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Abelian Integrals for Cubic Vector Fields (*).

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Abstract. – *It is proved in this paper that the lowest upper bound of the number of the isolated zeros of the Abelian integral*

$$I(h) = \oint_{\Gamma_h} (\alpha + \beta x + \gamma x^2)\, y\, dx$$

is two for $h \in (-1/12, 0)$, *where* Γ_h *is the compact component of* $H(x, y) = (1/2)\, y^2 + (1/3)\, x^3 + (1/4)\, x^4 = h$, *and* α, β, γ *are arbitrary constants.*

1. – Introduction.

Consider the Abelian integral

$$I(h) = \oint_{\Gamma_h} Y\, dx - X\, dy\,, \tag{1.1}$$

where H, X and Y are real polynomials of x and y, Γ_h is the compact component of $H = h$. Finding the lowest upper bound for the number of zeros of $I(h)$ is called the weakened Hilbert 16th problem, posed by V. I. Arnold [1], and this problem is closely related to determining the number of limit cycles of the perturbed system

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y} + \varepsilon X(x, y)\,, \\[2mm] \dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x} + \varepsilon Y(x, y)\,, \end{cases} \tag{1.2}$$

where $0 < |\varepsilon| \ll 1$.

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In particular, suppose [2]

$$H(x, y) = \frac{1}{2} y^2 + U(x) = h , \tag{1.3}$$

where $U(x)$ is a real polynomial of x with degree n. Finding the number of zeros of $I(h)$ is one of ten problems in [2]. When $n = 3$, this problem was solved by [9], [11], [13] etc. When $n = 4$, some results were given by [7], [12], [14], but this case is far from completely solved. In this paper, we consider the case $n = 4$ and the Hamiltonian vector field $dH = 0$ possesses two critical points, one of which is a center and the other is a cusp; then (1.3) can be reduced to

$$H(x, y) = \frac{1}{2} y^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 = h . \tag{1.4}$$

The perturbed system has the following form

$$\begin{cases} \dot{x} = y , \\ \dot{y} = - x^2 - x^3 + \varepsilon(\alpha + \beta x + \gamma x^2) y , \end{cases} \tag{1.5_ε}$$

where α, β, γ are arbitrary constants, $0 < |\varepsilon| \ll 1$.

The unperturbed system $(1.5)_0$ has the first integral (1.4) and the closed level sets $\Gamma_h = \{(x, y) \mid H = h, h \in (-1/12, 0) \cup (0, \infty)\}$ are shown in fig. 1.1. The origin (0,0) is a cuspidal point which has two hyperbolic sectors and two separatrices: unstable Γ_0^u and stable Γ_0^s. $\Gamma_{-1/12}$ and Γ_0 correspond to the center $(-1, 0)$ and cuspidal loop with $\Gamma_0^u = \Gamma_0^s$ respectively.

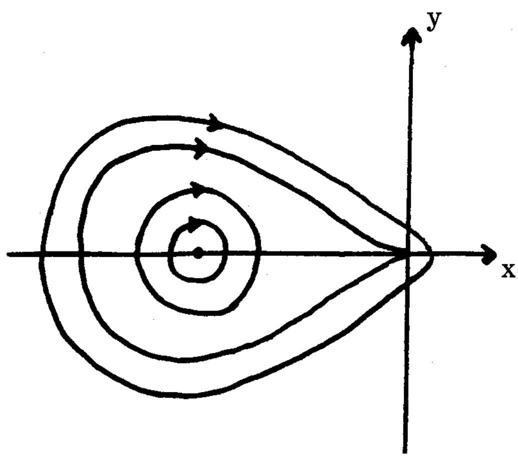

Fig. 1.1.

Denote

$$I_i(h) = \oint_{\Gamma_h} x^i y\, dx\,, \qquad i = 0, 1, 2\,, \tag{1.6}$$

$$I(h) = \alpha I_0 + \beta I_1 + \gamma I_2\,, \tag{1.7}$$

where $h \in (-1/12, 0) \cup (0, +\infty)$. The central result of this paper is the following:

THEOREM 1.1. – *Either the Abelian integral $I(h)$ vanishes identically, or the lowest upper bound of the number of its zeros is two in the interval $(-1/12, 0)$.*

For system $(1.5)_\varepsilon$, we have

THEOREM 1.1*. – *For all sufficiently small ε, either the system $(1.5)_\varepsilon$ is Hamiltonian, or the lowest upper bound for the number of limit cycles of $(1.5)_\varepsilon$ bifurcating from Γ_h is two for $h \in (-1/12, 0)$.*

For the homoclinic loop Γ_{h_0} of a hyperbolic saddle, there is the well known asymptotic expansion of R. Roussarie [15]

$$I(h) = c_0 + c_1(h - h_0)\ln(h - h_0) + c_2(h - h_0) + \dots\,.$$

However, it seems that no one has given an asymptotic expansion of $I(h)$ in the neighbourhood of a cuspidal loop. In this paper, using analytic theory of ordinary differential equations, we get

THEOREM 1.2. – *For system $(1.5)_\varepsilon$, near the value $h = 0$ corresponding to cuspidal loop, $I(h)$ has the following asymptotic expansion:*

$$I(h) = d_0 + d_1|h|^{5/6} + d_2 h + \sum_{k=1}^{\infty} a_k(\alpha, \beta, \gamma)|h|^{5/6}h^k \sum_{k=0}^{\infty} b_k(\alpha, \beta, \gamma)|h|^{7/6}h^k\,, \tag{1.8}$$

and the following statements are equivalent:

i) $d_0 = d_1 = d_2 = 0$,

ii) $\alpha = \beta = \gamma = 0$,

iii) $I(h) \equiv 0$,

where $|h| < 1/12$, $d_0 = (4/27)\sqrt{2}\pi\alpha - (10/81)\sqrt{2}\pi\beta + (28/243)\sqrt{2}\pi\gamma$, $d_1 = \alpha C_1$, $d_2 = -2\sqrt{2}\pi\beta + (4/3)\sqrt{2}\pi\gamma$, *and* $a_k(\alpha, \beta, \gamma)$, $b_k(\alpha, \beta, \gamma)$ *are linear functions of* α, β, γ, C_1 *is a constant with* $C_1 < 0$ *for* $h < 0$ *and* $C_1 > 0$ *for* $h > 0$.

The relationship between the expansion (1.8) and the number of limit cycles of $(1.5)_\varepsilon$ near Γ_0 is still open. We may conjecture as follows: If $d_0 = 0$ (resp. $d_0 = d_1 = 0$) and $d_1 \neq 0$ (resp. $d_2 \neq 0$), then there exists a neighbourhood of the loop Γ_0 containing at most 1 limit cycle of $(1.5)_\varepsilon$ (resp. 2) for $0 < |\varepsilon| \ll 1$.

