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On an Infinite Interval Boundary Value Problem (*).

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1. - Introduction.

We will present an existence result for a second order nonlinear differential equation

$$y''=f(t, y, y'), \qquad 0 \leqslant t < \infty ,$$

with the boundary condition

$$\alpha y(0) - \beta y'(0) = r$$

where $\alpha > 0$, $\beta \geqslant 0$, $r \in R$ are given constants and $f\colon R_+ \times R^2 \to R$ is continuous. Under certain growth conditions (suggested by the finite interval case [4]) on the nonlinearity f we establish that the considered boundary value problem—that we denote by (P)—has bounded solutions.

We apply our results to a plane membrane problem and to a boundary value problem which occurs in the theory of semiconductor devices.

Our approach is based on the topological transversality theorem. We briefly review the topological results we will use (for further details, see [6[).

Let C be a convex subset of a Banach space, X a metric space and $F\colon X \to C$ a continuous map. We say that F is compact if $F(X)$ is contained in a compact subset of C. A homotopy $\{\mathcal{H}_\lambda\colon X \to C\}_{\lambda \in [0,1]}$ is called compact if the map $\mathcal{H}\colon X \times [0, 1] \to C$ given by

$$\mathcal{H}(x, \lambda) = \mathcal{H}_\lambda(x), \qquad (x, \lambda) \in X \times [0, 1],$$

is compact.

Let now $U \subset C$ be open in C. We say that a compact (continuous) map $F\colon \overline{U} \to C$ is admissible if it is fixed point free on the boundary, ∂U, of U. It is called inessential if there is a fixed point free compact map from $\overline{U}$ to C such that its restriction to ∂U is the same as the restriction of F to ∂U. An admissible map which is not inessential is called

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essential. In this context, let us note that the constant map $F_u\colon \overline{U}\to C$, $F_u\equiv u$ on $\overline{U}$, where $u\in U$, is essential.

The main result we need is

TOPOLOGICAL TRANSVERSALITY THEOREM. – *Let $F, G\colon \overline{U}\to C$ be admissible and such that there exists a compact homotopy $\{\mathcal{H}_\lambda\colon \overline{U}\to C\}_{\lambda\in[0,1]}$ for which $F=\mathcal{H}_0$, $G=\mathcal{H}_1$, and $\mathcal{H}_\lambda$ is admissible for each $\lambda\in[0,1]$. Then one of the maps is essential if the other is.*

Before proceeding let us introduce the following notation: let $BC^2(R_+)$ be the space of all functions $u(t)$ on R_+ with $u^{(i)}(t)$ bounded and continuous on R_+ for $i=0, 1, 2$.

2. – Solutions to the boundary value problem.

The following lemma with the Arzela-Ascoli theorem will imply our basic existence theorem for (P):

LEMMA 1. – *Assume $f(t, x, y)$ is continuous on $R_+\times R^2$ and satisfies*

(i) *there is a constant $M>0$ such that $xf(t, x, 0)>0$ for $|x|\geq M$;*

(ii) *there are functions $A(t, x)$, $B(t, x)>0$ which are bounded when x varies in a bounded interval and t varies in R_+ such that*

$$|f(t, x, y)|\leq A(t, x)\,w(y^2)+B(t, x)$$

where $w\in C(R_+,(0,\infty))$ is nondecreasing and $\int_0^\infty ds/w(s)=\infty$.

Let n be a positive integer and consider the boundary value problem

$$\text{(2.1.}n\text{)}\quad y''=f(t, y, y'),\quad 0\leq t\leq n,\qquad \alpha y(0)-\beta y'(0)=r,\qquad y(n)=0.$$

Then (2.1.n) has at least one solution $y_n\in C^2[0, n]$ and there is a constant $K>0$ independent of n such that

$$\sup_{t\in[0,n]}\{|y_n(t)|, |y_n'(t)|, |y_n''(t)|\}\leq K.$$

PROOF. – Let us first show that there is an $K_1>0$ such that if $y\in C^2[0, n]$ is a solution of

$$y''=\lambda f(t, y, y'),\quad 0\leq t\leq n,\qquad \alpha y(0)-\beta y'(0)=r,\qquad y(n)=0.$$

for some $\lambda\in(0, 1)$, then

$$\sup_{t\in[0,n]}\{|y_n(t)|, |y_n'(t)|, |y_n''(t)|\}\leq K_1.$$

Observe that y^2 must have a maximum at a point $t_0\in[0, n]$. If $t_0=0$ then $0\geq y(0)\,y'(0)$ thus $0\geq \beta y(0)\,y'(0)=\alpha y^2(0)-y(0)\,r$ and we get $|y(t)|\leq |r|/\alpha$,

$t \in [0, n]$. If $t_0 = n$ we immediately have that $y(t) = 0$, $t \in [0, n]$. In the case $t_0 \in (0, n)$ we have $y(t_0)\, y'(t_0) = 0$ and $y(t_0)\, y''(t_0) \leq 0$ thus

$$0 \geq y(t_0) f(t_0, y(t_0), 0)$$

and by (i) we get $|y(t_0)| \leq M$. All this enables us to write

$$\sup_{t \in [0, n]} \{|y(t)|\} \leq \max \left\{ \frac{|r|}{\alpha}, M \right\} = M_1 .$$

