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RESEARCH ARTICLE

CORRECTION AND SUPPLEMENT TO "ON A PULL-BACK DIAGRAM FOR ORTHODOX SEMIGROUPS"

Mária B. Szendrei

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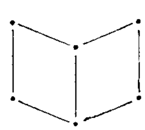
In my paper "On a pull-back diagram for orthodox semigroups" which appeared in Vol. 20(1980), 1-10, Corollary 2 is false and condition(C) in Theorem 3 is necessary but far from sufficient. Corollary 4 is also false. The present note contains counterexamples for the false statements and a correct form of Theorem 3 and its Corollary 4. As an application, we investigate which bands have the property that condition (C) is sufficient for every inverse semigroup S .

0. In this section we give counterexamples for Corollary 2, Theorem 3 and Corollary 4 of the original paper. Examples 1 and 2 are due to T. E. Hall (private communication).

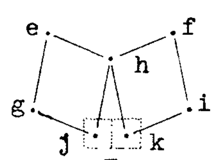
<u>EXAMPLE 1</u>. Let E be a non-trivial right zero semigroup X with an adjoined identity. Then W_E/γ is isomorphic to the full permutation group on X , with an adjoined zero. Therefore the least idempotent separating homomorphic image of W_E/γ is the two-element semilattice. On the other hand, W_E is fundamental, which shows that the "only if" part of Corollary 2 is false.

The next example shows that condition (C) in Theorem 3 of the original paper is not sufficient.

<u>EXAMPLE 2</u>. Let Y be the semilattice and E the band with $E/\mathcal{D} \cong Y$, given by the diagrams below:



Y



E

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The lowest $\mathcal{D}$-class in E is a right zero semigroup. The multiplication in E is uniquely determined. Denote by ν an isomorphism of $E/\mathcal{D}$ onto Y. Observe that $S=W_E\tau_\nu$ is an inverse semigroup satisfying condition (C) in Theorem 3. However, we claim that there is no orthodox semigroup T with band of idempotents E and with greatest inverse semigroup homomorphic image S.

Suppose that such a T exists. Since the only isomorphism of $\langle e\rangle$ onto $\langle f\rangle$ in E is the mapping Θ assigning f to e, h to h, i to g, k to j and j to k, there is an element t and an inverse t' of t in T with $\Theta_{t',t}=\Theta$. Then $\Theta_{t'h,ht}$ is the non-identical automorphism of $\langle h\rangle$, and so $ht\mathcal{H}h$ but $ht\not\gamma h$ in T. Hence we obtain that the $\mathcal{H}$-class of $h\gamma$ in T/γ has at least two elements. On the other hand, we easily see that S is combinatorial. This contradiction proves our claim.

Corollary 4 of the original paper is trivially false, as condition (C) is not sufficient by the previous example. The following examples show that conditions (i) and (ii) are neither necessary nor sufficient even if "(C)" is substituted by "(C) and (D)" in the statement of Corollary 4. Here (D) is the new condition in the corrected form of Theorem 3 (see below).

<u>EXAMPLE 3</u>. Let E be a band from Example 1 and let $S=Y=E/\mathcal{D}$. Then, clearly, we have $T_Y(S)\tau_\nu^{-1}=W_E$, and E is properly contained in W_E. However, E is the unique orthodox semigroup with band of idempotents E and greatest inverse semigroup homomorphic image $S=E/\mathcal{D}$. Thus (i) is not necessary in Corollary 4 of the original paper.

Note that if (i) is fulfilled for some E and S then (ii) is necessary for uniqueness.

<u>EXAMPLE 4</u>. Now let E be the band defined in Example 2. We will "stick" two semigroups to W_E so that the greatest inverse semigroup homomorphic images of the two orthodox semigroups obtained are isomorphic to each other and both have E as their bands of idempotents.

Let T and U be semigroups with a common subsemigroup $V=T\cap U$ and χ a homomorphism of T onto V. Suppose that V is an ideal in T and $\chi|V$ is identical. We say that we <u>stick</u> U <u>to</u> T <u>by means of</u> χ if we consider the underlying set $T\cup U$ and extend the multiplications in T and U to $T\cup U$ by setting $tu=(t\chi)u$ and $ut=u(t\chi)$ for $t\in T$, $u\in U$. This multiplication is well defined since, by assumption, if $t\in T$ and $v\in V$ then tv, $vt\in V$ and $tv=(tv)\chi=(t\chi)(v\chi)=(t\chi)v$, $vt=(vt)\chi=(v\chi)(t\chi)=v(t\chi)$. We also have $vu=(v\chi)u$ and $uv=u(v\chi)$ for every $v\in V$ and $u\in U$. One can easily see that this multiplication is associative and so $T\cup U$ is a semigroup with this operation.

Note that this construction is a very special way of embedding the semigroup amalgam $[T,U;V]$ when no extra elements are needed.

Now we give the semigroup which will be stuck to W_E. Let G be a group and N a normal subgroup of index 2 in G. Adjoin two right zero elements j and k to G and define

$$jg=\begin{cases} j & \text{if } g\in N \\ k & \text{otherwise}\end{cases} \quad \text{and} \quad kg=\begin{cases} k & \text{if } g\in N \\ j & \text{otherwise}\end{cases}$$

for every g in G. One can easily check that the groupoid $U(G,N)$ defined in this manner is a semigroup. Observe that $N=\{g\in G:\ jg=j\}=\{g\in G:\ kg=k\}$. Therefore $U(G,N_1)$ is not isomorphic to $U(G,N_2)$ provided N_1 and N_2 are non-isomorphic normal subgroups of index 2 in G.

