,6}, Y_{2,5,6}, Y_{3,4,5}, Y_{3,4,6}, Y_{3,5,6}, Y_{4,5,6}\}.$$

According to the special structure at infinity (see Fig. 2.1(a)) an extension plane hits $\mathbf{I}^{8,7}$ if it just hits $\hat{\mathbf{I}} := \{Y_{8,1,2}, Y_{8,3,4}, Y_{8,5,6}\}_7^+$ since these are extreme vertices in the contraction $X(8)/8$ on the positive side of element 7. We obtain:

Lemma 3.2. *Let $(X(8)\backslash 7) \cup f$ be an extension of $X(8)\backslash 7$. If f hits* $\mathbf{c}$, *then f hits $\hat{\mathbf{I}}$.*

We now consider the oriented matroid $X(8)_{(7,8,1,2)}$ as defined in the last section. For matters of symmetry, in the following arguments we relable the new elements in $X(8)_{(7,8,1,2)}$ by

$$7_1 \to 12, \quad 7_2 \to 11, \quad 8_1 \to 10, \quad 8_2 \to 9,$$

the resulting oriented matroid on $E := \{1, \ldots, 12\}$ is called $R(12)$. Figure 3.1 gives a description of $R(12)$ in terms of the induced pseudoline arrangements in every contraction. We define $E_1 := \{1, \ldots, 6\}$ and $E_2 := \{7, \ldots, 12\}$. Notice that, by our construction for $(i, j) \in \Lambda(E_2, 2)$, the oriented matroid program

$$(R(12)\backslash(E_2 - \{i, j\}), i, j)$$

is isomorphic to $(X(8), 8, 7)$. In this case we get a very strong component either by

$$\mathbf{c}_1 := \{Y_{1,2,3}, Y_{1,2,4}, Y_{1,3,4}, Y_{2,3,4}, Y_{1,2,5}, Y_{1,2,6}, Y_{1,5,6}, Y_{2,5,6}, Y_{3,4,5}, Y_{3,4,6}, Y_{3,5,6}, Y_{4,5,6}\}$$

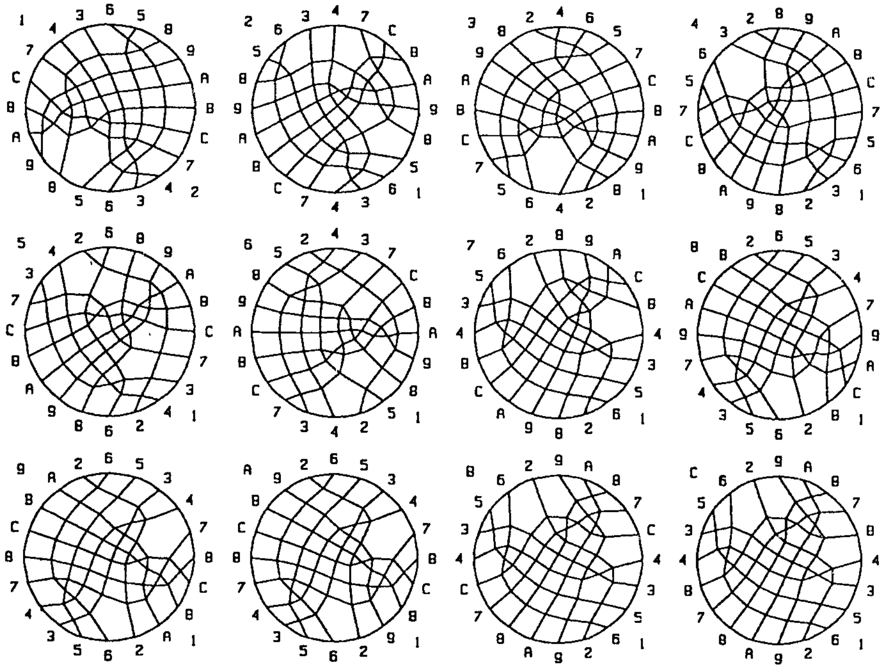

Fig. 3.1. Contractions of $R(12)$.

