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Erratum

Level and Pythagoras number of some geometric rings

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Statements 1) and 3) of Theorem 4.1 are false. The correct statement of the theorem is the following

Theorem 4.1 *Let A be a finitely generated R-algebra of Krull dimension $\leqq d$, and f a totally positive element of A. Then*

1) *if $d \leqq 3$, we have an identity*

$$f\boxed{2^{d+1}-1}=1+\boxed{2^{d+1}-1}$$

2) *if $d \leqq 4$ and f is a unit in A, we have*

$$f\boxed{2^d}=1+\boxed{2^d-1}$$

3) *if $d \geqq 4$, we have*

$$f\boxed{2^d+7}=1+\boxed{2^{d+1}+d-4}$$

4) *if $d \geqq 5$ and f is a unit in A, we have*

$$f\boxed{2^d}=1+\boxed{2^d+d-5}.$$

Subsequently, Corollary 4.4 on the level has to be modified as follows:

Corollary 4.4 *Let A be an R-algebra of transcendence degree d without any real point* (*i.e.* $\mathrm{Spec}_r A=\emptyset$), *then*

1) *if $d \leqq 4$ then* $-1=\boxed{2^{d+1}-1}$
2) *if $d \geqq 5$ then* $-1=\boxed{2^{d+1}+d-5}$.

Proof. Apply 2) or 4) of Proposition 4.3 to $f=-1$ which is a totally positive unit when $\mathrm{Spec}_r A=\emptyset$.

Also in Remark 4.5, if the level of a curve without any real point is actually less than 3 (because $2^1+1=2^2-1$!), the real bound for surfaces is 7 instead of 5. Nothing else in the paper is affected by this modification, except the proof of Theorem 4.1, which is the subject of Sect. 5.

We give below a complete replacement of the end of Sect. 5 starting from line 18 p. 624.

Let us prove part 1). Let $d=\mathrm{tr.d.}(A)$. If f is not a unit, it may be 0 in the function field K_i of level 2^d, of some complex irreducible component of A and in that case, we get $0=1+\boxed{2^d}$. Because 0 has the shape $f\boxed{2^{d-1}}$, we can

write $f\boxed{2^{d-1}}=1+\boxed{2^d}$ in every field K_i, and thus in the product $\prod K_i$.

In the 0-dimensional case, we have $f\boxed{1}=1+\boxed{1}$ in each K_i, and so in A.

In dimension 1, with the above arguments, we obtain an identity

$$(1+\boxed{1}-f\boxed{1})^2+\boxed{2}-f\boxed{1}=0$$

in A. Applying Lemma 3.1 with $X=1$ and $Y=\boxed{1}-f\boxed{1}$, we get

$$1+\boxed{1}-f\boxed{1}+(1+\boxed{1}-f\boxed{1})(\boxed{2}-f\boxed{1})=0$$

which gives

$$1+\boxed{1}-f\boxed{1}+\boxed{2}-f\boxed{2}=0$$

and eventually

$$f\boxed{3}=1+\boxed{3}.$$

The same method in dimension 2, starting from

$$(1+\boxed{3}-f\boxed{3})^2+\boxed{4}-f\boxed{2}=0$$

leads (using Lemma 3.1 and Lemma 3.3 part 1) to

$$f\boxed{7}=1+\boxed{7}.$$

For the dimension 3, we do as in dimension 4 when f is a unit, starting with a double reduction in the dimension to get an identity

$$((1+\boxed{3}-f\boxed{3})^2+\boxed{4}-f\boxed{2})^2+\boxed{8}-f\boxed{4}=0$$

and using Lemmas 3.1 with $X=(1+\boxed{3}-f\boxed{3})^2$ and $Y=\boxed{4}-f\boxed{2}$, we get

$$(1+\boxed{3}-f\boxed{3})^6+\boxed{4}-f\boxed{2}+(\boxed{5}-f\boxed{2})(\boxed{8}-f\boxed{4})=0.$$

With Lemmas 3.2 ($n=6$) and 3.3 part 2, this gives

$$(1+\boxed{3}-f\boxed{3})^2+\boxed{4}-f\boxed{2}+\boxed{8}-f\boxed{8}=0$$

and a last application of Lemma 3.1 gives

$$1+\boxed{3}-f\boxed{3}+\boxed{4}-f\boxed{4}+\boxed{8}-f\boxed{8}=1+\boxed{15}-f\boxed{15}=0.$$

In order to get the parts 3) and 4) of Theorem 4.1, we make an induction respectively for $d\geqq 3$ and $d\geqq 4$. Let $\mathrm{tr.d.}(A)=d+1$ and f a totally positive (resp. unit) in A. Letting $d'=d+1$ (resp. d), we have

$$g^2+\boxed{2^{d'}}-f\boxed{2^d}=0$$

in A for some non zero-divisor g. By induction hypothesis, we get

$$1+\boxed{s_d}-\bar{f}\boxed{t_d}=0$$

in A/g for $s_d=2^{d+1}+d-4$ (resp. $s_d=2^d+d-5$) and $t_d=2^d+7$ (resp. 2^d), and so

$$\lambda g=1+\boxed{s_d}-f\boxed{t_d}$$

in A, and

$$(\lambda g)^2=(1+x-f\,y)^2$$

where $x=\boxed{s_d}$ and $y=\boxed{t_d}$. This gives

$$(\lambda g)^2=1+(x-f\,y)^2+2(x-f\,y).$$

But because 2 is a square, we get

$$(\lambda g)^2=1+\boxed{s_d}+1-f\boxed{t_d}.$$

Returning to the original equation, we get

$$1+\boxed{s_d+1}-f\boxed{t_d}+\boxed{2^{d'}}-f\boxed{2^d}=1+\boxed{s_d+1+2^{d'}}-f\boxed{t_d}+\boxed{2^d}=0.$$

In the non unit case, putting $s_3=15=2^4+3-4$, $d'=d+1$ and $t_3=15=2^3+7$, the induction relations $s_{d+1}=1+s_d+2^{d+1}$ and $t_{d+1}=t_d+2^d$ give the result.

In the unit case, putting $s_4=15=2^4+4-5$ and $t_4=16=2^4$, the relations $s_{d+1}=1+s_d+2^d$ and $t_{d+1}=t_d+2^d$ give the result.