The paper is organized as follows: In sect. 2, the monotonicity of $P(h) = I_1(h)/I_0(h)$, $Q(h) = I_2(h)/I_0(h)$ and $R(h) = I_2(h)/I_1(h)$ is proved, which implies that the curve $\Omega = \{(P, Q) \mid P = P(h), Q = Q(h)\}$ can be defined in PQ-plane. In sect. 3, the asymptotic expansions of $I(h)$ near its endpoints are given and Theorem 1.2 is proved, which shows that the lowest upper bound of the number of zeros of $I(h)$ is at least 2. A simple but important fact is that the ratio $g(h) = I''(h)/I_0''(h)$ satisfies a Riccati equation and $I''(h)$ can be denoted as a linear combination of $I_0''(h)$ and $I_1''(h)$. This is crucial for our analysis. From the beginning of sect. 5, instead of estimating the number of zeros of $I(h)$, we will prove that $I(h)$ has at most two inflection points, i.e., $I''(h)$ has at most two zeros in $(-1/12, 0)$, which implies that the lowest upper bound of the number of zeros of $I(h)$ does not exceed three. Qualitative analysis of Riccati equation of $I_1''(h)/I_0''(h)$ yields the monotonicity of $I_1''(h)/I_0''(h)$, which gives the upper bound of the number of zeros of $I''(h)$ in most cases. In sect. 6, by applying the fact that the zeros of $g(h)$ equal the zeros of $I''(h)$, we estimate the upper bound of the number of zeros of $I''(h)$ in those cases which are not discussed in sect. 5. Finally, the asymptotic expansion of $I(h)$ near its endpoints shows the main results of this paper.

2. - Monotonicity of $P(h)$, $Q(h)$ and $R(h)$.

Let

$$P(h) = \frac{I_1(h)}{I_0(h)}, \qquad Q(h) = \frac{I_2(h)}{I_0(h)}, \qquad R(h) = \frac{I_2(h)}{I_1(h)}. \tag{2.1}$$

It follows from Green's formula that

LEMMA 2.1. - *For* $h \in (-1/12, 0)$,

i) $I_0(h) > 0,\ I_1(h) < 0,\ I_2(h) > 0,\ I_0'(h) > 0,\ I_2'(h) > 0,$

ii) $P(h) < 0,\ Q(h) > 0.$

Rewrite (1.4) in the form

$$\frac{1}{2} y^2 + \Phi(x) = h, \tag{2.2}$$

where $\Phi(x) = x^3/3 + x^4/4$ satisfying

$$\Phi'(x)(x+1) > 0, \qquad \text{for } x \in \left(-\frac{4}{3}, -1\right) \cup (-1, 0). \tag{2.3}$$

For any $x \in (-4/3, -1)$, there exists a unique $\tilde{x} \in (-1, 0)$, such that

$$\Phi(x) = \Phi(\tilde{x}), \qquad -\frac{4}{3} < x < -1 < \tilde{x} < 0. \tag{2.4}$$

Therefore, we can define a function $\tilde{x} = \tilde{x}(x)$ for $x \in (-4/3, -1)$ satisfying (2.4). The in-equality (2.3) implies

$$\frac{d\tilde{x}}{dx} = \frac{\Phi'(x)}{\Phi(\tilde{x})} < 0 . \tag{2.5}$$

Define

$$\xi_{km}(x) = \frac{x^k \Phi'(\tilde{x}) - \tilde{x}^k \Phi'(x)}{x^m \Phi'(\tilde{x}) - \tilde{x}^m \Phi'(x)} , \tag{2.6}$$

where $k = 1, 2,\ m = 0, 1.$

By Theorem 1 of [10], we have

LEMMA 2.2. – *If* $\xi'_{km}(x) > 0$ (resp. < 0), *then* $u'_{km}(h) < 0$ (resp. > 0), *where* $u_{km}(h) = I_k(h)/I_m(h)$.

THEOREM 2.3. – *For* $h \in (-1/12, 0)$, $P'(h) > 0$, $Q'(h) < 0$, $R'(h) > 0$.

PROOF. – For $P(h) = I_1(h)/I_0(h)$,

$$\xi_{10}(x) = \frac{x\Phi'(\tilde{x}) - \tilde{x}\Phi'(x)}{\Phi'(\tilde{x}) - \Phi'(x)} = \frac{x\tilde{x}(1 + x + \tilde{x})}{\tilde{x}^2 + x^2 + \tilde{x}x + x + \tilde{x}} ,$$

$$\xi'_{10}(x) = \frac{A(x, \tilde{x})}{(\tilde{x}^2 + x\tilde{x} + x^2 + x + \tilde{x})^2} ,$$

where

$$A(x, \tilde{x}) = \left[\tilde{x}(1 + x + \tilde{x}) + x\frac{d\tilde{x}}{dx}(1 + x + \tilde{x}) + x\tilde{x}\left(1 + \frac{d\tilde{x}}{dx}\right)\right](\tilde{x}^2 + x^2 + x\tilde{x} + x + \tilde{x}) -$$

$$-\left(2\tilde{x}\frac{d\tilde{x}}{dx} + 2x + \tilde{x} + x\frac{d\tilde{x}}{dx} + 1 + \frac{d\tilde{x}}{dx}\right)x\tilde{x}(1 + x + \tilde{x}) =$$

$$= \tilde{x}^2(\tilde{x} + 1)(\tilde{x} + 1 + 2x) + x^2(x + 1)(x + 1 + 2\tilde{x})\frac{d\tilde{x}}{dx} .$$

It follows from (2.4), (2.5) that $\tilde{x} + 1 > 0$, $x + 1 < 0$, $\tilde{x} + 2x + 1 = (x + \tilde{x}) + (x + 1) < 0$, $x + 1 + 2\tilde{x} < 0$, $d\tilde{x}/dx < 0$, which implies $A(x, \tilde{x}) < 0$, i.e., $\xi'_{10}(x) < 0$. Hence, by Lemma 2.2, $P'(h) > 0$.

For $Q(h)$ and $R(h)$,

$$\xi_{20}(x) = \frac{x^2\Phi'(\tilde{x}) - \tilde{x}^2\Phi'(x)}{\Phi'(\tilde{x}) - \Phi'(x)} = \frac{x^2\tilde{x}^2}{\tilde{x} + x + \tilde{x}^2 + x\tilde{x} + x^2} ,$$

$$\xi'_{20}(x) = \frac{B(x, \tilde{x})}{(x + \tilde{x} + \tilde{x}^2 + x\tilde{x} + x^2)^2},$$

and

$$\xi_{21}(x) = \frac{x^2 \Phi'(\tilde{x}) - \tilde{x}^2 \Phi'(x)}{x\Phi'(\tilde{x}) - \tilde{x}\Phi'(x)} = \frac{x\tilde{x}}{1 + x + \tilde{x}},$$

$$\xi'_{21}(x) = \frac{C(x, \tilde{x})}{(1 + x + \tilde{x})^2},$$

where

$$B(x, \tilde{x}) = x\tilde{x}\left[\tilde{x}(\tilde{x} + 1)(x + 2\tilde{x}) + x\frac{d\tilde{x}}{dx}(x + 1)(\tilde{x} + 2x)\right] > 0,$$

$$C(x, \tilde{x}) = \tilde{x}(1 + \tilde{x}) + x(1 + x)\frac{d\tilde{x}}{dx} < 0,$$

which implies $\xi'_{20}(x) > 0$, $\xi'_{21}(x) < 0$. Therefore, by Lemma 2.2, $Q'(h) < 0$ and $R'(h) > 0$.

3. – Picard-Fuchs equation and the asymptotic behaviour of $I(h)$ near its endpoints.

