Chaos in Singular Systems (*).

SANDRA CHIARELLI

Abstract. – *In this paper we consider singular systems of differential equations and we show that, under right conditions, the Poincaré map associated to those systems, and not just a suitable iterate, behaves chaotically. We use the notion of exponential dichotomies to prove the existence of a transverse homoclinic orbit of our system and after use the shadow lemma to show that the Poincaré map associated to its topologically conjugate to the Bernoulli shift on a set of two symbols.*

1. – Introduction.

Let F be a diffeomorphism on $\mathbb{R}^n$. It is known [8, 9], that, if F has a transverse homoclinic point, there exists an invariant Cantor-like set on which a suitable iterate F behaves like the Bernoulli shift on a finite number of symbols. On the other hand, when F is the Poincaré map associated to a nonlinear oscillator $\dot{x} = f(t, x)$, the existence of a transverse homoclinic point of F is equivalent to the existence of a «Transverse homoclinic orbit» $\gamma(t)$ of $\dot{x} = f(t, x)$, homoclinic to a hyperbolic periodic orbit $x_0(t)$, that is a solution $\gamma(t)$ such that

$$\lim_{t \rightarrow \pm \infty} |\gamma(t) - x_0(t)| = 0$$

and the variational system

$$\dot{x} = f_x(t, \gamma(t))x$$

has an exponential dichotomy on $\mathbb{R}$ [8].

However, recognizing the existence of such a kind of orbits is not always an easy task, and it is often reduced, when the system is

$$(1) \quad \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \quad x \in \mathbb{R}^n, \quad g(t+T, x, \varepsilon) = g(t, x, \varepsilon)$$

and $\dot{x} = f(x)$ has a (necessarily non-transverse) homoclinic orbit $\gamma_0(t)$, to the study of a Melnikov function $M(\alpha)$. Actually, it is known [3, 6, 8] that, if $M(\alpha)$ has a simple zero at

(*) Entrata in Redazione il 3 aprile 1997 e, in versione riveduta, il 30 ottobre 1997.

Indirizzo dell'A.: Via Bagnaturo 76, 67039, Sulmona, L'Aquila. Tel. 0864-251309 (Cell. 0347/3712671).

$\alpha = \alpha_0$ and $\gamma_0(t)$ satisfies a suitable non degenerateness condition, the Poincaré Map associated to (1) has a transverse homoclinic orbit with associated chaotic behavior. Both this property of the Melnikov function and Smale theorem were first proved by geometrical arguments. Starting from [3] analytical methods have been applied to this kind of problems and Palmer [8] in 1984 gave an analytical proof of Smale theorem making use of a kind of «Shadow lemma» for diffeomorphism with a hyperbolic invariant set.

Recently, investigators have addressed their attentions to singular systems of differential equations like:

$$(2) \quad \varepsilon \dot{x} = f(x) + \varepsilon^p g(t, x, \varepsilon)$$

where $x \in \mathbb{R}^n$, $g(t+1T, x, \varepsilon)$, $p > 0$.

It is assumed that f and g are sufficiently smooth ($\mathcal{C}^2$) in their arguments and that the «Boundary layer» system $\dot{x} = f(x)$ has a homoclinic orbit $\gamma_0(t)$ such that the variational system $\dot{x} = f'(\gamma_0(t))x$ has the unique bounded solution $\dot{\gamma}_0(t)$, up to a multiplicative constant (non degenerateness). As an example we mention the Duffing equation

$$(3) \quad \ddot{z} + \alpha^2 f_0(z) = h(t), \quad z \in \mathbb{R}, \quad h(t) = h(t+T)$$

where $\alpha \gg 1$ is a sufficiently large real parameter.

Setting $\varepsilon = \alpha^{-1} > 0$, $z = x_1$, $\varepsilon \dot{z} = x_2$, equation (3) reads:

$$\varepsilon \dot{x}_1 = x_2, \quad \varepsilon \dot{x}_2 = -f_0(x_1) + \varepsilon^2 h(t)$$

and we fall in the class of singular system as in (2).