In particular, let G be the group of symmetries of a square, N_1 the subgroup of all rotations and N_2 the subgroup generated by the reflections through the diagonals. Clearly, N_1 is a cyclic group of order 4 and N_2 is a four-group. Thus N_1 and N_2 are non-isomorphic normal subgroups of index 2 in G, and hence $U_1=U(G,N_1)$ and $U_2=U(G,N_2)$ are not isomorphic. Both U_1 and U_2 are orthodox semigroups with bands of idempotents isomorphic to the subband $\langle h\rangle$ in E. Moreover, a reflection t through an axis of symmetry parallel to an edge of the square is an element of order 2 belonging to $G\setminus N_1\cup N_2$ whence we can easily see by definition that $V=\{t,t^2,j,k\}$ is a full orthodox subsemigroup in U_1 and U_2 as well, and V is isomorphic to $W_{\langle h\rangle}$. Let us identify V with

$W_{\langle h\rangle}$. Hence $W_E \cap U_1 = W_E \cap U_2 = W_{\langle h\rangle}$.

Define a mapping χ of W_E by restricting the partial isomorphisms of E to $\langle h\rangle$, more precisely, let $(\rho_x \alpha_\ell, \lambda_y \alpha_r^{-1})\chi = (\rho_{hxh}(\Theta_h \alpha)_\ell, \lambda_{hyh}(\Theta_h \alpha)_r^{-1})$. Since $\langle h\rangle \alpha \subseteq \langle h\rangle$ for every partial isomorphism α occurring in the definition of W_E , the mapping χ is a homomorphism onto $W_{\langle h\rangle}$. Observe that $W_{\langle h\rangle}$ is an ideal in W_E and $\chi | W_{\langle h\rangle}$ is clearly identical.

Now we are ready to stick U_1 and U_2 to W_E by means of χ . Clearly, the semigroups T_1 and T_2 obtained in this way are orthodox semigroups with bands of idempotents E, and their greatest inverse semigroup homomorphic images are isomorphic to each other. It is not difficult to see that T_1 and T_2 are not isomorphic. For, if φ were an isomorphism of T_1 onto T_2 , then we would have $h\varphi = h$ and hence $U_1\varphi = U_2$, contradicting the fact that U_1 and U_2 are not isomorphic. Since the band E and the common inverse semigroup homomorphic image S of T_1 and T_2 satisfy conditions (i) and (ii) of Corollary 4 in the original paper it follows that these conditions are not sufficient either.

1. Before turning to the main point we make some remarks in connection with the result formulated in Section 1 of the original paper in the case when T is an inverse semigroup. The proof of the converse part is based on the following observation. Let T be an inverse semigroup with semilattice of idempotents Y and ρ an idempotent separating congruence on T . Define G to be the union of the idempotent ρ-classes. Clearly, G is a semilattice Y of groups G_α $(\alpha \in Y)$ where α is the identity in G_α . The set $X = \{x_s \in T : s \in T/\rho ,\ x_s\rho = s\}$ is termed a <u>cross-section of ρ-classes</u> provided $x_s x_{s^{-1}} = x_{ss^{-1}}$ for every s in T/ρ . One can easily verify that there exists a cross-section of ρ-classes in T and every cross-section has the property that $x_\alpha = \alpha$ for each α in Y and $x_{s^{-1}} = x_s^{-1}$ for each s in T/ρ . Given a cross-section X of ρ-classes we can define a $(T/\rho, G)$-pair h, χ as follows:

$$gh_s = x_{s^{-1}} g x_s \quad \text{and} \quad \chi_{s,\bar{s}} = x_{(s\bar{s})^{-1}} x_s x_{\bar{s}}$$

for every $s, \bar{s}$ in T/ϱ and g in G. Furthermore, the mapping $\iota : \mathcal{S}(T/\varrho, G; h, \chi) \to T$ assigning $x_s g$ to the pair (s, g) is an onto isomorphism which will be called the <u>canonical isomorphism determined by the cross-section</u> X.

If we choose another cross-section $X^* = \{ x_s^* \in T : s \in T/\varrho,\ x_s^* \varrho = s \}$ then, for every s in T/ϱ, there exists an element g_s in $G_{s^{-1}s}$ such that $x_s^* = x_s g_s$. By definition, we have $g_{s^{-1}} = g_s^{-1}$ for every s in T/ϱ. Put $\mathcal{G} = \{ g_s : s \in T/\varrho \}$ and call it a <u>translational basis for</u> T/ϱ <u>and</u> G. Clearly, given a cross-section $X = \{ x_s \in T : s \in T/\varrho,\ x_s \varrho = s \}$ and a set $\mathcal{G} = \{ g_s \in G_{s^{-1}s} : s \in T/\varrho \}$ with $g_{s^{-1}} = g_s^{-1}$ for every $s \in T/\varrho$, we can define a new cross-section $X^* = \{ x_s g_s : s \in T/\varrho \}$ such that $\mathcal{G}$ is the translational basis for T/ϱ and G corresponding to X and X^*. The $(T/\varrho, G)$-pair corresponding to the cross-section X^* is $(h)^{\mathcal{G}}, (\chi)^{\mathcal{G}}$ defined by

$$(h)_s^{\mathcal{G}} = h_s \alpha_{g_s} \quad \text{and} \quad (\chi)_{s,\bar{s}}^{\mathcal{G}} = g_{s\bar{s}}^{-1} \chi_{s,\bar{s}} (g_s h_{\bar{s}}) g_{\bar{s}}$$

for every $s, \bar{s}$ in T/ϱ where α_g is used to mean the inner endomorphism of G defined by g in G, that is, $g_1 \alpha_g = g^{-1} g_1 g$ for each g_1 in G.