In this section, we shall derive Picard-Fuchs equation of $I_i(h)$, $i = 0, 1, 2$ and describe the behaviour of $I(h)$ near $h = 0$ and $h = -1/12$.

LEMMA 3.1.

i) $I_i(-(1/12)^+) = 0$, $P(-(1/12)^+) = -1$, $Q(-(1/12)^+) = 1$,

ii) $I_0(0) = (4/27)\sqrt{2}\pi$, $I_1(0) = -(10/81)\sqrt{2}\pi$, $I_2(0) = (28/243)\sqrt{2}\pi$, $P(0) = -5/6$, $Q(0) = 7/9$,

iii) *For* $h \in [-1/12, 0]$, $-1 \leq P(h) \leq -5/6$, $7/9 \leq Q(h) \leq 1$.

PROOF. – The results i) and ii) are obtained by direct computation. The conclusion iii) follows from the results i), ii) and Theorem 2.3.

PROPOSITION 3.2. – *$I_0(h)$, $I_1(h)$, $I_2(h)$ satisfy the following Picard-Fuchs equation*

$$h(12h + 1)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 9h + 5/6 & -1/6 & -5/4 \\ h & 12h + 7/6 & 5/4 \\ -h & 2h & 15h \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix}, \tag{3.1}$$

which is equivalent to

$$
(3.2)\qquad \begin{cases} 3I_0 = 4hI_0' + \dfrac{1}{3}I_2' \,, \\[2mm] 4I_1 = 4hI_1' - \dfrac{1}{3}I_0 - \dfrac{1}{3}I_2' \,, \\[2mm] 5I_2 = (4h + \dfrac{1}{3})\, I_2' - \dfrac{2}{3}I_1 + \dfrac{1}{3}I_0 \,. \end{cases}
$$

PROOF. – It follows from (1.4) that

$$
(3.3)\qquad \frac{\partial y}{\partial h} = \frac{1}{y}
$$

and

$$
(3.4)\qquad y\,\frac{\partial y}{\partial x} + x^2 + x^3 = 0\,.
$$

Obviously, (3.3) implies that

$$
(3.5)\qquad I_i'(h) = \oint_{\Gamma_h} x^i y^{-1}\, dx\,, \qquad i = 0, 1, 2\,.
$$

Multiplying (3.4) by $x^{n-3}y^{-1}$ and integrating by parts over Γ_h give the following equality

$$
(3.6)\qquad \oint_{\Gamma_h} x^n y^{-1}\, dx = (n-3)\oint_{\Gamma_h} x^{n-4} y\, dx - \oint_{\Gamma_h} x^{n-1} y^{-1}\, dx\,.
$$

This implies

$$
(3.7)\qquad I_3'(h) = -I_2'(h),
$$

$$
(3.8)\qquad I_4'(h) = I_0(h) + I_2'(h),
$$

$$
(3.9)\qquad I_5'(h) = 2I_1(h) - I_0(h) - I_2'(h).
$$

Using (1.4) and (3.5) again,we get

$$
(3.10)\qquad I_k(h) = \oint_{\Gamma_h} \frac{x^k y^2}{y}\, dx =
$$

$$
= \oint_{\Gamma_h} \frac{x^k(2h - (2/3)\,x^3 - (1/2)\,x^4)}{y}\, dx = 2hI_k' - \frac{2}{3}\, I_{k+3}' - \frac{1}{2}\, I_{k+4}'\,.
$$

On the other hand, integrating by parts and using (3.4), (3.5), we have

$$I_k(h) = -\frac{1}{k+1}\oint_{\Gamma_h} x^{k+1}\,dy = \frac{1}{k+1}\oint_{\Gamma_h} x^{k+1}\frac{x^2+x^3}{y}\,dx = \frac{1}{k+1}(I'_{k+3}+I'_{k+4}). \tag{3.11}$$

Eliminating I'_{k+4} from (3.10) and (3.11), we obtain

$$(k+3)\,I_k = 4hI'_k - \frac{1}{3}I'_{k+3}. \tag{3.12}$$

Taking $k = 0, 1, 2$, we have

$$\begin{cases} 3I_0 = 4hI'_0 - \frac{1}{3}I'_3, \\ 4I_1 = 4hI'_1 - \frac{1}{3}I'_4, \\ 5I_2 = 4hI'_2 - \frac{1}{3}I'_5. \end{cases} \tag{3.13}$$

Substituting (3.7)~(3.9) into (3.13), we obtain (3.2), which implies (3.1).

PROPOSITION 3.3 (Behaviour near $h = 0$ and $h = -1/12$).
i)

$$\begin{pmatrix} I_0(h) \\ I_1(h) \\ I_2(h) \end{pmatrix} = C_1|h|^{5/6}\begin{pmatrix} 1-(35/88)\,h+\dots \\ (21/44)\,h+\dots \\ -(6/11)\,h+\dots \end{pmatrix} + \tag{3.14}$$

$$+C_2|h|^{7/6}\begin{pmatrix} 1-(385/208)\,h+\dots \\ -2+(55/26)\,h+\dots \\ -(30/13)\,h+\dots \end{pmatrix} + \begin{pmatrix} (4/27)\sqrt{2}\pi \\ -(10/81)\sqrt{2}\pi - 2\sqrt{2}\pi h \\ (28/243)\sqrt{2}\pi + (4/3)\sqrt{2}\pi h \end{pmatrix},$$

where C_i *is real constant,* $i = 1, 2$, $C_1 < 0$ *for* $h < 0$ *and* $C_1 > 0$ *for* $h > 0$, $|h| <$
$< 1/12$.

ii) *Let* $|h+1/12|<1/12$, *then* $I_i(h)$ $(i=0,1,2)$ *is holomorphic at* $h=-1/12$ *and*

$$
(3.15)\quad \begin{cases}
I_0(h) = I_0'\left(-\frac{1}{12}\right)\left[\left(h+\frac{1}{12}\right)+\frac{31}{24}\left(h+\frac{1}{12}\right)^2+\frac{10465}{1728}\left(h+\frac{1}{12}\right)^3+\dots\right],\\
I_1(h) = I_0'\left(-\frac{1}{12}\right)\left[-\left(h+\frac{1}{12}\right)-\frac{7}{24}\left(h+\frac{1}{12}\right)^2-\frac{2065}{1728}\left(h+\frac{1}{12}\right)^3+\dots\right],\\
I_2(h) = I_0'\left(-\frac{1}{12}\right)\left[\left(h+\frac{1}{12}\right)-\frac{5}{24}\left(h+\frac{1}{12}\right)^2-\frac{695}{1728}\left(h+\frac{1}{12}\right)^3+\dots\right].
\end{cases}
$$