Note that the «boundary layer» equation is, in this case, $\ddot{z} = f_0(z) = 0$ and our conditions are that this equation has a homoclinic orbit to a hyperbolic equilibrium point, the non-degenerateness hypothesis being automatically satisfied because the equation is of the 2nd order in $\mathbb{R}$. It is known [2] that, if a certain Melnikov function, associated to (3), has a simple zero at some point, equation (3) has a transverse homoclinic orbit.

This fact, together with Smale theorem implies chaotic behaviour of an iterate of the associated Poincaré Map. However, recently, BATTELLI and FECKMAN [1] have shown that, under the same conditions, there exists $\varepsilon_0 > 0$ such that equation (2) has periodic solutions of period mT , for any $m \in \mathbb{N}$ and $0 < \varepsilon < \varepsilon_0$.

This fact lead to think that the Poincaré map itself behaves chaotically and not just a suitable iterate.

The purpose of this paper is to show that the above conjecture is in fact true. To this end we will follow the approach in [8] and use the notion of exponential dichotomies and the shadow lemma as stated in [8]. Let us briefly explain the plan of this work.

In Section 2 we recall the notion of exponential dichotomies and prove the existence of a transverse heteroclinic orbit of (2). This kind of result was already proved in [2] where they used scales of Banach-spaces, in order to prove the smoothness with respect to ε of a «bifurcation function» from which the Melnikov condition is derived.

Our method of proof, instead, is inspired by [1] (where, however the proof of transversality was missed) and does not need to show the smoothness in ε . This fact has also the consequence that the method can be applied even when $p > 0$ is real. In Section 3 we recall the shadow-lemma proved in [8] and show the main result of this paper stat-

ing that the Poincaré Map of system (2) is homeomorphic to the Bernoulli shift on a set of two symbols.

2. – Existence of transverse homoclinic orbits.

Let $A(t)$ a $n \times n$ bounded and continuous matrix on an interval $I \subseteq \mathbb{R}$. We will say that the linear system on $\mathbb{R}^n$

$$(4) \quad \dot{x}(t) = A(t) x(t), \quad x \in \mathbb{R}^n,$$

has an exponential dichotomy on I , if there exists $K \geq 1$, $\delta > 0$ and a projection P such that the fundamental matrix $X(t)$ of (4), which satisfies $X(0) = I$, satisfies

$$\begin{aligned} \|X(t) P X^{-1}(s)\| &\leq K e^{-\delta(t-s)}, \quad \forall s, t \in I, \quad s \leq t, \\ \|X(t)(I - P) X^{-1}(s)\| &\leq K e^{+\delta(t-s)}, \quad \forall s \geq t, \quad s, t \in I. \end{aligned}$$

When (4) has an exponential dichotomy on $I \subseteq \mathbb{R}$ we set $\mathcal{P}(t) = X(t) P X^{-1}(t)$, $t \in I$.

The projections P , $\mathcal{P}(t)$ are not uniquely determined however, if $I = \mathbb{R}_+$ the range of $\mathcal{P}(0)$, $\mathcal{R}\mathcal{P}(0)$, must be the stable subspace

$$\left\{ \xi \in \mathbb{R}^n : \sup_{t \geq 0} |X(t) X^{-1}(0) \xi| < \infty \right\}$$

but the kernel may be any complementary subspace; while, if $I = \mathbb{R}_-$ the kernel of $\mathcal{P}(0)$, $\mathcal{K}\mathcal{P}(0)$, must be the unstable subspace

$$\left\{ \xi \in \mathbb{R}^n : \sup_{t \leq 0} |X(t) X^{-1}(0) \xi| < \infty \right\}$$

but the range may be any complementary subspace [4, 8].

As a consequence, if (4) has an exponential dichotomy on $\mathbb{R}$, both $\mathcal{R}P$ and $\mathcal{K}P$ are uniquely determined and hence so are P , $\mathcal{P}(t)$.