By a <u>trivial</u> $(T/\varrho, G)$-<u>pair</u> we mean a $(T/\varrho, G)$-pair h, χ such that $\chi_{s,\bar{s}} = \ell(s\bar{s})$ for every $s, \bar{s}$ in T/ϱ.

In the proof of Theorem 3 we need the following lemma.

<u>LEMMA 2</u>. <u>For</u> $i = 1, 2$, <u>let</u> T_i <u>be an inverse semigroup and</u> ϱ_i <u>an idempotent separating congruence on it. Let</u> $X_i = \{ x_s^{(i)} : s \in T_i/\varrho_i,\ x_s^{(i)} \varrho_i = s \}$ <u>be a cross-section of</u> ϱ_i-<u>classes. Denote by</u> G_i <u>the union of idempotent</u> ϱ_i-<u>classes and by</u> $h^{(i)}, \chi^{(i)}$ <u>the</u> $(T_i/\varrho_i, G_i)$-<u>pair determined by</u> X_i. <u>Assume that</u> ϑ <u>is an isomorphism of</u> T_1/ϱ_1 <u>onto</u> T_2/ϱ_2. <u>Then, if</u> Φ <u>is a homomorphism [isomorphism] of</u> T_1 <u>into [onto]</u> T_2 <u>such that</u> $x_s^{(1)} \Phi = x_{s\vartheta}^{(2)}$ <u>for every</u> s <u>in</u> T_1/ϱ_1 <u>then</u> $\varphi = \Phi \restriction G_1$ <u>is a homomorphism [isomorphism] of</u> G_1 <u>into [onto]</u> G_2 <u>with the property that</u>

(i) $h_s^{(1)} \varphi = \varphi h_{s\vartheta}^{(2)}$ <u>for every</u> s <u>in</u> T_1/ϱ_1, <u>and</u>

(ii) $\chi^{(1)}_{s,\bar{s}}\varphi = \chi^{(2)}_{s\vartheta,\bar{s}\vartheta}$ <u>for each pair</u> $s,\bar{s}$ <u>in</u> T_1/ρ_1.
<u>Conversely, if</u> φ <u>is a homomorphism [isomorphism] of</u> G_1 <u>into [onto]</u> G_2 <u>such that</u> (i) <u>and</u> (ii) <u>are satisfied then the mapping</u> Φ <u>of</u> T_1 <u>into [onto]</u> T_2 <u>defined by</u> $(x^{(1)}_s g)\Phi = x^{(2)}_{s\vartheta} g\varphi$ $(g \in G_{s^{-1}s})$ <u>is a homomorphism [isomorphism]</u>. <u>Clearly,</u> Φ <u>separates idempotents</u>.

One can verify this lemma by a straightforward calculation. Therefore it is left to the reader.

For brevity, we say that ϑ <u>and</u> φ <u>commute with the</u> $(T_1/\rho_1, G_1)$<u>-pair</u> $h^{(1)}, \chi^{(1)}$ <u>and the</u> $(T_2/\rho_2, G_2)$<u>-pair</u> $h^{(2)}, \chi^{(2)}$ if conditions (i) and (ii) in Lemma 2 are satisfied. If ϑ is the identity automorphism then we simply say that φ commutes with $h^{(1)}, \chi^{(1)}$ and $h^{(2)}, \chi^{(2)}$.

2. Now we turn to M. Yamada's problems. The reader is assumed to be familiar with the original paper. Definitions, notations and arguments are not repeated here. Before establishing the correct form of Theorem 3 we make several remarks which will be applied in the proof.

The kernel of the homomorphism τ_ν is an inverse semigroup congruence on W_E but not the least one in general. Let γ be the least inverse semigroup congruence on W_E . Then there exists a unique homomorphism ω_ν of W_E/γ into T_Y such that $\gamma^{\natural}\omega_\nu = \tau_\nu$. Obviously, $\omega_\nu \circ \omega_\nu^{-1}$ is the greatest idempotent separating congruence on W_E/γ as T_Y is fundamental. The latter observation makes it possible to consider W_E/γ as an idempotent separating extension of $W_E\tau_\nu$. By Section 1, there exists a canonical isomorphism ι_ν of $\mathcal{S}(W_E\tau_\nu, \tilde{G}; \tilde{h}, \tilde{\chi})$ onto W_E/γ with $\iota_\nu\omega_\nu$ the first projection where $\tilde{G}$ is the union of the idempotent $\omega_\nu \circ \omega_\nu^{-1}$-classes and $\tilde{h}, \tilde{\chi}$ is the $(W_E\tau_\nu, \tilde{G})$-pair defined by a cross-section $\tilde{X} = \{\tilde{x}_\sigma : \sigma \in W_E\tau_\nu , \tilde{x}_\sigma\omega_\nu = \sigma\}$. Here $\tilde{G}$ is a semilattice $\bar{Y}$ of groups $\tilde{G}_{\tilde{\alpha}}$ $(\tilde{\alpha} \in \bar{Y})$. It is not difficult to show that, up to isomorphism, $\tilde{G}_{\tilde{\alpha}}$ is just the group $\mathrm{Aut}_{\mathcal{B}}\langle e\rangle$ of all automorphisms of $\langle e\rangle$ preserving $\mathcal{B}$-classes where $e \in E_{\tilde{\alpha}}$. One has to check only that an isomorphism of $\tilde{G}_{\tilde{\alpha}}$ onto $\mathrm{Aut}_{\mathcal{B}}\langle e\rangle$ is defined by assigning