PROOF. – (i) Consider the analytic continuation of $I_i(h)$ from $(-1/12, 0)\cup(0, +\infty)$ to the complex domain $\mathbb{C}$. Using analytic theory of ordinary differential equation [5], [6], we obtain asymptotic expansions of $I_i(h)$, $i=0,1,2$,

$$
(3.16)\quad \begin{pmatrix} I_0(h)\\ I_1(h)\\ I_2(h)\end{pmatrix} = \overline{C}_1 h^{5/6}\begin{pmatrix} 1-(35/88)\,h+\dots\\ (21/44)\,h+\dots\\ -(6/11)\,h+\dots\end{pmatrix} + \overline{C}_2 h^{7/6}\begin{pmatrix} 1-(385/208)\,h+\dots\\ -2+(55/26)\,h+\dots\\ -(30/13)\,h+\dots\end{pmatrix} + \overline{C}_3\begin{pmatrix} 18\\ -15-243h\\ 14+162h\end{pmatrix}
$$

where $\overline{C}_i$ $(i=0,1,2)$ is a complex or real constant, $h\in\mathbb{C}$, $0<|h|<1/12$. We notice that $I_i(h)$ is real for $h\in(-1/12, 0)$, hence the imaginary part of $I_i(h)$ equals zero in (3.16), which implies $\overline{C}_1 = e^{-(5/6)\pi i}C_1$, $\overline{C}_2 = e^{-(7/6)\pi i}C_2$ (C_1, C_2 are real constants) for $h<0$, and $\overline{C}_i$ is real for $h>0$. From Lemma 3.1 and $I_0'(h)>0$, we have $\overline{C}_3 = 2\sqrt{2}\pi/243$ and $\overline{C}_1>0$ for $h>0$. Therefore, $I_i(h)$ is denoted as (3.14).

For $h<0$, it follows from (3.14) that

$$
P(h) = -\frac{5}{6}+\frac{45}{16\pi}\sqrt{2}C_1|h|^{5/6}-\frac{27}{2}h-\frac{63}{16\pi}\sqrt{2}C_2|h|^{7/6}+\dots.
$$

Therefore,

$$
P'(h) = -\frac{75}{32}\sqrt{2}C_1|h|^{-1/6}-\frac{27}{2}+o(1).
$$

If $C_1=0$, then $P'(h) = -27/2+o(1)<0$ as $h\to 0^-$. This contradicts Theorem 2.3. Using Theorem 2.3 again, we have $C_1<0$ for $h<0$.

(ii) Using the same argument as above, we can get ii).

PROOF OF THEOREM 1.2. – (1.8) follows from Proposition 3.3. $d_0 = d_1 = d_2 = 0$ if and only if

$$\begin{cases} \frac{4}{27}\sqrt{2}\pi\alpha - \frac{10}{81}\sqrt{2}\pi\beta + \frac{28}{243}\sqrt{2}\pi\gamma = 0\,, \\ \alpha C_1 = 0\,, \\ -2\sqrt{2}\pi\beta + \frac{4}{3}\sqrt{2}\pi\gamma = 0\,. \end{cases}$$

Noting $C_1 < 0$ and

$$\begin{vmatrix} \frac{4}{27}\sqrt{2}\pi & -\frac{10}{81}\sqrt{2}\pi & \frac{28}{243}\sqrt{2}\pi \\ C_1 & 0 & 0 \\ 0 & -2\sqrt{2}\pi & \frac{4}{3}\sqrt{2}\pi \end{vmatrix} = -\frac{32}{243}\pi^2 C_1 \neq 0\,,$$

therefore, $d_0 = d_1 = d_2 = 0$ if and only if $\alpha = \beta = \gamma = 0$, which implies $I(h) \equiv 0$. Conversely, if $I(h) \equiv 0$, then $d_i (i = 0, 1, 2)$ must equal zero. The theorem is proved.

The intersection points of the lines $\alpha + \beta P + \gamma Q = 0$ with the curve Ω correspond to the zeros of $I(h)$, where

$$\Omega = \left\{ (P, Q) \,|\, P = P(h),\ Q = Q(h),\ h \in \left(-\frac{1}{12}, 0 \right) \right\}, \tag{3.17}$$

or $Q = Q(h(P))$, and $h = h(P)$ is the inverse function of $P = P(h)$, cf. Theorem 2.3.

PROPOSITION 3.4. – *The lowest upper bound N for the number of zeros of $I(h)$ is at least two for $h \in (-1/12, 0)$.*

PROOF. – Theorem 2.3 yields $N \geq 1$. Suppose $N = 1$, which implies that Ω is a straight line in $(-1/12, 0)$. Assume Ω: $\tilde{\alpha} + \tilde{\beta} P + \tilde{\gamma} Q = 0$ and $|\tilde{\alpha}| + |\tilde{\beta}| + |\tilde{\gamma}| \neq 0$, then $I(h) = \tilde{\alpha} I_0 + \tilde{\beta} I_1 + \tilde{\gamma} I_2 \equiv 0$, which contradicts Theorem 1.2. Therefore, $N \geq 2$.

4. – Riccati equation of $I''(h)/I_0''(h)$.

From this section, we begin to estimate the number of inflection points of $I(h)$, i.e. the zeros of $I''(h)$, which determine the upper bound of the number of zeros of $I(h)$. We will derive the Riccati equation satisfied by $I''(h)/I_0''(h)$, which is based on $I_0''(h) \neq 0$, $h \in (-1/12, 0)$, proved by S.N. Chow and L. Gavrilov in [3] [8]. Theorem 2.3 implies that $I(h)$ has at most one zero for $h \in (-1/12, 0)$ if $\alpha\beta\gamma = 0$, $\alpha^2 + \beta^2 + \gamma^2 \neq 0$. Hence, from now on, without loss of generality, we suppose $\beta = 1$ unless the opposite is claimed.

LEMMA 4.1.

$$\text{i) } 4hI_1'' = -\frac{2}{3}I_0' - 4hI_0'', \tag{4.1}$$

$$\text{ii) } I_2'' = 6hI_0'' + 18hI_1''. \tag{4.2}$$

PROOF. – Differentiate once the first two equations in (3.2),

$$4hI_0'' + \frac{1}{3}I_2'' = -I_0', \tag{4.3}$$

$$4hI_1'' - \frac{1}{3}I_0' - \frac{1}{3}I_2'' = 0. \tag{4.4}$$

Eliminating $I_2''(h)$ (resp. I_0') from (4.3) and (4.4) yields (4.1) (resp. (4.2)).

LEMMA 4.2. – *The integral I_0, I_1 satisfy the following equation*

$$2h(12h+1)\begin{pmatrix} I_0''' \\ I_1''' \end{pmatrix} = \begin{pmatrix} -21h - 7/3 & 27h - 1/3 \\ -7h & -51h - 5/3 \end{pmatrix}\begin{pmatrix} I_0'' \\ I_1'' \end{pmatrix}. \tag{4.5}$$

PROOF. – Differentiate twice the second and third equation of (3.2) to get

$$4hI_1''' = \frac{1}{3}I_0'' - 4I_1'' + \frac{1}{3}I_2''', \tag{4.6}$$

$$\left(4h + \frac{1}{3}\right)I_2''' = -3I_2'' + \frac{2}{3}I_1'' - \frac{1}{3}I_0''. \tag{4.7}$$

Differentiating (4.3) once, we have

$$I_2'''(h) = -15I_0'' - 12hI_0'''. \tag{4.8}$$

Substituting (4.2) and (4.8) into (4.6) and (4.7), we have

$$2hI_1''' = -\frac{7}{3}I_0'' - 2I_1'' - 2hI_0''', \tag{4.9}$$

$$2h(12h+1)I_0''' = \left(-21h - \frac{7}{3}\right)I_0'' + \left(27h - \frac{1}{3}\right)I_1'', \tag{4.10}$$

which implies (4.5).