or by $-\mathbf{c}_1$. Symmetrically, for every pair $(i,j) \in \Lambda(E_1, 2)$ the oriented matroid program $(R(12)\backslash(E_1 - \{i,j\}), i, j)$ is isomorphic to $(X(8), 8, 7)$ with the very strong component either

$$\mathbf{c}_2 := \{Y_{7,8,9}, Y_{7,8,10}, Y_{7,9,10}, Y_{8,9,10}, Y_{7,8,11}, Y_{7,8,12}, Y_{7,11,12}, Y_{8,11,12}, Y_{9,10,11}, Y_{9,10,12}, y_{9,11,12}, Y_{10,11,12}\}$$

or $-\mathbf{c}_2$. We now prove:

Theorem 3.3. *Let $\mathcal{M} := R(12) \cup f$ be a one-element extension of $R(12)$. The element f cannot hit $\mathbf{c}_1$ and $\mathbf{c}_2$ simultaneously.*

Proof. Assume on the contrary that f hits both $\mathbf{c}_1$ and $\mathbf{c}_2$. By Lemma 3.1, for every $(i,j) \in \Lambda(E_1, 2) \cup \Lambda(E_2, 2)$ the element f hits the infinite vertices $\mathbf{I}^{i,j}$ of the oriented matroid program $(R(12), i, j)$. For every $i \in E_2$ we pick one $j \in E_2\backslash\{i\}$ and define

$$\hat{\mathbf{I}}^i := \{Y_{1,2,i}, Y_{3,4,i}, Y_{5,6,i}\}_j^+.$$

Up to a common sign the set $\hat{\mathbf{I}}^i$ is independent of the special choice of j and

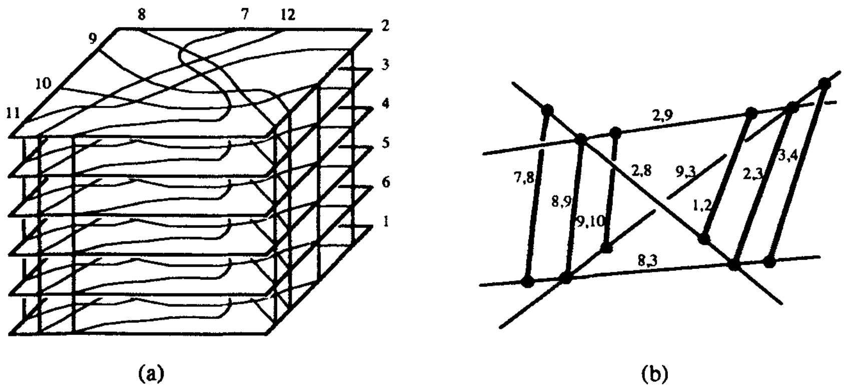

Fig. 3.2. Subconfigurations of $R(12)$.

therefore the expression "f hits $\hat{\mathbf{I}}^i$" is well defined. Likewise for $i \in E_1$ and $j \in E_1 \setminus \{i\}$ we define

$$\hat{\mathbf{I}}^i := \{Y_{7,8,i}, Y_{9,10,i}, Y_{11,12,i}\}_j^+.$$

Applying the isomorphism between the substructures of $(R(12), i, j)$ and $(X(8), 8, 7)$ and Lemma 3.2 we obtain *f hits $\hat{\mathbf{I}}^i$ for all $i \in E$.* The geometric situation in the corresponding pseudoarrangement for $i \in E_1$ is illustrated in Fig. 3.2(a). Since all vertices formed by intersections of elements in E_2 lie in the same region with respect to the planes E_1, the lines $Y_{7,8}$, $Y_{9,10}$, and $Y_{11,12}$ intersect the planes in E_1 all in the same order $(2, 3, 4, 5, 6, 1)$. We refer to the open line segment on $Y_{7,8}$ joining vertex $Y_{1,7,8}$ and vertex $Y_{2,7,8}$ while passing the other planes of E_1 by $S_{7,8}$. We define $S_{9,10}$ and $S_{11,12}$ similarly. Each of the lines $Y_{7,8}$, $Y_{9,10}$, and $Y_{11,12}$ is cut by the plane f exactly once. Since f hits $\hat{\mathbf{I}}^i$ for every "level" $i \in \{2, 3, 4, 5, 6, 1\}$ at most one segment in $\{S_{7,8}, S_{9,10}, S_{11,12}\}$ lies strictly on the positive side of f and at most one segment lies strictly on the negative side of f.

A similar construction makes sense if we consider the situation at the lines spanned by elements of E_1. The lines $Y_{1,2}$, $Y_{3,4}$, and $Y_{5,6}$ are all cut by the planes of E_2 in the order $(8, 9, 10, 11, 12, 7)$. We define open line segments $S_{1,2}$, $S_{3,4}$, and $S_{5,6}$, respectively. At least two of them are separated by f.