$(\Theta_f|\langle e\rangle)\pi(\Theta_e|\langle g\rangle)$ to $(\varrho_f\pi_\ell, \lambda_g\pi_r^{-1})\gamma \in \tilde{G}_{\bar\alpha}$. In what follows, for every $\bar\alpha$ in $\bar{Y}$, we choose and fix an element $e_{\bar\alpha}$ in $E_{\bar\alpha}$ and identify $\tilde{G}_{\bar\alpha}$ with $\mathrm{Aut}_{\mathcal{B}}\langle e_{\bar\alpha}\rangle$ under this isomorphism. Note that the structure homomorphisms of $\tilde{G}=\bigcup_{\bar\alpha\in\bar{Y}}\tilde{G}_{\bar\alpha}$ are $\Gamma_{\bar\alpha,\bar\beta}$ $(\bar\alpha\geq\bar\beta)$ defined by

$$g\,\Gamma_{\bar\alpha,\bar\beta}=(\Theta_f|\langle e_{\bar\beta}\rangle)\,(g|\langle f\rangle)(\Theta_{e_{\bar\beta}}|\langle e_{\bar\alpha}\,e_{\bar\beta}\,e_{\bar\alpha}\rangle)$$

for all g in $\tilde{G}_{\bar\alpha}$ where $f=(e_{\bar\alpha}e_{\bar\beta}e_{\bar\alpha})g^{-1}$.

Observe that if a full orthodox subsemigroup W_o of W_E is considered then $\omega_o=\omega_\nu|W_o/\gamma_o$ is the unique homomorphism of W_o/γ_o onto $T_o=W_o\tau_\nu$ such that $\gamma_o^\natural\omega_o=\tau_o$ where $\tau_o=\tau_\nu|W_o$ and γ_o is used to mean $\gamma|W_o$ which is the least inverse semigroup congruence on W_o . Moreover, $\omega_o\circ\omega_o^{-1}$ is the greatest idempotent separating congruence on W_o/γ_o . Thus, if $\tilde{X}_o$ is a cross-section of $\omega_o\circ\omega_o^{-1}$-classes then there exists a canonical isomorphism ι_o of $\mathcal{S}(T_o,\tilde{G}^o;\tilde{h}^o,\tilde{\chi}^o)$ onto W_o/γ_o determined by $\tilde{X}_o$ where $\tilde{G}^o$ is a subsemigroup in $\tilde{G}$ and $\tilde{h}^o,\tilde{\chi}^o$ is the $(T_o,\tilde{G}^o)$-pair defined by $\tilde{X}_o$. Clearly, we have $\iota_o=\iota_\nu|\mathcal{S}(T_o,\tilde{G}^o;\tilde{h}^o,\tilde{\chi}^o)$ provided $\tilde{X}_o=\{\tilde{x}_\sigma\in\tilde{X}:\ \sigma\in T_o\}$ and $\tilde{X}_o\subseteq W_o/\gamma_o$.

According to the identification of $\tilde{G}_{\bar\alpha}$ with $\mathrm{Aut}_{\mathcal{B}}\langle e_{\bar\alpha}\rangle$, the definition of a $(T_o,\tilde{G}^o)$-pair $\tilde{h}^o,\tilde{\chi}^o$ determined by a cross-section $\tilde{X}_o$ can be modified as follows. Let $W^o_{e,f}=\{\vartheta\in W_{e,f}:(\varrho_e\vartheta_\ell\ ,\lambda_f\vartheta_r^{-1})\in W_o\}$ for each pair e,f in E and let $W^o=\bigcup\{W^o_{e,f}: e,f\in E\}$. Assume that $\Sigma=\{\hat\sigma\ :\ \sigma\in T_o\}$ where $\hat\sigma\in W_{e_{\alpha\nu^{-1}},e_{\beta\nu^{-1}}}$ with the property that $(e_{\alpha\nu^{-1}}E_{\delta\nu^{-1}}e_{\alpha\nu^{-1}})\hat\sigma\subseteq E_{\delta\sigma\nu^{-1}}$ for every δ in Y with $\delta\leq\alpha$ provided $\sigma\in T_{\alpha,\beta}$. Moreover, we require that $\hat\sigma\,\hat\sigma^{-1}=\widehat{\sigma\sigma^{-1}}$ for all σ in T_o . Under these conditions, Σ is termed a T_o-system of E over ν . Clearly, if $\Sigma\subseteq W^o$ then $\tilde{X}_o=\{\tilde{x}^o_\sigma :\ \sigma\in T_o\}$ where $\tilde{x}^o_\sigma=(\varrho_{e_{\bar\alpha}}\hat\sigma_\ell\ ,\lambda_{e_{\bar\beta}}\hat\sigma_r^{-1})\gamma$ provided $\hat\sigma\in\Sigma\cap W_{e_{\bar\alpha},e_{\bar\beta}}$ is a cross-section of $\omega_o\circ\omega_o^{-1}$-classes. One has to observe only that $\tilde{x}^o_\sigma\,\omega_o=(\varrho_{e_{\bar\alpha}}\hat\sigma_\ell\ ,\lambda_{e_{\bar\beta}}\hat\sigma_r^{-1})\tau_\nu$ is just σ . Conversely, a straightforward calculation shows that if $\tilde{X}_o=\{\tilde{x}^o_\sigma :\ \sigma\in T_o,\ \tilde{x}^o_\sigma\,\omega_o=\sigma\}$ is a cross-section of $\omega_o \circ \omega_o^{-1}$-classes then we obtain a T_o-system of E over ν with $\Sigma \subseteq W^o$ by defining $\Sigma = \{\hat{\sigma} : \sigma \in T_o\}$ where $\hat{\sigma} = (\Theta_f \mid \langle e_{\bar{\alpha}} \rangle)\, \tilde{\sigma}\, (\Theta_{e_{\bar{\beta}}} \mid \langle g \rangle)$ provided $\tilde{x}^o_\sigma = (\rho_f \tilde{\sigma}_\ell \, , \lambda_g \tilde{\sigma}_r^{-1})\gamma$ and $f \in E_{\bar{\alpha}}$, $g \in E_{\bar{\beta}}$. One can immediately verify that $\tilde{x}^o_\sigma = (\rho_{e_{\bar{\alpha}}} \hat{\sigma}_\ell \, , \lambda_{e_{\bar{\beta}}} \hat{\sigma}_r^{-1})\gamma$. Hence we infer that the cross-section determined by Σ according to the above definition is just $\tilde{X}_o$. Thus there is a one-to-one correspondence between the T_o-systems Σ of E over ν with $\Sigma \subseteq W^o$ and the cross-sections of $\omega_o \circ \omega_o^{-1}$-classes. Hence we can work with a $(T_o, \tilde{G}^o)$-pair defined by a T_o-system of E over ν contained in W^o instead of a $(T_o, \tilde{G}^o)$-pair defined by a cross-section of $\omega_o \circ \omega_o^{-1}$-classes. Given a T_o-system Σ of E over ν with $\Sigma \subseteq W^o$, the $(T_o, \tilde{G}^o)$-pair $\tilde{h}^o, \tilde{\chi}^o$ determined by Σ is defined in the following way: if $\sigma \in T_{\alpha,\beta}$, $\bar{\sigma} \in T_{\bar{\alpha},\bar{\beta}}$ and $g \in \tilde{G}^o_\delta$ then