Let $C_b^0(\mathbb{R}_+, \mathbb{R}^n)$ the Banach space of all $\mathbb{R}^n$ -valued functions which are continuous in $\mathbb{R}_+$ and bounded, $C_b^0(\mathbb{R}_-, \mathbb{R}^n)$ the Banach space of all $\mathbb{R}^n$ -valued functions which are continuous in $\mathbb{R}_-$ and bounded.

The following lemma [8] is of basic importance:

LEMMA 2.1. – Let (4) have an exponential dichotomy on $\mathbb{R}_+$ with constants $k \geq 1$, $\delta > 0$ and projection P_+ .

Then for any $h \in C_b^0(\mathbb{R}_+, \mathbb{R}^n)$, $\forall \xi \in \mathcal{R}P_+$, there exists a unique solution $x(t)$ of the system

$$\begin{cases} \dot{x}(t) = A(t) x(t) + h(t), \\ P_+ x(0) = \xi, \end{cases}$$

$x(t) \in C_b^0[\mathbb{R}_+]$ given by

$$x(t) = X(t) \xi + \int_0^t X(t) P_+ X^{-1}(s) h(s) ds - \int_t^\infty X(t) Q_+ X^{-1}(s) h(s) ds$$

where $Q_+ = \mathbb{I} - P_+$.

Moreover $\|x\| \leq \mathcal{C}(\|h\| + |\xi|)$.

REMARK. – We have a similar result when the dichotomy is on $\mathbb{R}_-$. In this case the system

$$\begin{cases} \dot{x}(t) = A(t) x(t) + h(t), \\ Q_- x(0) = \eta, \end{cases} \quad \eta \in \mathcal{H}P_- ,$$

has the unique solution bounded on $\mathbb{R}_-$

$$x(t) = X(t) \eta + \int_{-\infty}^t X(t) P_- X^{-1}(s) h(s) ds - \int_t^0 X(t) [\mathbb{I} - P_-] X^{-1}(s) h(s) ds$$

with $\|x(t)\| \leq \mathcal{C}(\|h\| + |\eta|)$.

In order to apply the above lemma and remark to our problem, we simply need to note the following facts. As p is a hyperbolic equilibrium point, the linear system $\dot{x} = f'(p)x$ has an exponential dichotomy on $\mathbb{R}$ with projection P that can be chosen to commute with $X(t)$ (hence $\mathcal{P}(t) = P$, $\forall t \in \mathbb{R}$).

Now, $\gamma_0(t)$ is homoclinic to p , thus it follows from the roughness of exponential dichotomies [4] that the linear system $\dot{x} = f'(\gamma_0(t))x$ has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, with constants k , δ and projection P_+ , P_- , which satisfy

$\dim \mathcal{R}P_+ = \dim \mathcal{R}P =$ number of eigenvalues of $f'(p)$ with negative real part counted with their multiplicities,

$\dim \mathcal{H}P_- = \dim \mathcal{H}P =$ number of eigenvalues of $f'(p)$ with positive real part counted with their multiplicities.

Moreover [8] we have

$$\begin{aligned} |P_+(t) - P| &\rightarrow 0 \quad \text{as } t \rightarrow +\infty, \\ |P_-(t) - P| &\rightarrow 0 \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

Finally, as $\gamma_0(t)$ is homoclinic, the system $\dot{x} = f'(\gamma_0(t))x$ and its adjoint $\dot{x} = -f'(\gamma_0(t))^*x$ (here and in the following $*$ means transpose) have the same number of (independent) bounded solutions. So, as we assume that the space of bounded solutions of the variational system $\dot{x} = f'(\gamma_0(t))x$ is spanned by $\dot{\gamma}_0(t)$, there exists a unique, up to a multiplicative constant, non-zero bounded solution $\psi(t)$ of $\dot{x} = -f'(\gamma_0(t))^*x$.