If y^2 has a maximum in $(0, n)$ then y' vanishes at least once in $(0, n)$. If the maximum of y^2 on $[0, n]$ is at $t = 0$ and $\beta \neq 0$, then $|y'(0)| \leq |r - \alpha y(0)|/\beta \leq \leq (|r| + \alpha M_1)/\beta$, whereas if $\beta = 0$ by the mean value theorem there is a point $t_0 \in \in (0, n)$ with $|y'(t_0)| = |y(n) - y(0)|/n \leq M_1$. Clearly if the maximum of y^2 on $[0, n]$ occurs at $t = n$, then $y(t) = 0$, $t \in [0, n]$. In any case, there is a constant $M_2 > 0$ (independent of n) such that there is a point $t_0 \in [0, n]$ with $|y'(t_0)| \leq M_2$. We deduce that each $t \in [0, n]$ with $|y'(t)| > M_2$ belongs to some interval $[a, b] \subset [0, n]$ with $|y'(s)| > M_2$ for $a < s < b$ and $|y'(a)| = M_2$ or $|y'(b)| = M_2$. Suppose $y'(a) = M_2$ and $y'(t) > M_2$ on (a, b)—the other cases are similar. If

$$A = \sup_{t \in R_+,\, |x| \leq M_1} \{A(t, x)\}, \qquad B = \sup_{t \in R_+,\, |x| \leq M_1} \{B(t, x)\},$$

we obtain

$$y''(t) \leq A\omega((y'(t))^2) + B, \qquad a < t < b,$$

thus

$$\frac{2y'(t)\, y''(t)}{w((y'(t))^2) + 1} \leq 2(A + B)\, y'(t), \qquad a < t < b,$$

and an integration on $[a, t]$ yields

$$\int_{M_2^2}^{(y'(t))^2} \frac{ds}{w(s) + 1} \leq 2(A + B) \int_a^t y'(s)\, ds \leq 4(A + B)\, M_1, \qquad a \leq t \leq b .$$

Now, the hypotheses on w imply (see [3]) that

$$\int_0^{\infty} \frac{ds}{w(s) + 1} = \infty$$

thus there is a constant $M_3 > 0$ (independent of a, b, n) such that

$$|y'(t)| \leq M_3, \qquad a \leq t \leq b .$$

We get

$$\sup_{t \in [0, n]} \{|y'(t)|\} \leq \max \{M_2, M_3\} = M_4 .$$

By (ii) we obtain

$$|y''(t)| \leq Aw(M_4^2) + B, \qquad 0 \leq t \leq n,$$

thus if $K_1 = \max\{M_1, M_4, Aw(M_4^2) + B\}$ then

$$\sup_{t \in [0,n]} \{|y_n(t)|, |y_n'(t)|, |y_n''(t)|\} \leq K_1.$$

Let now

$$C_n^2 = \{u \in C^2[0,n]: \alpha u(0) - \beta u'(0) = r, u(n) = 0\},$$

and

$$U_n = \left\{u \in C_n^2: \sup_{t \in [0,n]} \{|u(t)|, |u'(t)|, |u''(t)|\} < K_1 + 1\right\} \subset C_n^2.$$

The operator L_n: $C_n^2 \to C[0,n]$ defined by $Lu = u''$ is one-to-one and onto. Let us define

$$F: C^1[0,n] \to C[0,n], \quad Fy(t) = f(t, y(t), y'(t)), \qquad 0 \leq t \leq n,$$

and let j_n: $C_n^2 \to C^1[0,n]$ be the natural embedding (which is completely continuous). We also consider the function $l_n \in U$,

$$l_n(x) = \frac{-rx}{n\alpha + \beta} + \frac{nr}{n\alpha + \beta'}, \qquad 0 \leq x \leq n.$$

One can see that

$$\mathcal{H}: \overline{U}_n \times [0,1] \to C_n^2, \qquad \mathcal{H}(u, \lambda) = \lambda L_n^{-1} F j_n(u) + (1-\lambda)\, l_n,$$

is a compact homotopy. A fixed point u of $\mathcal{H}_\lambda$ must satisfy $\lambda F_{j_n}(n) = L_n u$ because $L_n l_n = 0$ and therefore, by the choice of K_1, the map $\mathcal{H}_\lambda$ is fixed point free on ∂U_n. Since $l_n \in U_n$, we have that $\mathcal{H}_0 \equiv l_n$ is essential and by topological transversality, $\mathcal{H}_1$ will be essential, so $\mathcal{H}_1$ has a fixed point. This fixed point is a solution to (1.2.n). ■

THEOREM 1. – *If f satisfies the conditions of Lemma* 1 *then the problem* (P) *has at least one solution in* $BC^2(R_+)$.