$$g\tilde{h}^o_\sigma = (\Theta_i \mid \langle e_{(\alpha\delta)\sigma\nu^{-1}} \rangle)\, (\hat{\sigma}^{-1} \mid \langle i \rangle)\, (\Theta_j \mid \langle i\hat{\sigma}^{-1} \rangle) \,.$$
$$\cdot (g \mid \langle j \rangle)\, (\Theta_k \mid \langle jg \rangle)\, (\hat{\sigma} \mid \langle k \rangle)\, (\Theta_{e_{(\alpha\delta)\sigma\nu^{-1}}} \mid \langle k\hat{\sigma} \rangle),$$

where $i = e_{\beta\nu^{-1}} e_{(\alpha\delta)\sigma\nu^{-1}} e_{\beta\nu^{-1}}$, $j = e_{\delta\nu^{-1}} (i\hat{\sigma}^{-1}) e_{\delta\nu^{-1}}$, $k = e_{\alpha\nu^{-1}} (jg) e_{\alpha\nu^{-1}}$ and

$$\tilde{\chi}^o_{\sigma,\bar{\sigma}} = (\widehat{\sigma\bar{\sigma}})^{-1} (\Theta_i \mid \langle e_{(\bar{\alpha}\beta)\sigma^{-1}\nu^{-1}} \rangle)\, (\hat{\sigma} \mid \langle i \rangle) \,.$$
$$\cdot (\Theta_j \mid \langle i\hat{\sigma} \rangle)\, (\hat{\bar{\sigma}} \mid \langle j \rangle)\, (\Theta_{e_{(\bar{\alpha}\beta)\bar{\sigma}\nu^{-1}}} \mid \langle j\hat{\bar{\sigma}} \rangle) \quad ,$$

where $i = e_{\alpha\nu^{-1}} e_{(\bar{\alpha}\beta)\sigma^{-1}\nu^{-1}} e_{\alpha\nu^{-1}}$, $j = e_{\bar{\alpha}\nu^{-1}} (i\hat{\sigma}) e_{\bar{\alpha}\nu^{-1}}$.

Observe that W_o can be reobtained by means of $\tilde{G}^o$ and Σ as follows. Consider the canonical isomorphism ι_o of $\mathcal{S}(T_o, \tilde{G}; \tilde{h}^o, \tilde{\chi}^o)$ into W_E/γ determined by Σ . Here $\tilde{h}^o, \tilde{\chi}^o$ is the $(T_o, \tilde{G})$-pair defined by Σ . Let $\mathcal{W}(\tilde{G}^o, \Sigma) = \mathcal{S}(T_o, \tilde{G}^o; \tilde{h}^o, \tilde{\chi}^o)\iota_o (\gamma^\natural)^{-1}$. Clearly, we have $W_o = \mathcal{W}(\tilde{G}^o, \Sigma)$.

Note that if $T_o \subseteq W_E \tau_\nu$, $\tilde{G}^o$ is a subsemigroup in $\tilde{G}$ and Σ is a T_o-system of E over ν such that the $(T_o, \tilde{G})$-pair $\tilde{h}^o, \tilde{\chi}^o$ defined by Σ possesses the property that $\tilde{G}^o \tilde{h}^o_\sigma \subseteq \tilde{G}^o$ and $\tilde{\chi}^o_{\sigma,\bar{\sigma}} \in \tilde{G}^o$ for every $\sigma, \bar{\sigma}$ in T_o then $\tilde{h}^o, \tilde{\chi}^o$ is a $(T_o, \tilde{G}^o)$-pair and $W_o = \mathcal{W}(\tilde{G}^o, \Sigma)$ is a full orthodox subsemigroup in W_E.