Without loss of generality we can also assume that $\psi = \psi(0)$ is such that $|\psi(0)| = 1$. We are now able to prove the main result of this section:

THEOREM 2.1. – Given the singular system

$$(5) \quad \varepsilon \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

$f(x)$, $g(t, x, \varepsilon)$ being regular ($\mathcal{C}^r$, $r \geq 1$) and bounded in their arguments and g , T -periodic in t , suppose that the boundary layer problem

$$\dot{x} = f(x)$$

has a solution $\gamma_0(t)$ homoclinic to a hyperbolic equilibrium point p .

Furthermore, assume that the variational system

$$\dot{x} = \frac{\partial f}{\partial x}(\gamma_0(t)) \cdot x$$

has the unique bounded solution $\dot{\gamma}(t)$.

Then, if there exists α_0 such that

$$\int_{-\infty}^{+\infty} \psi^*(t) g(\alpha_0, \gamma_0(t), 0) dt = 0$$

and

$$\int_{-\infty}^{+\infty} \psi^*(t) g(\alpha_0, \gamma_0(t), 0) dt \neq 0,$$

there exists a unique $\alpha(\varepsilon)$ with, $\alpha(0) = \alpha_0$ such that the system

$$\varepsilon \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

has a unique solution $\gamma(t, \varepsilon)$ satisfying

$$\sup_{t \in \mathbb{R}} |\gamma(t, \varepsilon) - \gamma_0(\varepsilon^{-1}(t - \alpha(\varepsilon)))| = O(\varepsilon).$$

Moreover the variational system

$$\varepsilon \dot{x} = [f_x(\gamma(t, \varepsilon)) + \varepsilon g_x(t, \gamma(t, \varepsilon), \varepsilon)] \cdot x$$

has an exponential dichotomy on $\mathbb{R}$ with constants $k, \delta/\varepsilon$.

PROOF. – The proof of existence of $\gamma(t, \varepsilon)$ is known. However we will give it here for completeness and mainly because we need it to show the «transversality condition».

Put $u(t) = x(\varepsilon t + \alpha) - \gamma_0(t)$. Finding a solution of (5) is equivalent to the study of

$$\dot{u}(t) - A(t) u(t) = f(u(t) + \gamma_0(t)) - f(\gamma_0(t) - A(t) u(t) + \varepsilon g(\varepsilon t + \alpha, u(t) + \gamma_0(t), \varepsilon))$$

or

$$(6) \quad \dot{u}(t) = A(t) u(t) + h(t, u, \alpha, \varepsilon)$$

with

$$A(t) = f'(\gamma_0(t)),$$

$$h(t, u, \alpha, \varepsilon) = f(u(t) + \gamma_0(t)) - f(\gamma_0(t) - A(t) u(t) + \varepsilon g(\varepsilon t + \alpha, u(t) + \gamma_0(t), \varepsilon)).$$

Applying Lemma 2.1 and the contraction mapping principle we have: for all $\xi \in \mathcal{R}P_+$, such that $\langle \xi, \dot{\gamma}(0) \rangle = 0$, and $\eta \in \mathcal{R}P_-$ there exist unique bounded solutions u_+ of

$$\begin{cases} \dot{u}(t) - A(t) u(t) = h(t, u, \alpha, \varepsilon), \\ P_+ u(0) = \xi, \quad t \geq 0, \end{cases} \quad \text{and } u_- \text{ of } \begin{cases} \dot{u}(t) - A(t) u(t) = h(t, u, \alpha, \varepsilon), \\ Q_- u(0) = \eta, \quad t \leq 0, \end{cases}$$

satisfying:

$$\|u_+(\cdot, \xi, \alpha, \varepsilon)\| \leq 2K\{|\xi| + 2\delta^{-1}M|\varepsilon|\},$$

$$\|u_-(\cdot, \eta, \alpha, \varepsilon)\| \leq 2K\{|\eta| + 2\delta^{-1}M|\varepsilon|\},$$