PROOF. – Let K be the constant from Lemma 1 and let n be a positive integer, Consider the problem (2.1.n). By Lemma 1 there exists a solution $u_n \in C^2[0,n]$ to (2.1.n) with $|u^{(i)}(t)| \leq K$, $t \in [0,n]$, $0 \leq i \leq 2$. Define $y_n(t) = u_n(t)$, $t \in [0,n]$, and $y_n(t) = 0$, $t \geq n$. By the Arzela-Ascoli theorem there exists a sequence $n_j \to \infty$ and a continuously differentiable function z_1 on $[0,1]$ such that $y_{n_j}^{(i)}(t) \to z_1^{(i)}(t)$ uniformly on $[0,1]$ as $n_j \to \infty$ for $i = 0, 1$. Again by the Arzela-Ascoli theorem there is a subsequence $n_j^1 \to \infty$ of $\{n_j\}$ and a continuously differentiable function $z_2 \in C^1[0,2]$ such that $y_{n_j^1}^{(i)}(t) \to z_2^{(i)}(t)$ uniformly on $[0,2]$ as $n_j^1 \to \infty$ for $i = 0, 1$. Note that $z_2(t) = z_1(t)$ for $t \in [0,1]$. Inductively we can define z_k on $[0,k]$ for every integer k.

Now define $y(t) = z_k(t)$ for $t \in R_+$ where $k = [t] + 1$. From the above construction

$y \in C^1(R_+)$ is well defined and $\alpha y(0) - \beta y'(0) = r$. Fix $t \in R_+$ and let $k = [t] + 1$. We have

$$y_n'(t) - y_n'(0) = \int_0^t f(s, y_n(s), y_n'(s))\, ds$$

and since there is a sequence $n_j^k \to \infty$ such that $y_{n_j^k}^{(i)}(t) \to z_k^{(i)}(t)$ uniformly on $[0, 1]$ as $n_j^k \to \infty$ for $i = 0, 1$, we have

$$z_k'(t) - z_k'(0) = \int_0^t f(x, z_k(s), z_k'(s))\, ds$$

that is

$$y'(t) - y'(0) = \int_0^t f(s, y(s), y'(s))\, ds\,.$$

Since $t \in R_+$ was arbitrary we deduce that $y \in C^2(R_+)$ is a solution to (P). Also, by construction,

$$\sup_{t \in R_+} \{ |y(t)|,\ |y'(t)|,\ |y''(t)| \} \leq K\,,$$

thus $y \in BC^2(R_+)$. ■

As a particular case of Theorem 1 (for w linear) we obtain a result of GRANAS, GUENTHER, LEE and O'REGAN [8]. The following example shows that our result has a wider applicability than the results from [8].

EXAMPLE. – Consider the boundary value problem

$$x'' = x + (x')^2 \ln(1 + (x')^2)\,, \qquad 0 \leq t < \infty\,, \qquad x(0) - x'(0) = 0\,.$$

By Theorem 1 this problem has at least one solution in $BC^2(R_+)$ but we cannot apply the results of [8]. Since $\lim_{y \to \infty} |f(0, 0, y)| / (y \ln(y)) = \infty$ the result of [1] is also not applicable be the considered problem. ■

THEOREM 2. – *If f satisfies the conditions of Lemma* 1 *and if $u \in BC^2(R_+)$ is a solution of* (P) *with $\lim_{t \to \infty} u(t) = 0$ then $\lim_{t \to \infty} u'(t) = 0$.*

PROOF. – Let us define

$$\psi(t) = \sup_{s \geq t} \{ |u(t)| \}\,, \qquad t \geq 0\,.$$

Then ψ is a decreasing continuous function with $|u(t)| \leq \psi(t)$, $t \geq 0$, and $\lim_{t \to \infty} \psi(t) = 0$.

Since $\lim_{t \to \infty} u(t) = 0$ we have that for each $\varepsilon > 0$ and for each $s > 0$ there exists $t > s$ such that $|u'(t)| \leq \varepsilon$ (apply the mean-value theorem). Fix $\varepsilon > 0$ and let $s \in R_+$ be such that $|u'(s)| \leqslant \varepsilon$. In view of what we just said there is an interval $[s, s_0]$ such that u' has a fixed sign on $[s, s_0]$ and $|u'(s_0)| \leqslant \varepsilon$. Without loss of generality we assume $u'(t) \geqslant 0$ on $[s, s_0]$. Define

$$A = \sup_{t \in R_+,\, |x| \leqslant \psi(0)} \{A(t, x)\}, \qquad B = \sup_{t \in R_+,\, |x| \leqslant \psi(0)} \{B(t, x)\},$$

We obtain

$$-\frac{2u'(t)\, u''(t)}{w((u'(t))^2) + 1} \leqslant 2(A + B)\, u'(t), \qquad s \leqslant t \leqslant s_0,$$

and integrating from t to s_0 we get

$$-H((u'(s_0))^2) + H((u'(t))^2) \leqslant 2(A + B)(\psi(t) + \psi(s_0)) \leqslant 4(A + B)\, \psi(t)$$

because ψ is decreasing, where we denoted $H(x) = \int_0^x ds/(w(s) + 1)$, $x \geqslant 0$. Since $w \in C(R_+, (0, \infty))$ is such that $\int_0^\infty ds/w(s) = \infty$ we have (see [3]) that $\int_0^\infty ds/(w(s) + 1) = \infty$ thus $H\colon R_+ \to R_+$ is a homeomorphism.