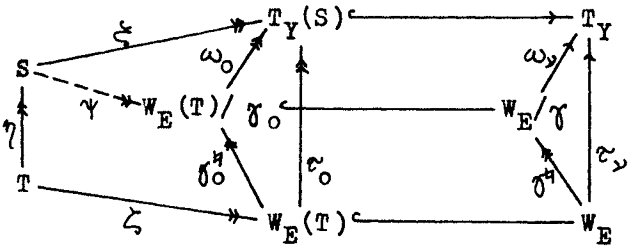
THEOREM 3. Let S be an inverse semigroup with semilattice of idempotents Y and let E be a band which is a semilattice $\bar{Y}$ of rectangular bands $E_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$. Denote by ξ the Munn homomorphism of S onto $T_Y(S)$. For every α in Y, let G_α stand for the $\xi \circ \xi^{-1}$-class containing α. Their union G is a semilattice Y of groups G_α $(\alpha \in Y)$. Consider the $(T_Y(S),G)$-pair h, χ defined by a cross-section X of $\xi \circ \xi^{-1}$-classes. Moreover, for every $\bar{\alpha}$ in $\bar{Y}$, let us choose and fix an element $e_{\bar{\alpha}}$ in $E_{\bar{\alpha}}$ and denote by $\tilde{G}_{\bar{\alpha}}$ the group of all automorphisms of $\langle e_{\bar{\alpha}} \rangle$ preserving $\mathcal{D}$-classes. Equip their union $\tilde{G}$ with a multiplication defined by means of the structure homomorphisms $\Gamma_{\bar{\alpha},\bar{\beta}}$ $(\bar{\alpha} \geq \bar{\beta})$ which let $\tilde{G}$ become a semilattice $\bar{Y}$ of groups $\tilde{G}_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$. Then there exists an orthodox semigroup T with band of idempotents E and with greatest inverse semigroup homomorphic image S if and only if there exists an isomorphism ν of $\bar{Y}$ onto Y such that

(C) for every s in S, there exist elements e in $E_{(ss^{-1})\nu^{-1}}$, f in $E_{(s^{-1}s)\nu^{-1}}$ and an isomorphism ϑ of eEe onto fEf such that $(eE_{\bar{\alpha}}e)\vartheta \subseteq E_{(s^{-1}(\bar{\alpha}\nu)s)\nu^{-1}}$ for each $\bar{\alpha}$ in $\bar{Y}$ with $\bar{\alpha} \leq (ss^{-1})\nu^{-1}$, and

(D) there exists a $T_Y(S)$-system Σ of E over ν and a homomorphism φ of G into $\tilde{G}$ such that $G_\alpha\varphi \subseteq \tilde{G}_{\alpha\nu^{-1}}$ for every α in Y and φ commutes with the $(T_Y(S),G)$-pair h, χ and the $(T_Y(S),G\varphi)$-pair $\tilde{h}^{\sigma}, \tilde{\chi}^{\sigma}$ defined by Σ.

REMARK. One can immediately see that condition (C) is equivalent to the inclusion $T_Y(S) \subseteq W_E\tau_\nu$ and it is equivalent to requiring the existence of a $T_Y(S)$-system of E over ν. The latter fact means that condition (C) is implicit in (D).

Proof. Let T be an orthodox semigroup with band of idempotents E and greatest inverse semigroup homomorphic image S. The necessity of (C) is verified as in the original paper. Denote by γ the least inverse semigroup congruence on W_E and by ω_ν the unique homomorphism of W_E/γ into T_Y with the property that $\gamma^\natural\omega_\nu=\tau_\nu$. Let $\gamma_0=\gamma|W_E(T)$ and $\omega_0=\omega_\nu|W_E(T)/\gamma_0$. Since $W_E(T)$ is a full orthodox subsemigroup in W_E therefore γ_0 is the least inverse semigroup congruence on $W_E(T)$ and ω_0 is the unique homomorphism of $W_E(T)/\gamma_0$ onto $T_Y(S)$ such that $\gamma_0^\natural\omega_0=\tau_0$. Now we define a homomorphism ψ



of S onto $W_E(T)/\gamma_0$ such that $\eta\psi=\zeta\gamma_0^\natural$ and $\psi\omega_0=$ $=\xi$. Let $s\psi=t\zeta\gamma_0^\natural$ for every s in S where t is an element in T with $t\eta=s$. Since both S and $W_E(T)/\gamma_0$ are inverse semigroups and $\eta\circ\eta^{-1}$ is the least inverse semigroup congruence on T the definition of ψ is independent of the choice of t. It can be immediately checked that ψ is a homomorphism and it is onto as both ζ and $\gamma_0^\natural$ are. The equality $\eta\psi=\zeta\gamma_0^\natural$ holds by definition. On the other hand, we have $s\psi\omega_0=$ $=t\zeta\gamma_0^\natural\omega_0=t\zeta\tau_0=t\eta\xi=s\xi$ for every s in S where $t\in T$ with $t\eta=s$. Thus equality $\psi\omega_0=\xi$ is also verified. Let Σ be the $T_Y(S)$-system of E over ν corresponding to the cross-section $\tilde{X}_0=X\psi$. Putting ψ for φ in Lemma 2, we infer the necessity of condition (D).

In order to prove sufficiency, suppose that there exists an isomorphism ν of $\mathbf{Y}$ onto Y such that (C) and (D) are fulfilled. Consider the full orthodox subsemigroup $W_0=\mathcal{W}(G\varphi,\Sigma)$ in W_E. Let $\gamma_0=\gamma|W_0$ and define a mapping ψ of S onto W_0/γ_0 by $(x_\sigma g)\psi=\tilde{x}_\sigma^0 g\varphi$ where $\sigma\in T_Y(S)\cap T_{\alpha,\beta}$, $x_\sigma\in X$, $g\in G_\beta$ and $\tilde{x}_\sigma^0=$ $=(\rho_{e_{\alpha\nu^{-1}}}\hat{\sigma}_\ell, \lambda_{e_{\alpha\nu^{-1}}}\hat{\sigma}_r^{-1})\gamma$. By Lemma 2, ψ is an idempotent separating homomorphism. By applying Theorem 1, we obtain an orthodox semigroup T and homomorphisms η and ζ of T onto S and W_0, respectively, such that $\eta \circ \eta^{-1}$ is the least inverse semigroup congruence and $\zeta \circ \zeta^{-1}$ is the greatest idempotent separating congruence on T. Thus the band of idempotents in T is isomorphic to E and S is the greatest inverse semigroup homomorphic image of T which completes the proof of the theorem.