LEMMA 4.3. – *Assume $v = I_1 + \gamma I_2$, $w = v''/I_0''$, then $w(h)$ satisfies the following Ric-*

cati equation

$$(4.11)\qquad 2h(12h+1)(1+18\gamma h)\,w' = -\left(27h-\frac{1}{3}\right)w^2 + \left[216\gamma h^2+(44\gamma-30)\,h+\frac{2}{3}\right]w + (-60\gamma^2+72\gamma)\,h^2+(8\gamma-7)\,h\,.$$

PROOF. – Substituting $I_1 = \nu - \gamma I_2$ into (4.5), we get

$$(4.12)\qquad \begin{cases} 2h(12h+1)\,I_0''' = -\left(21h-\frac{7}{3}\right)I_0'' + \left(27h-\frac{1}{3}\right)(\nu''-\gamma I_2''),\\ 2h(12h+1)\nu''' = 2\gamma h(12h+1)\,I_2''' - 7hI_0'' + \left(-51h-\frac{5}{3}\right)(\nu''-\gamma I_2''). \end{cases}$$

Eliminating I_1'' from $I_1''=\nu''-\gamma I_2''$ and (4.2), we have

$$(4.13)\qquad I_2'' = \frac{1}{1+18\gamma h}\,(6hI_0''+18h\nu'').$$

Substituting (4.13) into (4.12) yieids

$$(4.14)\qquad \begin{cases} 2h(12h+1)(1+18\gamma h)I_0''' =\\ \qquad = \left[-540\gamma h^2+(-40\gamma-21)\,h-\frac{7}{3}\right]I_0'' + \left(27h-\frac{1}{3}\right)\nu'',\\ 2h(12h+1)(1+18\gamma h)\nu''' =\\ \qquad = [(-60\gamma^2+72\gamma)\,h^2+(8\gamma-7)h]I_0'' + \left[-324\gamma h^2+(4\gamma-51)\,h-\frac{5}{3}\right]\nu''. \end{cases}$$

Noticing $w' = (\nu''' I_0'' - \nu'' I_0''')/(I_0'')^2$, the equation (4.11) follows from (4.14).

Define

$$(4.15)\qquad g(h) = \frac{I''(h)}{I_0''(h)}\,,\qquad h\in\left(-\frac{1}{12},0\right).$$

Obviously, $g=\alpha+w$. Theorem 4.4 follows from Lemma 4.3:

THEOREM 4.4. – *$g(h)$ satisfies the following Riccati equation*

$$2h(12h+1)(1+18\gamma h)\,g' = \tag{4.16}$$

$$= -\left(27h - \frac{1}{3}\right)g^2 + \left[216\gamma h^2 + (54\alpha + 44\gamma - 30)\,h + \frac{2}{3}(1-\alpha)\right]g + F(h)$$

where

$$F(h) = (-60\gamma^2 - 216\gamma\alpha + 72\gamma)\,h^2 + \tag{4.17}$$

$$+[-27\alpha^2 + (-44\gamma + 30)\,\alpha + 8\gamma - 7]\,h + \frac{1}{3}\,\alpha(\alpha - 2).$$

5. – Monotonicity of I_1''/I_0'' and relevant results.

Define

$$r(h) = \frac{I_1''}{I_0''}, \qquad h \in \left(-\frac{1}{12}, 0\right).$$

LEMMA 5.1. – *$r(h)$ satisfies the following Riccati equation*

$$2h(12h+1)\,r' = -\left(27h - \frac{1}{3}\right)r^2 + \left(-30h + \frac{2}{3}\right)r - 7h\,. \tag{5.1}$$

PROOF. – (5.1) follows from (4.11) by taking $\gamma = 0$.

Consider the system

$$\begin{cases} \dot{h} = 2h(12h+1), \\ \dot{r} = -\left(27h - \dfrac{1}{3}\right)r^2 + \left(-30h + \dfrac{2}{3}\right)r - 7h\,. \end{cases} \tag{5.2}$$

The zero isocline $r^{\pm}(h)$ of system (5.2) is determined by the algebraic curve

$$G(h, r) = -\left(27h - \frac{1}{3}\right)r^2 + \left(-30h + \frac{2}{3}\right)r - 7h = 0\,, \tag{5.3}$$

where

$$r^+(h) = \frac{45h - 1 + 3\sqrt{36h^2 - (23/3)h + 1/9}}{1 - 81h}, \tag{5.4}$$

$$r^-(h) = \frac{45h - 1 - 3\sqrt{36h^2 - (23/3)h + 1/9}}{1 - 81h}. \tag{5.5}$$

Obviously, $36h^2 - (23/3)h + 1/9 > 0$ for $h \in (-\infty, 0)$.

LEMMA 5.2.

i) $r^+\left(-\frac{1}{12}\right) = -\frac{7}{31}$, $r^-\left(-\frac{1}{12}\right) = -1$, $r^+(0) = 0$, $r^-(0) = -2$,

ii) $\left.\frac{dr^+(h)}{dh}\right|_{h=-1/12} = \frac{770}{961}$.

PROOF. – Direct computation.

LEMMA 5.3. – $(dr^+(h))/dh > 0$, $(dr^-(h))/dh < 0$ *for* $h \in (-\infty, 0)$.

PROOF. – Assume $(dr^{\pm}(h))/dh = 0$ at $h = \bar{h}$. Differentiating (5.3) with respect h, we have

$$r^{\pm}(\bar{h}) = -\frac{1}{3} \quad \text{or} \quad r^{\pm}(\bar{h}) = -\frac{7}{9}. \tag{5.6}$$

However, $G(\bar{h}, -1/3) = -5/27 \neq 0$, $G(\bar{h}, -7/9) = -77/243 \neq 0$, which contradicts to the definition of $r^{\pm}(h)$ (see (5.3)). This implies $(dr^{\pm}(h))/dh \neq 0$. By Lemma 5.2 i), we get $(dr^+(h))/dh > 0$, $(dr^-(h))/dh < 0$.

THEOREM 5.4. – *For* $h \in (-1/12, 0)$,

i) $\frac{d}{dh}\left(\frac{I_1''}{I_0''}\right) > 0$,

ii) $-\frac{7}{31} < \frac{I_1''}{I_0''} < 0$.