$$\left\| \frac{\partial u_+}{\partial \xi}(\cdot, \xi, \alpha, \varepsilon) \right\| \leq \mathcal{C}, \quad \left\| \frac{\partial u_-}{\partial \eta}(\cdot, \eta, \alpha, \varepsilon) \right\| \leq \mathcal{C},$$

$$\left\| \frac{\partial u_{\pm}}{\partial \alpha}(\cdot, \xi, \eta, \alpha, \varepsilon) \right\| \leq \mathcal{C}|\varepsilon|,$$

because of the definition of h and the conditions on f and g (see also [1]).

Moreover if $u(t)$ is a bounded solution of (6), $u(t)$ satisfies both the first and the second system with $\xi = P_+ u(0)$ and $\eta = Q_- u(0)$.

So, the problem of existence of bounded solutions of (6) is equivalent to solving

$$u_+(0) - u_-(0) = 0.$$

We have

$$\begin{aligned} u_+(0) - u_-(0) &= X(0) \xi - \int_0^{\infty} X(0) Q_+ X^{-1}(s) h(s, u_+(s), \alpha, \varepsilon) ds - X(0) \eta + \\ &- \int_{-\infty}^0 X(0) P_- X^{-1}(s) h(s, u_-(s), \alpha, \varepsilon) ds = \\ &= \xi - \eta - \int_0^{\infty} Q_+ X^{-1}(s) h(s, u_+(s), \alpha, \varepsilon) ds + \end{aligned}$$

$$\begin{aligned}
& - \int_{-\infty}^0 P_- X^{-1}(s) h(s, u_-(s), \alpha, \varepsilon) ds u_+(0) - u_-(0) = 0 \Leftrightarrow \xi - \eta = \\
& = \int_0^{\infty} Q_+ X^{-1}(s) h(s, u_+(s), \alpha, \varepsilon) ds + \int_{-\infty}^0 P_- X^{-1}(s) h(s, u_-(s), \alpha, \varepsilon) ds = H(\xi, \eta, \alpha, \varepsilon).
\end{aligned}$$

Notice that if $\xi \in \mathcal{RP}_+$ and $\eta \in \mathcal{XP}_-$ are such that $\xi - \eta = H(\xi, \eta, \alpha, \varepsilon)$, we have

$$\psi^* \cdot H = 0,$$

because $\xi - \eta \in [\mathcal{RP}_+ + \mathcal{XP}_-] = \{\psi\}^\perp$, then we can study

$$(7) \quad \begin{cases} \xi - \eta = H(\xi, \eta, \alpha, \varepsilon) + \psi^* \cdot H(\xi, \eta, \alpha, \varepsilon) \psi, \\ \psi^* \cdot H(\xi, \eta, \alpha, \varepsilon) = 0, \end{cases}$$

(this procedure is called the «*Lyapunov-Schmidt procedure*»).

It follows from the definition of $H(\xi, \eta, \alpha, \varepsilon)$ that $|H(\xi, \eta, \alpha, \varepsilon)| \leq 2K\delta^{-1}\|h\|$ and because of $h(t, u, \alpha, \varepsilon) = f(u(t) + \gamma_0(t)) - f(\gamma_0(t)) - A(t)u(t) + \varepsilon g(\varepsilon t + \alpha, u(t) + \gamma_0(t), \varepsilon)$ we get

$$(8) \quad |H(\xi, \eta, \alpha, \varepsilon)| \leq \mathcal{C}(|\xi| + |\eta| + |\varepsilon|)$$

uniformly with respect to α .