As an application, we can easily answer the question which bands have the property that condition (C) is sufficient for every inverse semigroup S.

COROLLARY 4a. *Let a band E be given which is a semilattice $\bar{Y}$ of rectangular bands. For every inverse semigroup S with semilattice of idempotents Y which satisfies condition (C), there exists an orthodox semigroup with band of idempotents E and greatest inverse semigroup homomorphic image S if and only if there exists an isomorphism ν of $\bar{Y}$ onto Y and a $W_E\tau_\nu$-system Σ of E over ν which defines a trivial $(W_E\tau_\nu, \tilde{G})$-pair or, equivalently, for every isomorphism ν of $\bar{Y}$ onto Y there exists a $W_E\tau_\nu$-system Σ of E over ν which defines a trivial $(W_E\tau_\nu, \tilde{G})$-pair.*

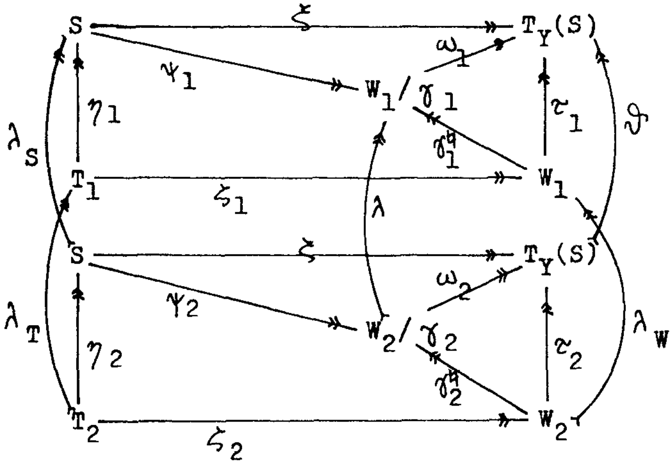
Proof. Suppose first that, for every inverse semigroup S satisfying condition (C), there exists an orthodox semigroup with the required properties. Then, in particular, there exists an orthodox semigroup T with band of idempotents E and greatest inverse semigroup homomorphic image $W_E\tau_{\nu_0}$ where ν_0 is an isomorphism of $\bar{Y}$ onto Y. Since $W_E\tau_{\nu_0}$ is fundamental the necessity of condition (D) in Theorem 3 implies that there exists an isomorphism ν of $\bar{Y}$ onto Y and a $W_E\tau_{\nu_0}$-system Σ of E over ν which defines a trivial $(W_E\tau_{\nu_0}, \tilde{G})$-pair. Observe that $\tau_{\nu_0} \circ \tau_{\nu_0}^{-1} = \tau_\nu \circ \tau_\nu^{-1}$ as each of them is just the greatest congruence on W_E with the property that its restriction to E is $\mathcal{D}$. Moreover, we have $(\varrho_{e_{\bar{\alpha}}}\hat{\sigma}_\ell, \lambda_{e_{\bar{\beta}}}\hat{\sigma}_r^{-1})\tau_\nu = \sigma$ for every $\hat{\sigma}$ in Σ provided $\hat{\sigma} \in W_{e_{\bar{\alpha}}, e_{\bar{\beta}}}$. Thus Σ can be considered to be a $W_E\tau_{\nu_0}$-system of E over ν_0. This completes the proof of the "only if" part.

Conversely, assume that there exists ν and Σ with the required properties. Then, for every inverse semigroup S, the homomorphism φ assigning the identity of $\tilde{G}_{\alpha\nu^{-1}}$ to all elements in G_α and the $T_Y(S)$-system $\Sigma_0=\{\hat{\sigma}\in\Sigma : \sigma\in T_Y(S)\}$ satisfy condition (D). Thus Theorem 3 implies the "if part", too.

As far as the problem of uniqueness is concerned we can formulate the following assertion.

COROLLARY 4b. *Let S be an inverse semigroup with semilattice of idempotents Y and let E be a band which is a semilattice $\bar{Y}$ of rectangular bands $E_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$. Define G, h, χ and $\tilde{G}$ as in Theorem 3. Suppose that conditions (C) and (D) are fulfilled. Then the orthodox semigroup with band of idempotents E and greatest inverse semigroup homomorphic image S is unique up to isomorphism if and only if the following is satisfied: whenever ν_i $(i=1,2)$ is an isomorphism of $\bar{Y}$ onto Y, φ_i $(i=1,2)$ is a homomorphism of G into $\tilde{G}$ and Σ_i $(i=1,2)$ is a $T_Y(S)$-system of E over ν_i such that the triple $\nu_i, \varphi_i, \Sigma_i$ $(i=1,2)$ fulfils condition (D) then there exist automorphisms ε, $\bar{\varepsilon}$ and $\varkappa$ of Y, E and G, respectively, such that ε induces an automorphism ϑ of $T_Y(S)$, $\bar{\varepsilon}$ induces an isomorphism μ of $G\varphi_2$ onto $G\varphi_1$, the equality $\varkappa\varphi_1=\varphi_2\mu$ holds and ϑ and $\varkappa$ commute with h, χ and $(h)\mathcal{G}$, $(\chi)\mathcal{G}$ where $\mathcal{G}$ is a translational basis for $T_Y(S)$ and G with the property that $\mathcal{G}\varphi_1$ is the translational basis for $T_Y(S)$ and $G\varphi_1$ corresponding to Σ_1 and $\Sigma_2^{\bar{\varepsilon}}$. Here $\Sigma_2^{\bar{\varepsilon}}$ is used to mean the $T_Y(S)$-system of E induced from Σ_2 by $\bar{\varepsilon}$.*