PROOF. – It follows from Proposition 3.3 that

$$r(h) = \frac{I_1''}{I_0''} = -\frac{7}{31} + \frac{385}{961}\left(h + \frac{1}{12}\right) + o\left(h + \frac{1}{12}\right) + \dots$$

as $h \to -(1/12)^+$, which implies

(5.7) $$r\left(-\frac{1}{12}\right) = -\frac{7}{31}, \quad r'\left(-\frac{1}{12}\right) = \frac{385}{961}.$$

Similarly, Proposition 3.3 gives

(5.8) $$\lim_{h \to 0^-} r(h) = \lim_{h \to 0^-} \frac{I_1''(h)}{I_0''(h)} = 0.$$

It is known that $r(h) = (I_1''(h))/(I_0''(h))$ satisfies (5.2), which has four critical points: two saddles at $B(0, -2)$ and $D(-1/12, -7/31)$, an unstable node at $A(0, 0)$, a stable node at $C(-1/12, 1)$. From (5.7) and (5.8), the graph $r(h) = (I_1''(h))/(I_0''(h))$ is the trajectory of (5.2) starting from the unstable node A to the saddle point D. On the other hand, the zero isoclines $r^+(h)$ and $r^-(h)$ are monotonically increasing and decreasing respectively (cf. Lemma 5.3). In the phase plane of system (5.2), the region $\{(h, r) \mid -1/12 \leq h \leq 0\}$ is divided into three parts by the curve $r^+(h)$,$r^-(h)$ and the invariant lines $h = 0$, $h = -1/12$. It follows from Lemma 5.2 ii) and (5.7) that $(dr^+(h))/dh|_{h=-1/12} > (dr(h))/dh|_{h=-1/12}$. Hence, the graph of $r(h) = I_1''/I_0''$ must stay in the region $\{(h, r) \mid -1/12 \leq h \leq 0, r^-(h) < r < r^+(h)\}$, which implies

$$\frac{d}{dh}\left(\frac{I_1''}{I_0''}\right) = \frac{dr(h)}{dh} = \frac{-(27h - 1/3)(r - r^+)(r - r^-)}{2h(12h + 1)} > 0$$

for $h \in (-1/12, 0)$, see fig. 5.1. The conclusion ii) follows from i), (5.7) and (5.8).

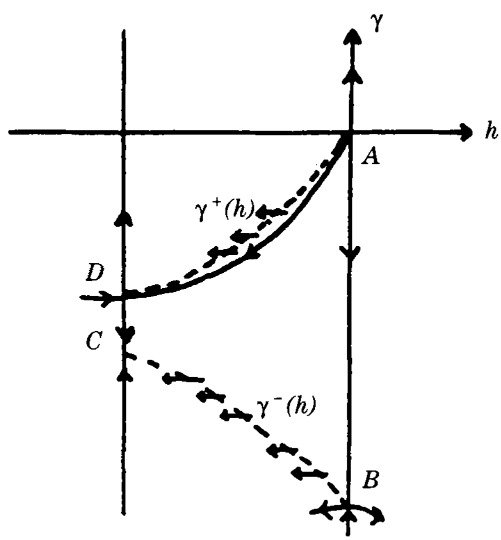

Fig. 5.1.

Denote

$$q(h) = f(h) + \frac{I_1''}{I_0''}, \tag{5.9}$$

where $h \in (-1/12, 0)$ and

$$f(h) = \frac{\alpha + 6\gamma h}{1 + 18\gamma h}. \tag{5.10}$$

Since $f'(h) = (-18\gamma(\alpha - 1/3))/(1 + 18\gamma h)^2$, we have $f'(h) > 0$ for $\gamma(\alpha - 1/3) < 0$ and $f'(h) < 0$ for $\gamma(\alpha - 1/3) > 0$

LEMMA 5.5.

i) $I''(h) = (\alpha + 6\gamma h) I_0'' + (1 + 18\gamma h) I_1''$,

ii) $I''(-1/18\gamma) \neq 0$ *for* $\alpha \neq 1/3$, *which implies that the number of zeros of* $I(h)$ *equals the number of zeros of* $q(h)$ *for* $\alpha \neq 1/3$.

PROOF. – The equality i) follows from Lemma 4.1 ii). The conclusion ii) follows from i) and $I_0''(h) \neq 0$ for $h \in (-1/12, 0) \cup (0, +\infty)$.

REMARK. – Lemma 5.5 is important for our analysis. We may estimate the number of zeros of $q(h)$ instead of the number of zeros of $I''(h)$.

COROLLARY 5.6. – *If* $\alpha = 1/3$, $h \in (-1/12, 0)$, *then*

i) $h = -1/18\gamma$ *is the unique zero of* $I''(h)$ *for* $\gamma > 2/3$,

ii) $I''(h)$ *has no zero for* $\gamma \leq 2/3$.

PROOF. – When $\alpha = 1/3$, $I(h) = (1 + 18\gamma h)(1/3 + I_1''/I_0'')$. It follows from Theorem 5.4 ii) that $1/3 + I_1''/I_0'' > 0$. Noting that $-1/18\gamma \in (-1/12, 0)$ for $\gamma > 2/3$ and $-1/18\gamma \notin (-1/12, 0)$ for $\gamma \leq 2/3$, the results are proved.

COROLLARY 5.7. – i) *If* $\gamma > 2/3$, *then* $I''(h)$ *has at most two zeros for* $0 < \alpha < 1/3$ *and at most one zero for* $\alpha \leq 0$, $h \in (-1/12, 0)$.

ii) If $\gamma < 0$, $\alpha > 1/3$, then $I''(h)$ has at most one zero in $(-1/12, 0)$.

PROOF. – i) It follows from Theorem 5.4 and $f'(h) > 0$ that $q'(h) > 0$, i.e., $q(h)$ is monotonically increasing function in $(-1/12, -1/18\gamma) \cup (-1/18\gamma, 0)$, so $q(h)$ has at most two zeros in $(-1/12, 0)$. However, if $\alpha \leq 0$, then $f(h) = (\alpha + 6\gamma h)/(1 + 18\gamma h) < 0$ for $h \in (-1/18\gamma, 0)$. Theorem 5.4 ii) yields $q(h) < 0$ for $h \in (-1/18\gamma, 0)$. Therefore, $q(h)$ has at most one zero in $(-1/12, -1/18\gamma)$. The result follows from Lemma 5.5 ii).

ii) In this case, $-1/18\gamma \notin (-1/12, 0)$ and $q'(h) > 0$, which implies ii).

COROLLARY 5.8. – *If* $0 < \gamma \leqslant 2/3$, $h \in (-1/12, 0)$, *then*

i) $I''(h)$ *has at most one zero for* $0 < \alpha < 1/3$,

ii) $I''(h)$ *has no zero for* $\alpha \leqslant 0$.

PROOF. – In the case of $0 < \gamma \leqslant 2/3$, $\alpha < 1/3$, we have $-1/18\gamma \leqslant -1/12, f'(h) > 0$. It follows from Theorem 5.4 that $q'(h) > 0$ for $h \in (-1/12, 0)$, which implies i). However, if $\alpha \leqslant 0$, noting $(\alpha + 6\gamma h)/(1 + 18\gamma h) < 0$ and Theorem 5.4 ii), we conclude that $q(h) < 0$ for $h \in (-1/12, 0)$, the result ii) follows.

COROLLARY 5.9. – i) $I''(h)$ *has no zero in* $(-1/12, 0)$ *for* $0 < \gamma \leqslant 2/3$, $\alpha > 1/3$.

ii) $I''(h)$ *has no zero in* $(-1/18\gamma, 0)$ *for* $\gamma > 2/3$, $\alpha > 1/3$.

PROOF. – i) In this case, $f(h)$ is a continuous and monotonically decreasing function in $(-1/12, 0)$, hence $f(h) > f(0)$, i.e. $f(h) > \alpha > 1/3$ for $h \in (-1/12, 0)$. By Theorem 5.4 ii), we obtain $q(h) = f(h) + I_1''/I_0'' > 0$, i.e., $I''(h)$ has no zero in $(-1/12, 0)$.

ii) Using the same argument as above.

6. – Proof of Theorem 1.1.