In the same way we get

$$(9) \quad \left| \frac{\partial H}{\partial \xi}(\xi, \eta, \alpha, \varepsilon) \right| \leq \mathcal{C}(|\xi| + |\eta| + |\varepsilon|),$$

$$(10) \quad \left| \frac{\partial H}{\partial \eta}(\xi, \eta, \alpha, \varepsilon) \right| \leq \mathcal{C}(|\xi| + |\eta| + |\varepsilon|), \quad \left| \frac{\partial H}{\partial \alpha}(\xi, \eta, \alpha, \varepsilon) \right| \leq \mathcal{C}|\varepsilon|.$$

Hence, for ε sufficiently small, the uniform contraction theorem gives the existence of a unique pair of functions $\xi(\alpha, \varepsilon), \eta(\alpha, \varepsilon)$ such that $\xi(\alpha, \varepsilon) \leq \mathcal{C}|\varepsilon|$, $\eta(\alpha, \varepsilon) \leq \mathcal{C}|\varepsilon|$ satisfying the first equation of (7).

Moreover it follows from (8), (9), (10) that

$$\left| \frac{\partial \xi}{\partial \alpha}(\alpha, \varepsilon) \right| \leq \mathcal{C}|\varepsilon|, \quad \left| \frac{\partial \eta}{\partial \alpha}(\alpha, \varepsilon) \right| \leq \mathcal{C}|\varepsilon|,$$

$\mathcal{C}$ being independent of α .

The problem of finding solutions $\gamma(t, \varepsilon)$ «near» $\gamma_0(t)$ is then equivalent to solving the bifurcation equation $G(\alpha, \varepsilon) = 0$ where

$$G(\alpha, \varepsilon) = \psi^* \cdot H(\xi(\alpha, \varepsilon), \eta(\alpha, \varepsilon), \alpha, \varepsilon).$$

Define now $\tilde{G}(\alpha, \varepsilon) = \varepsilon^{-1}G(\alpha, \varepsilon)$.

As

$$\psi^* Q_+ X^{-1}(s) = [X^{-1}(s)^* Q_+^* \psi]^* = \{X^{-1}(s)^* (I - P_+^*) \psi\}^* = \{X^{-1}(s)^* \psi\}^*$$

and

$$\psi^* P_- X^{-1}(s) = [X^{-1}(s)^* P_-^* \psi]^* = \{X^{-1}(s)^* \psi\}^*$$

we get

$$\begin{aligned} \tilde{G}(\alpha, \varepsilon) &= \varepsilon^{-1} \int_{-\infty}^{+\infty} \psi^*(t) h(t, u_{\pm}(t, \xi(\alpha, \varepsilon), \eta(\alpha, \varepsilon), \alpha, \varepsilon)) dt = \\ &= \varepsilon^{-1} \int_{-\infty}^{+\infty} \psi^*(t) \{f(u_{\pm} + \gamma_0(t)) - f(\gamma_0(t)) - A(t) u_{\pm}(t) + \varepsilon g(\varepsilon t + \alpha, u_{\pm} + \gamma_0(t), \varepsilon)\} dt. \end{aligned}$$

As

$$f(u_{\pm} + \gamma_0(t)) - f(\gamma_0(t)) - A(t) u(t) = O(\|u_{\pm}\|_0^2) = O(|\varepsilon|^2)$$

we have

$$\tilde{G}(\alpha, \varepsilon) = \int_{-\infty}^{+\infty} \psi^*(t) g(\varepsilon t + \alpha, u_{\pm}(t) + \gamma_0(t), \varepsilon) dt + O(|\varepsilon|)$$

which tends to $M(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t) g(\alpha, \gamma_0(t), 0) dt$ as $\varepsilon \rightarrow 0^+$.

Moreover this convergence holds together with the derivatives with respect to α . In fact for example

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} \psi^*(t) \left\{ [f_x(u_{\pm}(t, \alpha) + \gamma_0(t)) - A(t)] \frac{\partial u_{\pm}}{\partial \alpha}(t, \alpha) \right\} dt \right| &\leq \\ &\leq \int_{-\infty}^{+\infty} k e^{-\delta|t|} \left\{ N^0 u_{\pm}(\cdot, \alpha, \varepsilon)^0 \left\| \frac{\partial u_{\pm}}{\partial \alpha}(\cdot, \alpha, \varepsilon) \right\| \right\} dt. \end{aligned}$$