Proof. Assume that the triple $\nu_i, \varphi_i, \Sigma_i$ $(i=1,2)$ fulfils condition (D). By means of the triple $\nu_i, \varphi_i, \Sigma_i$ $(i=1,2)$, construct the idempotent separating homomorphism ψ_i of S onto W_i/γ_i where $W_i=\mathcal{W}(G\varphi_i,\Sigma_i)$ and $\gamma_i=\gamma|W_i$ and construct an orthodox semigroup T_i with homomorphisms η_i and ζ_i as in the proof of sufficiency in Theorem 3. In order to prove necessity suppose that the orthodox semigroup with band of idempotents E and greatest inverse semigroup homomorphic image S is unique up to isomorphism. In this case, there exists an isomorphism λ_T of T_2 onto T_1. It is easily seen that λ_T determines an automorphism λ_S of S for which we have $\lambda_T \eta_1 = \eta_2 \lambda_S$ as $\eta_1 \circ \eta_1^{-1}$ and $\eta_2 \circ \eta_2^{-1}$ are the least inverse semigroup congruences on T_1 and T_2, respectively. Let $\varepsilon = \lambda_S|Y$ and $\bar{\varepsilon} = \lambda_T|E$. Clearly, ε and $\bar{\varepsilon}$ induce an automorphism ϑ of $T_Y(S)$ and an isomorphism λ_W of W_2



onto W_1, respectively. The equalities $\lambda_S \xi = \xi \vartheta$ and $\lambda_T \zeta_1 = \zeta_2 \lambda_W$ are immediately deduced. Moreover, λ_W determines an isomorphism λ of W_2/γ_2 onto W_1/γ_1 with $\lambda_W \gamma_1^\natural = \gamma_2^\natural \lambda$ since γ_1 and γ_2 are the least inverse semigroup congruences on W_1 and W_2, respectively. Then we have $\eta_2 \lambda_S \psi_1 = \lambda_T \eta_1 \psi_1 = \lambda_T \zeta_1 \gamma_1^\natural = \zeta_2 \lambda_W \gamma_1^\natural = \zeta_2 \gamma_2^\natural \lambda = \eta_2 \psi_2 \lambda$ whence we infer $\lambda_S \psi_1 = \psi_2 \lambda$ as η_2 is onto. Similarly, since ψ_2 is also onto the equalities $\psi_2 \lambda \omega_1 = \lambda_S \psi_1 \omega_1 = \lambda_S \xi = \xi \vartheta = \psi_2 \omega_2 \vartheta$ imply $\lambda \omega_1 = \omega_2 \vartheta$. By applying Lemma 2 for λ_S and λ and taking into consideration the equalities $\lambda_S \psi_1 = \psi_2 \lambda$ and $\lambda \omega_1 = \omega_2 \vartheta$, we see that $\varepsilon = \lambda_S|Y$, $\bar{\varepsilon} = \lambda_T|E$, $\varkappa = \lambda_S|G$ and the translational basis $\mathcal{G}$ for $T_Y(S)$ and G corresponding to X and $X\lambda_S$ satisfy the conditions required which proves necessity.

Conversely, now assume that there exist ε, $\bar{\varepsilon}$, $\varkappa$ and $\mathcal{G}$ with the properties formulated in Corollary 4b.

Then, by Lemma 2, we obtain an automorphism λ_S of S with $\lambda_S\xi = \xi\vartheta$ and an isomorphism λ of W_2/γ_2 onto W_1/γ_1 with $\lambda\omega_1 = \omega_2\vartheta$ such that the equality $\lambda_S\psi_1 = \psi_2\lambda$ is valid. Moreover, the automorphism of W_E induced by $\tilde{\varepsilon}$ maps W_2 onto W_1, that is, $\tilde{\varepsilon}$ induces an isomorphism λ_W of W_2 onto W_1. One can easily see that $\lambda_W\gamma_1^\natural = \gamma_2^\natural\lambda$. Thus we obtain that both of the pairs of mappings η_1, ζ_1 and $\eta_2\lambda_S, \zeta_2\lambda_W$ are pull-backs of the homomorphisms ψ_1 and $\gamma_1^\natural$. Therefore Theorem 1 implies T_1 and T_2 to be isomorphic which completes the proof of the Corollary.

Note that the isomorphism problem of Hall-Yamada semigroups was solved by T. E. Hall (private communication) as follows: Let S, S' be inverse semigroups, E, E' bands and $\varphi: S \to W_E/\gamma$, $\varphi': S' \to W_{E'}/\gamma'$ idempotent separating homomorphisms whose ranges contain all the idempotents of W_E/γ and $W_{E'}/\gamma'$, respectively. Then $\mathcal{H}(S,E,\varphi)$ and $\mathcal{H}(S',E',\varphi')$ are isomorphic to each other if and only if there exist isomorphisms α of S onto S' and β of E onto E' such that we have $\varphi' = \alpha^{-1}\varphi\hat{\hat{\beta}}$ where $\hat{\hat{\beta}}$ is the unique isomorphism of W_E/γ onto $W_{E'}/\gamma'$ that makes the diagram

$$\begin{array}{ccc} W_E & \xrightarrow{\hat{\beta}} & W_{E'} \\ \gamma^\natural \downarrow & & \downarrow \gamma'^\natural \\ W_E/\gamma & \xrightarrow[\hat{\hat{\beta}}]{} & W_{E'}/\gamma' \end{array}$$

commute. Here $\hat{\beta}$ is the unique isomorphism of W_E onto $W_{E'}$ extending β.

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
H-6720 SZEGED, ARADI VÉRTANUK TERE 1
HUNGARY

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