PROPOSITION 6.1. – i) *If* $\alpha \leqslant 0$, $\gamma \leqslant 0$, *then* $I(h)$ *has no zero in* $(-1/12, 0)$,

ii) *If* $\gamma > 2/3$, $\alpha \geqslant 3/8$, *then* $I(h)$ *has no zero in* $(-1/12, 0) \cup (0, +\infty)$.

PROOF. – i) Lemma 2.1 and $\alpha \leqslant 0$, $\gamma \leqslant 0$ implies $I(h) < 0$.

ii) Recall system $(1.5)_\varepsilon$ (taking $\beta = 1$)

$$\begin{cases} \dot{x} = y = \tilde{X}(x, y), \\ \dot{y} = -x^2 - x^3 + \varepsilon(\alpha + x + \gamma x^2) y = \tilde{Y}(x, y). \end{cases}$$

It follows from $\gamma > 2/3$, $\alpha \geqslant 3/8$ that $1 - 4\gamma\alpha < 0$, which implies $\operatorname{div}(\tilde{X}, \tilde{Y}) = \varepsilon(\alpha + x + \gamma x^2) \neq 0$. Therefore, system $(1.5)_\varepsilon$ has no limit cycle in the phase plane, i.e., $I(h)$ has no zero in $(-1/12, 0) \cup (0, +\infty)$.

By Theorem 4.4, $g(h) = I''/I_0'''$ satisfies the following equation

$$(6.1) \qquad \begin{cases} \dot{h} = 2h(12h + 1)(1 + 18\gamma h), \\ \dot{g} = -\left(27h - \dfrac{1}{3}\right) g^2 + \left[216\gamma h^2 + (54\alpha + 44\gamma - 30) h + \dfrac{2}{3}(1 - \alpha)\right] g + F(h). \end{cases}$$

LEMMA 6.2. – i) *In the case of* $\gamma > 2/3$, *system* (6.1) *has six critical points: two unstable nodes at* $A(0, \alpha)$ *and* $C(-1/12, \alpha + \gamma - 1)$, *a stable node at* $E(-1/18\gamma, \alpha - 1/3)$, *three saddles at* $B(0, \alpha - 2)$, $D(-1/12, \alpha - (5\gamma + 7)/31)$ *and* $F(-1/18\gamma, \alpha + (-33 +$

$+34\gamma)/(3(9+2\gamma)))$. *The ordinates of critical points C, D, E, F satisfy*

$$\alpha - \frac{5\gamma+7}{31} < \alpha - \frac{1}{3} < \alpha + \gamma - 1 \leqslant \alpha + \frac{-33+34\gamma}{3(9+2\gamma)}$$

for $2/3 < \gamma \leqslant 3/2$, *and*

$$\alpha - \frac{5\gamma+7}{31} < \alpha - \frac{1}{3} < \alpha + \frac{-33+34\gamma}{3(9+2\gamma)} < \alpha + \gamma - 1$$

for $\gamma > 3/2$.

ii) *In the case of* $\gamma < 2/3$, *system* (6.1) *has four critical points in the region* $\{(h, g) \mid -1/12 \leqslant h \leqslant 0\}$: *an unstable node at* $A(0, \alpha)$, *two saddle points at* $B(0, \alpha - -2)$ *and* $D(-1/12, \alpha - (5\gamma+7)/31)$, *a stable node at* $C(-1/12, \alpha+\gamma-1)$. *The ordinates of* C, D *satisfy*

$$\alpha + \gamma - 1 < \alpha - \frac{5\gamma+7}{31} \, .$$

iii) *If* $\gamma \neq 0$, *then system* (6.1) *has three invariant straight lines* $h = 0$, $h = -1/12$ *and* $h = -1/18\gamma$.

LEMMA 6.3. – i) *In the case of* $\gamma > 2/3$, *the critical points A, E, D are on the curve* $g(h) = I''/I_0''$, $h \in [-1/12, 0]$.

ii) *In the case of* $\gamma < 2/3$, *the critical points A and D are on the curve* $g(h) = = I''/I_0''$, $h \in [-1/12, 0]$.

PROOF. – From Lemma 5.5 i), we have

$$g(h) = \frac{I''}{I_0''} = \alpha + 6\gamma h + (1+18\gamma h)\frac{I_1''}{I_0''} \, . \tag{6.2}$$

It follows from (5.7), (5.8) and (6.2) that

$$g\left(-\frac{1}{12}\right) = \alpha - \frac{5\gamma+7}{31}\,, \qquad g(0) = \alpha\,, \qquad g\left(-\frac{1}{18\gamma}\right) = \alpha - \frac{1}{3}\,,$$

which implies i) and ii).

Obviously, since $I_0''(h) \neq 0$, the number of zeros of $g(h)$ equals the number of zeros of $I''(h)$. It follows from (6.1) that

$$\dot{g}\,|_{g=0} = F(h)\,, \tag{6.3}$$

which implies that the trajectories of system (6.1) contact the h-axis at most at two points, on which the vector field is horizontal. On the other hand, the abscissa of the intersection point of curve $g(h)$ and the h-axis is the zero of $g(h)$. We will use these facts and system (6.1) to estimate the upper bound of the number of zeros of $I''(h)$.

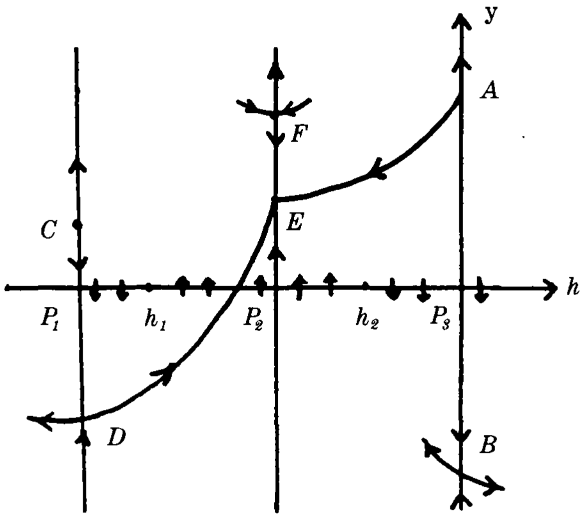

Fig. 6.1.

PROPOSITION 6.4. – *Suppose* $\gamma > 2/3$, $1/3 < a < 3/8$, *then*

i) $I''(h)$ *has one zero in* $(-1/12, 0)$ *for* $a - (5\gamma + 7)/31 < 0$,

ii) $I''(h)$ *has no zero in* $(-1/12, 0)$ *for* $a - (5\gamma + 7)/31 \geqslant 0$.