We have thus

$$|u'(t)|^2 \leqslant H^{-1}[4(A + B)\, \psi(t) + H(|u'(s_0)|^2)] \leqslant H^{-1}[4(A + B)\, \psi(t) + H(\varepsilon^2)], \qquad s \leqslant t \leqslant s_0 .$$

Repeating the argument we deduce that

$$|u'(t)|^2 \leqslant H^{-1}[4(A + B)\, \psi(t) + H(\varepsilon^2)], \qquad t \geqslant s .$$

Since $\lim_{t \to \infty} H^{-1}[4(A + B)\, \psi(t)] = 0$ we deduce that $\lim_{t \to \infty} u'(t) = 0$. ∎

To show that the problem (P) has a solution $y \in BC^2(R_+)$ satisfying $\lim_{t \to \infty} y(t) = 0$ further more delicate considerations are needed. We show how to do this in Section 4 for a nonlinear semiconductor problem.

LEMMA 3. – *Assume $f(t, x, y)$ is continuous on $R_+ \times R^2$ and satisfies*

(i) *there is a constant $M > 0$ such that $xf(t, x, 0) > 0$ for $|x| \geqslant M$;*

(ii) *there are continuous functions $A(t, x)$, $B(t, x) > 0$ such that*

$$|f(t, x, y)| \leqslant A(t, x)\, w(y^2) + B(t, x)$$

where $w \in C(R_+, (0, \infty))$ is nondecreasing and $\int_0^\infty ds/w(s) = \infty$.

Then (P) *has at least a bounded solution $y \in C^2(R_+)$.*

PROOF. – Similar to the proof of Lemma 1 one can show that for any $n \in N$ the problem (2.1.n) has at least one solution $y_n \in C^2[0, n]$ such that

$$\sup_{t \in [0, n]} \{ |y_n(t)| \} \leq \max \left\{ M, \frac{|r|}{\alpha} \right\}$$

and

$$\sup_{t \in [0, n]} \{ |y_n'(t)|, |y_n''(t)| \} \leq K_n$$

where $K_n > 0$ is a constant depending on n. A construction similar to the one made in the proof of Theorem 1 works with the difference that the obtained function is no more in $BC^2(R_+)$ but only in $C^2(R_+)$ and bounded by $\max\{M, |r|/\alpha\}$. ∎

In [2] the problem of the existence of a bounded solution for the boundary value problem

$$y'' = f(t, y), \qquad 0 \leq t < \infty, \qquad y(0) = r,$$

where $r \in R$ is given, was considered. To establish the consistency of our result with respect to [2] we give the following

EXAMPLE. – The boundary value problem

$$y'' = \frac{1}{t+1} y^3, \qquad 0 \leq t < \infty, \qquad y(0) = 1,$$

has a bounded solution by Theorem 3. Since $\lim_{t \to \infty} |f(t, y) - f(t, 0)|/|y| = 0$ we cannot apply the result of [2]. ∎

3. – Applications to nonlinear mechanics.

Consider a circular membrane of radius R and thickness r subjected to a normal uniform pressure p. We assume that the deformation is rotationally symmetric. Let x be the radial coordinate. The membrane equations can be reduced (see [9]) to the nonlinear ordinary differential equation for the dimensionless radial stress $y(x)$,

$$y'' = -\frac{k}{y^2} - \frac{3}{x} y', \qquad 0 < x \leq 1, \tag{3.1}$$

where $k > 0$ is a constant. To complete the formulation, conditions are required at the center $x = 0$ and at the edge of the plate, $x = 1$. The assumed symmetry and regularity imply that $y(0)$ is regular and $y'(0) = 0$. At the edge we consider the condition $y(1) = \lambda > 0$ where the prescribed constant λ is proportional to the radial stress applied to the boundary. We have thus the boundary conditions

$$y(1) = \lambda, \tag{3.2}$$

$$y'(0) = 0, \tag{3.3}$$

and $y(0)$ should be regular.

Making a change of variables we replace the problem (3.1)-(3.2) with a boundary value problem on $0 \leq t < \infty$ and using our results we prove the existence of a bounded solution. Returning to the initial variable it turns out that the solution satisfies also the condition (3.3) and $y(0)$ is regular.

Let us first prove

LEMMA 2. – *Suppose $f \in C(R_+ \times R^2, R)$ is such that $f(t, x, y)$ is strictly increasing as a function of x and nondecreasing as a function of y for fixed t. Let $\alpha > 0$, $\beta \geq 0$, $r \in R$. If $u_1, u_2 \in C^2(R_+)$ satisfy*

$$\alpha u_1(0) - \beta u_1'(0) \geq r, \qquad \alpha u_2(0) - \beta u_2'(0) \leq r,$$

and on the set $\{t \in R_+ : u_1(t) \leq u_2(t)\}$ we have

$$u_1'' \leq f(t, u_1, u_1'), \qquad u_2'' \geq f(t, u_2, u_2'),$$

then $u_2(t_0) > u_1(t_0)$ for some $t_0 \geq 0$ implies $\lim_{t \to \infty} (u_2(t) - u_1(t)) = \infty$.

PROOF. – Define $u(t) = u_2(t) - u_1(t)$, $t \geq 0$.

Let us first show that $u'(t) > 0$ for $t \geq t_0$.