It remains to prove that all these separations cannot take place in the same arrangement. Essentially we have to explore all nine combinations of possible separations of the sets $\{S_{1,2}, S_{3,4}, S_{5,6}\}$ and $\{S_{7,8}, S_{9,10}, S_{11,12}\}$. We restrict ourselves to considering only one of these cases. The remaining cases can be solved analogously. Assume that f separates $S_{1,2}$ and $S_{3,4}$ and that f also separates $S_{7,8}$ and $S_{9,10}$. If we consider planes 2, 3, 8, and 9, they form a tetrahedron as given in Fig. 3.2(b) (the lines of this picture are labeled by the corresponding two planes of intersection). The line segments are also indicated in this picture. Observe that

f hits the three-vertex sets:

$$\{Y_{1,2,8}, Y_{2,3,8}, Y_{3,4,8}\}, \{Y_{1,2,9}, Y_{2,3,9}, Y_{3,4,9}\},$$
$$\{Y_{7,8,2}, Y_{8,9,2}, Y_{9,10,2}\}, \{Y_{7,8,3}, Y_{8,9,3}, Y_{9,10,3}\}.$$

Therefore, for instance, f has to cross the edge path $Y_{1,2,8} \to Y_{2,3,8} \to Y_{3,4,8}$. Hence f either hits $\{Y_{1,2,8}, Y_{2,3,8}\}$ or $\{Y_{2,3,8}, Y_{3,4,8}\}$. Similar statements hold for the other three sets. Assume that f hits $\{Y_{1,2,8}, Y_{2,3,8}\}$. Since each of the lines $Y_{2,8}$, $Y_{2,9}$, $Y_{3,8}$, $Y_{3,9}$ is crossed only once by f, the plane f also has to hit $\{Y_{8,9,2}, Y_{9,10,2}\}$, $\{Y_{2,3,9}, Y_{3,4,9}\}$, and $\{Y_{7,8,3}, Y_{7,9,3}\}$. This forces an impossible configuration. A similar contradiction occurs if we assume that f does not hit $\{Y_{1,2,8}, Y_{2,3,8}\}$. In this case f has to hit $\{Y_{2,3,8}, Y_{3,4,8}\}$, $\{Y_{8,9,3}, Y_{9,10,3}\}$, $\{Y_{1,2,9}, Y_{2,3,9}\}$, and $\{Y_{7,8,2}, Y_{8,9,2}\}$, which is also impossible.

This proves the impossibility of separating $S_{1,2}$, $S_{3,4}$ and $S_{7,8}$, $S_{9,10}$ simultaneously. By choosing suitable planes of reference the other eight cases can be concluded similarly. This proves the claim. □

We finally translate Theorem 3.3 into two corollaries about extensions and strong maps of $R(12)$. The first result improves a theorem of Goodman and Pollack [10] about prescribed points in extensions. They proved that there is an oriented matroid $GP(8)$ containing three points that cannot be simultaneously intersected by an extending pseudoplane. We prove:

Corollary 3.4. *In the pseudoarrangement corresponding to* $R(12)$ *let* T_1 *and* T_2 *be the simplicial regions corresponding to the mutations* $(1, 2, 3, 4)$ *and* $(7, 8, 9, 10)$, *respectively. No extending pseudoplane of* $R(12)$ *intersects both* T_1 *and* T_2.

Proof. Observe that for $i = 1, 2$ the vertices of T_i lie all in the very strong component $\mathbf{C}_i$. Hence, every pseudoplane intersecting T_i hits $\mathbf{c}_i$. Applying Theorem 3.3 immediately proves the claim. □

Finally, we disprove the *strong-map conjecture* of Las Vergnas, stating that every strong map $\mathcal{M}_1 \to \mathcal{M}_2$ factors in an extension followed by a contraction. We prove:

Corollary 3.5. *There is a strong map* $R(12) \to \mathcal{M}$ *such that no extension of* $R(12)$ *intersects all covectors of* $\mathcal{M}$.

Proof. We have simply to embed a line l in general position in $R(12)$ intersecting the topes T_1 and T_2 of Corollary 3.4. Such a line always exists by Proposition 4.2.3 of [3]. The rank 2 oriented matroid $\mathcal{M}$ induced on l by Corollary 3.4 cannot be contained in an extension plane of $R(12)$. This proves the claim. □

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