PROOF. – Assume that h-axis intersects the lines $h = -1/12$, $h = -1/18\gamma$, $h = 0$ at P_1, P_2 and P_3 respectively. Lemma 6.2 shows that A, F, C and E are in the upper half-plane. By Corollary 5.9 ii) and Lemma 6.3, $g(h)$ does not intersect the h-axis at $(-1/18\gamma, 0)$ and $g(h)$ consists of $\widehat{AE}$ and $\widehat{DE}$, where $\widehat{AE}$ and $\widehat{DE}$ are trajectories of the system (6.1) in the phase-plane.

i) In the case of $a - (5\gamma + 7)/31 < 0$, D and B are under the h-axis, and other critical points are in the upper half-plane. We can determine the directions of three invariant straight lines at P_1, P_2 and P_3 by Lemma 6.2 i) (see fig. 6.1), which implies $F(0) < 0$, $F(-1/18\gamma) > 0$, $F(-1/12) < 0$ (cf. formula (6.1),(6.3), (4.17)). Noticing that $F(h)$ has at most two zeros in $(-\infty, +\infty)$, there must exist $h_1 \in (-1/12, -1/18\gamma)$ and $h_2 \in (-1/18\gamma, 0)$, such that $F(h_i) = 0$, $i = 1, 2$. It follows that the trajectories of the system (6.1) cross the h-axis from the lower half-plane to the upper one for $h \in (h_1, h_2)$ and go in the opposite direction for $h \in (-\infty, h_1) \cup (h_2, +\infty)$. Hence $\widehat{DE}$ must intersect the h-axis only once and the result follows.

ii) Using the same arguments as above, one gets the result ii).

PROPOSITION 6.5. – *Suppose* $0 < a < 1/3$, $\gamma < 0$, *then*

i) $I''(h)$ *has one zero for* $a - (5\gamma + 7)/31 < 0$,

ii) $I''(h)$ *has at most two zeros for* $a - (5\gamma + 7)/31 \geqslant 0$.

PROOF. – Using the same arguments as for Proposition 6.4.

LEMMA 6.6 (Behaviour near the endpoints).

i) $I'\left(-\frac{1}{12}\right) = I_0'\left(-\frac{1}{12}\right)(\alpha+\gamma-1)$, $I''\left(-\frac{1}{12}\right) = \frac{31}{12} I_0'\left(-\frac{1}{12}\right)\left(\alpha - \frac{5\gamma+7}{31}\right)$,

ii) $I(0^-) = \frac{4}{27}\left(\alpha - \frac{5}{6} + \frac{7}{9}\gamma\right)\sqrt{2}\pi$,

$$I'(h) = -\frac{5}{6}C_1\alpha|h|^{-1/6} + \left(-2\sqrt{2}\pi + \frac{4}{3}\sqrt{2}\pi\gamma\right) + o(1),$$

$$I''(h) = -\frac{5}{36}C_1\alpha|h|^{-7/6} + o(|h|^{-7/6})$$

as $h \to 0^-$, *where* $h \in (-1/12, 0)$, $I(h) = \alpha I_0 + I_1 + \gamma I_2$.

PROOF. – It follows from Proposition 3.3 ii) and Theorem 1.2.

PROOF OF THEOREM 1.1. – Let $I(h) = \alpha I_0 + \beta I_1 + \gamma I_2$. Suppose $I(h) \not\equiv 0$, which implies $\alpha^2 + \beta^2 + \gamma^2 \neq 0$.

i) Assume $\alpha\beta\gamma = 0$. It follows from Lemma 2.1 and Theorem 2.3 that $I(h)$ has at most one zero in $(-1/12, 0)$.

ii) Assume $\alpha\beta\gamma \neq 0$. Without loss of generality, suppose $\beta = 1$. It follows from Corollaries 5.6-5.9 and Proposition 6.1, 6.4, 6.5 that either $I(h)$ has no zero or $I''(h)$ has at most two zeros, i.e., $I(h)$ has at most two inflection points for $h \in (-1/12, 0)$. Since $I(-1/12) = 0$, this implies that $I(h)$ has at most three zeros in $(-1/12, 0)$.

Assuming that $I(h)$ has three zeros in $(-1/12, 0)$, the graph of $I(h)$ must be one of curves drawn in fig. 6.2.

(a) In the case of fig. 6.2 (a), it follows from convexity, monotonicity and function value of $I(h)$ near its endpoints $h = 0$, $h = -1/12$ that

$$\begin{cases} I'\left(-\frac{1}{12}\right) \geq 0, \quad \lim_{h\to 0^-} I'(h) < 0, \quad I_0''\left(-\frac{1}{12}\right) \leq 0, \\ \lim_{h\to 0^-} I''(0) < 0, \quad I(0) < 0. \end{cases} \tag{6.4}$$

By Lemma 6.6, we have

$$\alpha + \gamma - 1 \geq 0, \quad \alpha < 0, \quad \alpha - \frac{5\gamma+7}{31} \leq 0, \quad \alpha - \frac{5}{6} + \frac{7}{9}\gamma < 0.$$

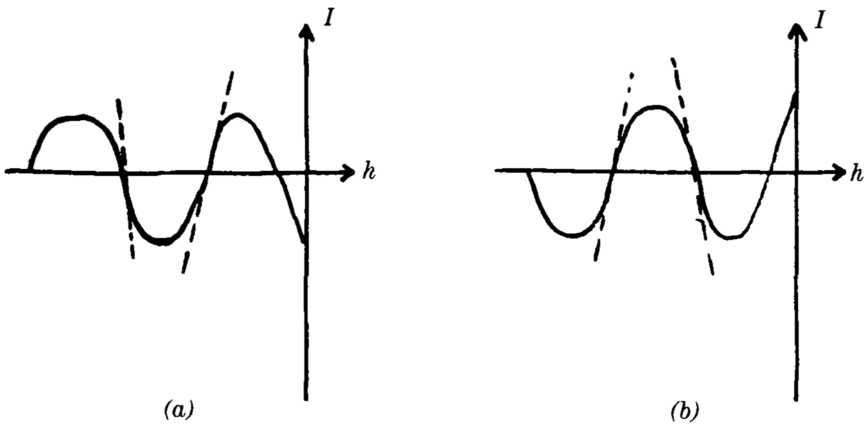

Fig. 6.2.

The first two inequalities imply $\alpha < 0$, $\gamma > 0$. When $\alpha < 0$, $\gamma > 0$, it follows from Corollary 5.7 i) and Corollary 5.8 that $I(h)$ has at most one inflection point in $(-1/12, 0)$, which contradicts to the assumption.

(*b*) In the case of fig. 6.2(b), using the same argument as (*a*), we have

$$\alpha + \gamma - 1 \leqslant 0\,, \qquad \alpha > 0\,, \qquad \alpha - \frac{5\gamma + 7}{31} \geqslant 0\,, \qquad \alpha - \frac{5}{6} + \frac{7}{9}\gamma > 0\,,$$

which implies

$$\gamma < \frac{2}{3}\,, \qquad \alpha > 0\,, \qquad \alpha > \frac{5}{6} - \frac{7}{9}\gamma\,.$$

In the case of $0 < \gamma \leqslant 2/3$, $\alpha > 0$, it follows from Corollary 5.8 i), Corollary 5.9 i) and Corollary 5.6 ii) that $I''(h)$ has at most one zero in $(-1/12, 0)$. This contradicts to the assumption again.

In the case of $\gamma < 0$, $\alpha > 5/6 - (7/9)\,\gamma > 1/3$, Corollary 5.7 ii) implies that $I''(h)$ has at most one zero for $h \in (-1/12, 0)$, which contradicts to the assumption, too.

Summing up the above discussion, we conclude that $I(h)$ has at most two zeros in $(-1/12, 0)$. Theorem 1.1 follows from this result and Proposition 3.4.

PROOF OF THEOREM 1.1*. – It follows from Theorem 1.1 and Theorem 1.2.

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