Fix $t_1 \geq t_0$ and suppose $u'(t_1) \leq 0$. Since $u(t_0) > 0$ we have that u attains a positive maximum on $[0, t_1]$ at some point $t_2 \in [0, t_1]$. If $t_2 = 0$ we would have $u'(0) \leq 0$ thus, since $\alpha u(0) - \beta u'(0) \leq 0$, we obtain $u(0) \leq 0$, contradicting the positivity of the maximum. We deduce that $t_2 \in (0, t_1]$ with $u'(t_2) = u_2'(t_2) - u_1'(t_2) = 0$ so that $u(t_2) = u_2(t_2) - u_1(t_2) > 0$ implies

$$0 \geq u''(t_2) = u_2''(t_2) - u_1''(t_2) \geq f(t_2, u_2(t_2), u_2'(t_2)) - f(t_2, u_1(t_2), u_1'(t_2)) > 0$$

contradiction.

Thus $u'(t) > 0$, $t \geq t_0$, so $u(t) = u_2(t) - u_1(t) \geq u(t_0) > 0$, $t \geq t_0$. We obtain that

$$u''(t) \geq f(t, u_2(t), u_2'(t)) - f(t, u_1(t), u_1'(t)) > 0, \qquad t \geq t_0,$$

thus $u'(t) \geq u'(t_0) > 0$ for $t \geq t_0$ and we get

$$u(t) - u(t_0) \geq u'(t_0)(t - t_0) \to \infty$$

as $t \to \infty$. ■

THEOREM 4. – *The problem* (3.1)-(3.2) *has a solution $y \in C^2[0, 1]$ satisfying*

$$\lambda \leq y(x) \leq \lambda + \frac{k}{8\lambda^2}(1 - x^2), \qquad 0 \leq x \leq 1.$$

PROOF. – Let us change independent variables with $x = 1/(t+1)$, $0 \leq t < \infty$. The problem (3.1)-(3.2) is transformed to

$$y'' = -\frac{k}{(t+1)^4}\frac{1}{y^2} + \frac{1}{t+1}y', \qquad 0 \leq t < \infty, \tag{3.4}$$

$$y(0) = \lambda, \tag{3.5}$$

We cannot apply Theorem 1 immediately because of the singularity of (3.4) at $y=0$.

Let $y_2(t)=\lambda$, $t\geqslant 0$. Observe that

$$y_2''\geqslant -\frac{k}{(t+1)^4}\frac{1}{y_2^2}+\frac{1}{t+1}y_2', \qquad t\geqslant 0.$$

Also, if we denote $y_1(t)=\lambda+(k/8\lambda^2)(1-1/(t+1)^2)$, $t\geqslant 0$, we see that $y_1(0)=\lambda$ and

$$y_1''\leqslant -\frac{k}{(t+1)^4}\frac{1}{y_1^2}+\frac{1}{t+1}y_1', \qquad t\geqslant 0.$$

Let us consider the modified problem of (3.4)-(3.5),

$$y''=f(t,y,y'), \qquad 0\leqslant t<\infty, \tag{3.6}$$

$$y(0)=\lambda, \tag{3.7}$$

where

$$f(t,y,z)=\begin{cases} -\dfrac{k}{(t+1)^4}\dfrac{1}{y^2}+\dfrac{1}{t+1}z+y-y_1(t), & \text{if } y>y_1(t),\\ -\dfrac{k}{(t+1)^4}\dfrac{1}{y^2}+\dfrac{1}{t+1}z, & \text{if } y_1(t)>y\geqslant\lambda,\\ -\dfrac{k}{(t+1)^4}\dfrac{1}{\lambda^2}+\dfrac{1}{t+1}z+y-\lambda, & \text{if } y\leqslant\lambda. \end{cases}$$

One can easily verify that f satisfies the conditions of Theorem 1 and for each fixed t we have that $f(t,y,z)$ is strictly increasing as a function of y and z.

Theorem 1 now applies to the problem (3.6)-(3.7) to guarantee the existence of a solution $y\in BC^2(R_+)$.

By Lemma 2 we have that $y(t)\leqslant y_1(t)$, $t\geqslant 0$ (otherwise we would have that $\lim_{t\to\infty}(y(t)-y_1(t))=\infty$ which contradicts the boundedness of y). The same argument and Lemma 2 shows that $y(t)\geqslant y_2(t)=\lambda$, $t\geqslant 0$, and so y is a solution to the unmodified problem (3.4)-(3.5).

Returning to the problem (3.1)-(3.2) we obtain a solution $y\in C^2(0,1]$ with

$$\lambda\leqslant y(x)\leqslant\lambda+\frac{k}{8\lambda^2}(1-x^2), \qquad 0<x\leqslant 1. \tag{3.8}$$

Multiplying now (3.1) by x^3 and integrating from 1 to x, we find

$$x^3y'(x)-y'(1)=-k\int_1^x\frac{s^3\,ds}{y^2(s)}, \qquad 0<x\leqslant 1.$$

Dividing by x^3 and integrating again from 1 to x, we get (in view of (3.2))

$$y(x) = \lambda + \frac{y'(1)}{2}\left(1 - \frac{1}{x^2}\right) - k\int_1^x \frac{1}{t^3}\left\{\int_1^t \frac{s^3\,ds}{y^2(s)}\right\} dt\,, \qquad 0 < x \leq 1\,.$$

At this point an integration by parts yields

$$(3.9) \quad y(x) = \lambda + \frac{y'(1)}{2}\left(1 - \frac{1}{x^2}\right) + \frac{k}{2x^2}\int_1^x \frac{s^3\,ds}{y^2(s)} - \frac{k}{2}\int_1^x \frac{s\,ds}{y^2(s)}\,, \qquad 0 < x \leq 1\,.$$

Multiplying by x^2 we get

$$x^2 y(x) = \lambda x^2 + \frac{y'(1)}{2}(x^2 - 1) + \frac{k}{2}\int_1^x \frac{s^3\,ds}{y^2(s)} - \frac{k}{2}x^2\int_1^x \frac{s\,ds}{y^2(s)}\,, \qquad 0 < x \leq 1\,.$$

Letting $x \to 0$ (in view of (3.8)) we find that

$$y'(1) = -k\int_0^1 \frac{s^3\,ds}{y^2(s)}$$

and replacing this in (3.9) yields

$$(3.10) \qquad y(x) = \lambda - \frac{k}{2}\int_0^1 \frac{s^3\,ds}{y^2(s)} + \frac{k}{2x^2}\int_0^x \frac{s^3\,ds}{y^2(s)} + \frac{k}{2}\int_x^1 \frac{s\,ds}{y^2(s)}\,, \qquad 0 < x \leq 1\,.$$

so that $y(0)$ is regular.

The fact that $y \in C^2[0, 1]$ and $y'(0) = 0$ follows upon successive differentiation of (3.10) and use of l'Hôpital's rule. ■

The problem (3.1)-(3.3) has been treated in [5]. The method of DICKEY [5] consisted in using an iterative scheme for the integral equation (3.10). Our method shows the existence of a solution for all $\lambda > 0$ whereas the method of Dickey applies only to $\lambda^3 > 4/j_{11}^2$ (here j_{11} is the smallest root of the Bessel function of first order).

4. – Applications to semiconductor devices.

In studying the theory of semiconductor devices one is led to the boundary value problem

$$(4.1) \qquad y'' = f(t, y)\,, \qquad 0 \leq t < \infty\,,$$

$$(4.2) \qquad y'(0) - ay(0) = r\,,$$

$$(4.3) \qquad \lim_{t \to \infty} y(t) = 0\,,$$

where $a > 0$ and $r \in R$ are given constants.

Examples of functions f which appear in semiconductor applications are $f(t, y) = \lambda y$ and $f(t, y) = A \sinh(\lambda y)$ with A, $\lambda > 0$ constants. For a physical discussion of the problem (4.1)-(4.3) we refer to [10].

Our results enable us to give an existence theorem for the problem (4.1)-(4.3). Our hypotheses guarantee also uniqueness of the solution.

THEOREM 5. – *Assume that* $f \in C(R_+ \times R, R)$ *satisfies*:

(i) $(\partial f/\partial y)(t, y)$ *exists and there is a constant* $k > 0$ *such that* $(\partial f/\partial y)(t, y) \geq k$ *on* $R_+ \times R$;

(ii) $\lim_{t \to \infty} f(t, 0) = 0$.

Then the problem (4.1)-(4.3) *has a unique solution.*

PROOF. – By the mean-value theorem we have

$$xf(t, x) = x[f(t, x) - f(t, 0)] + xf(t, 0) \geq kx^2 + xf(t, 0), \; t \in R_+, \quad x \in R.$$

Since $\lim_{t \to \infty} f(t, 0) = 0$ we have that the hypotheses of Lemma 3 are satisfied and so there exists a bounded function $y \in C^2(R_+)$ which satsfies (4.1), (4.2).

We will show that $\lim_{t \to \infty} y(t) = 0$.

By quadrature we see that the boundary value problem

$$x'' = kx + g(t), \quad 0 \leq t < \infty, \qquad x'(0) - ax(0) = r_0, \quad \lim_{t \to \infty} x(t) = 0, \tag{4.4}$$

where $g \in C(R_+)$ is such that $\lim_{t \to \infty} g(t) = 0$, has a unique solution given by

$$x(t) = -\frac{r_0 + \int_0^\infty e^{-\sqrt{k}s} g(s)\, ds}{a + \sqrt{k}} e^{-\sqrt{k}t} - \int_0^t e^{-\sqrt{k}(t-s)} \left[\int_s^\infty e^{-\sqrt{k}(\tau - s)} g(\tau)\, d\tau \right] ds, \qquad t \in R_+. \tag{4.5}$$

The above formula shows that x is nonnegative (nonpositive) on R_+ if $r_0 \leq 0$ and $g(t) \leq 0$, $t \in R_+$ (respectively $r_0 \geq 0$ and $g(t) \geq 0$, $t \in R_+$).

Let u_1 be the solution of (4.4) with $r_0 = r - |r|$, $g(t) = f(t, 0) - |f(t, 0)|$, $t \in R_+$. We have that $u_1(t) \geq 0$, $t \in R_+$, and since $f(t, x)$ is strictly increasing in x for fixed t and

$$f(t, x) \geq kx + f(t, 0) \geq kx + f(t, 0) - |f(t, 0)|, \qquad t \in R_+, \quad x \in R_+,$$

(by the mean-value theorem) we deduce by Lemma 2 (u_1 and y being bounded on R_+) that

$$y(t) \leq u_1(t), \qquad t \in R_+ .$$

Let now u_2 be the solution of (4.4) with $r_0 = r + |r|$, $g(t) = f(t, 0) + |f(t, 0)|$, $t \in R_+$. We have that $u_2(t) \leq 0$, $t \in R_+$, and since

$$f(t, x) \leq kx + f(t, 0) \leq kx + f(t, 0) + |f(t, 0)|, \qquad t \in R_+ , \quad x \leq 0 ,$$

we deduce by Lemma 2 that

$$u_2(t) \leq y(t), \qquad t \in R_+ ,$$

thus

$$u_2(t) \leq y(t) \leq u_1(t), \qquad t \in R_+ .$$

Now, since $\lim_{t \to \infty} u_1(t) = \lim_{t \to \infty} u_2(t) = 0$ we obtain that $\lim_{t \to \infty} y(t) = 0$, thus y satisfies (4.3).

In order to complete the proof we have to show that y is the unique solution of (4.1)-(4.3).

Suppose y_1 is another solution of (4.1)-(4.3) and define $z(t) = (y(t) - y_1(t))^2$, $t \in R_+$. In view of (4.3), if z is not identically zero on R_+ it must have a positive maximum at some point $t_0 \in R_+$. We cannot have $t_0 = 0$ (this would imply $0 \geq z'(0) = 2az(0) > 0$) thus $t_0 > 0$ and we obtain $z'(t_0) = 0$ and

$$0 \geq z''(t_0) = 2(y(t_0) - y_1(t_0))[f(t_0, y(t_0)) - f(t_0, y_1(t_0))] \geq 2kz(t_0) > 0$$

contradiction and this shows that the solution is unique. ■

If in Theorem 5 we assume the additional condition that $f(t, y)$ is bounded when y varies in a bounded interval and $t \in R_+$, we can apply Theorem 1 (instead of Theorem 3) to deduce that (4.1)-(4.3) has a unique solution y and $y \in BC^2(R_+)$. Moreover, by Theorem 2 we have that $\lim_{t \to \infty} y'(t) = 0$.

COROLLARY [8]. – *In addition to the hypotheses of Theorem* 5 *assume that* $(\partial f / \partial y)(t, y)$ *is bounded for* $t \in R_+$ *and* y *varying in bounded intervals. Then* (4.1)-(4.3) *has a unique solution and*

$$\lim_{t \to \infty} y'(t) = \lim_{t \to \infty} y''(t) = 0 .$$

PROOF. – An application of the mean-value theorem shows that $f(t, y)$ is bounded for $t \in R_+$ and y varying in bounded intervals since

$$f(t, y) - f(t, 0) = y \left[\frac{\partial f}{\partial y}(t, \xi) \right]$$

for some $|\xi| \geq |y|$ and $\lim_{t \to \infty} f(t, 0) = 0$.

In view of the previous remarks all we have to show is that $\lim_{t\to\infty} y''(t) = 0$ holds.

The differential equation yields

$$y''(t) = [f(t, y(t)) - f(t, 0)] + f(t, 0) = y(t)\left[\frac{\partial f}{\partial y}(t, \xi(t))\right] + f(t, 0),$$

where $(\partial f/\partial y)(t, \xi(t))$ is obtained by the mean-value theorem, and this proves our claim. ■

EXAMPLE. – Consider the problem

$$y'' = (t+1)(y+y^3), \quad 0 \le t < \infty, \qquad y'(0) = y(0), \qquad \lim_{t\to\infty} y(t) = 0.$$

By Theorem 5 we have that there is a unique solution to this problem. The results from [8] are not applicable. ■

THEOREM 6. – *Assume that* $f \in C(R_+ \times R, R)$ *satisfies:*

(i) $(\partial f/\partial y)(t, y)$ *exists and there is a constant* $k > 0$ *such that* $(\partial f/\partial y)(t, y) \ge k$ *on* $R_+ \times R$;

(ii) $f(t, 0) \in L^1(R_+)$.

Then the problem (4.1)-(4.3) *has a unique solution.*

PROOF. – In order to repeat the steps of the proof of Theorem 5 all we have to show is that if $g \in L^1(R_+) \cap C(R_+)$ then $\lim_{t\to\infty} x(t) = 0$ where x is given by (4.5).

It is clear that if $g \in L^1(R_+)$ then x is well defined and bounded on R_+. Let us first prove that $x \in L^1(R_+)$.

An integration by parts shows that

$$e^{-\sqrt{k}t}\int_0^t e^{2\sqrt{k}s}\left[\int_s^\infty e^{-\sqrt{k}\tau} g(\tau)\, d\tau\right] ds =$$

$$= -\frac{1}{2\sqrt{k}} e^{-\sqrt{k}t}\int_0^\infty e^{-\sqrt{k}s} g(s)\, ds + \frac{1}{2\sqrt{k}} e^{\sqrt{k}t}\int_t^\infty e^{-\sqrt{k}s} g(s)\, ds +$$

$$+ \frac{1}{2\sqrt{k}} e^{-\sqrt{k}t}\int_0^t e^{\sqrt{k}s} g(s)\, ds, \qquad t \ge 0.$$

Again by integration by parts we see that

$$\int_0^T e^{\sqrt{k}t}\left[\int_t^\infty e^{-\sqrt{k}s} |g(s)|\, ds\right] dt \le \frac{1}{\sqrt{k}}\int_0^\infty |g(s)|\, ds, \qquad T \ge 0,$$

and

$$\int_0^T e^{-\sqrt{k}t}\left[\int_0^t e^{-\sqrt{k}s}\,|g(s)|\,ds\right]dt \leqslant \frac{1}{\sqrt{k}}\int_0^\infty |g(s)|\,ds\,, \qquad T \geqslant 0\,,$$

This shows that $x \in L^1(R_+)$.

Suppose that

$$\limsup_{t\to\infty} |x(t)| = A > 0\,.$$

Since $x \in L^1(R_+)$ we have that

$$\liminf_{t\to\infty} |x(t)| = 0$$

thus there is a strictly increasing sequence $t_n \to \infty$ as $n \to \infty$ such that

$$x(t_{2n}) > \frac{A}{2}\,, \qquad |x(t_{2n+1})| < \frac{A}{2} \qquad n \geqslant 1\,,$$

or

$$x(t_{2n}) < -\frac{A}{2}\,, \qquad |x(t_{2n+1})| < \frac{A}{2} \qquad n \geqslant 1\,,$$

and $\lim_{n\to\infty} |x(t_{2n})| = A$. Let us assume that $x(t_{2n}) > A/2$, $n \geqslant 1$ (the other case is similar).

By the continuity of x we have that each t_{2n} belongs to some maximal closed interval $I_n = [a_n, b_n]$ such that $x(t) > A/2$, $a_n < t < b_n$. Moreover, since $\{t_n\}_{n\geqslant 1}$ is strictly increasing we have that I_i and I_j are disjoint if $i \neq j$.

On I_n the function x has a maximum at some point $c_n \in (a_n, b_n)$ since $x(a_n) = x(b_n) = A/2$. We have that

$$x(c_n) - x(a_n) = (c_n - a_n)\,x'(c_n) - \int_{a_n}^{c_n} (t - a_n)\,x''(t)\,dt$$

and since $x'(c_n) = 0$ we get

$$|x(c_n) - x(a_n)| \leqslant \int_{a_n}^{c_n} (t - a_n)\,|x''(t)|\,dt \leqslant 2(b_n - a_n)^2 \sup_{t\in[a_n,\, b_n]} \{|x'(t)|\}\,.$$

The boundedness of x and the fact that $g \in L^1(R_+) \cap C(R_+)$ show that x' is also bounded on R_+. Let $M > 0$ be such that $|x'(t)| \leqslant M/2$, $t \in R_+$. We have

$$M(b_n - a_n)^2 \geqslant |x(c_n) - x(a_n)|\,, \qquad n \geqslant 1\,.$$

Since $\lim_{n\to\infty} x(t_{2n}) = A$ and $x(a_n) = A/2$ we deduce that $\lim_{n\to\infty} x(c_n) = A$ (it can't be more than A) so that

$$\liminf_{n\to\infty}(b_n - a_n)^2 \geq \frac{A}{2M} > 0 . \tag{4.6}$$

On the other hand we have that $x(t) > A/2$, $a_n < t < b_n$, $n \geq 1$, thus

$$\int_0^\infty |x(t)|\,dt \geq \sum_{n\geq 1}\int_{I_n}|x(t)|\,dt \geq \frac{A}{2}\sum_{n\geq 1}(b_n - a_n)$$

and since $x \in L^1(R_+)$ we deduce that $\lim_{n\to\infty}(b_n - a_n) = 0$ which is in contradiction with (4.6).

We proved so that $\lim_{t\to\infty} x(t) = 0$.

A repetition of the arguments of the proof of Theorem 5 enables us to conclude. ■

If in Theorem 6 we assume the additional condition that $f(t, y)$ is bounded when y varies in a bounded interval and $t \in R_+$, we obtain (applying Theorem 1 instead of Theorem 3) that (4.1)-(4.3) has a unique solution y and $y \in BC^2(R_+)$. Moreover, by Theorem 2 we get $\lim_{t\to\infty} y'(t) = 0$.

EXAMPLE. – Let $h \in C(R_+) \cap L^1(R_+)$. The problem

$$y'' = \frac{1}{(t+1)}(y^5 + y + 1) + h(t), \quad 0 \leq t < \infty , \qquad y'(0) - y(0) = 1 , \qquad \lim_{t\to\infty} y(t) = 0 ,$$

has a unique solution y and $y \in BC^2(R_+)$ with $\lim_{t\to\infty} y'(t) = 0$. ■

Observe that the common examples

$$f(t, y) = \lambda y , \qquad \text{and} \qquad f(t, y) = A \sinh(\lambda y)$$

with A, $\lambda > 0$ satisfy the conditions of Theorem 5 and Theorem 6.

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