But

$$\|u_{\pm}(\cdot, \alpha, \varepsilon)\| = O(|\varepsilon|)$$

and

$$\left\| \frac{\partial u_{\pm}}{\partial \alpha}(\cdot, \alpha, \varepsilon) \right\| = O(|\varepsilon|)$$

so

$$\int_{-\infty}^{+\infty} \left| \psi^*(t) \left\{ [f_x(u_{\pm}(t, \alpha) + \gamma_0(t)) - A(t)] \frac{\partial u_{\pm}}{\partial \alpha}(t, \alpha) \right\} \right| dt = O(|\varepsilon|^2).$$

In the same way we see that

$$\int_{-\infty}^{+\infty} \psi^*(t) \cdot \frac{\partial g}{\partial x}(\varepsilon t + \alpha, u_{\pm}(t) + \gamma_0(t), \varepsilon) \cdot \frac{\partial u_{\pm}}{\partial \alpha}(t, \alpha) dt = O(|\varepsilon|).$$

$M(\alpha)$ is our «Melnikov function».

Now, we know there exists α_0 such that $M(\alpha_0) = 0 \neq M'(\alpha_0)$, then from the uniform contraction theorem it follows that there exists a unique $\alpha(\varepsilon)$ such that $\tilde{G}(\alpha(\varepsilon), \varepsilon) = 0$ with $0 < \varepsilon < \varepsilon_0 \ll 1$.

So there exists a unique solution $u(t, \varepsilon)$ of the system

$$\dot{u}(t) = A(t) u(t) + h(t, u, \alpha(\varepsilon), \varepsilon).$$

Changing t with $\varepsilon t + \alpha$ and setting $\gamma(t, \varepsilon) = u(\varepsilon^{-1}(t - \alpha(\varepsilon), \varepsilon)) + \gamma_0(\varepsilon^{-1}(t - \alpha(\varepsilon)))$, we get

$$\sup_{t \in \mathbb{R}} |\gamma(t, \varepsilon) - \gamma_0(\varepsilon^{-1}(t - \alpha(\varepsilon)))| \leq C|\varepsilon|.$$

To prove that $\gamma(t, \varepsilon)$ is a traverse solution we need the following theorem about the local invertibility for operators [5].

THEOREM 2.2. – Let X, Y be Banach spaces.

Consider a C^r -smooth map $F: X \rightarrow Y$ satisfying $F(0) = 0$ and assume $DF(0)$ is Fredholm with index zero.

If there exist a neighborhood $U \subseteq X$ of 0 and real numbers $k > 0, \delta > 0$ such that for any linear bounded mapping $B: X \rightarrow Y, \|B\| \leq k$, the perturbations $\varepsilon B + F, 0 \leq \varepsilon < \delta$, has the only zero point 0 in U , then $DF(0)$ is invertible.

Notice that putting $y(t) = x(\varepsilon t + \alpha(\varepsilon))$, the system, (5) is equivalent to system

$$(11) \quad \dot{y} = f(y) + \varepsilon g(\varepsilon t + \alpha(\varepsilon), y, \varepsilon).$$

Furthermore if the system

$$\dot{y} = [f_x(u(t, \varepsilon) + \gamma_0(t)) + \varepsilon g_x(\varepsilon t + \alpha, u(t, \varepsilon) + \gamma_0(t), \varepsilon)] \cdot y$$

has an exponential dichotomy with constants k, δ then the system $\varepsilon \dot{x} [f_x(\gamma(t, \varepsilon)) + \varepsilon g_x(t, \gamma(t, \varepsilon), \varepsilon)] \cdot x$ has an exponential dichotomy with constants k and δ/ε .

So, we may study the system (12).

Consider the map

$$\mathcal{F}_\varepsilon: C_b^1[\mathbb{R}, \mathbb{R}^n] \rightarrow C_b^0[\mathbb{R}, \mathbb{R}^n]$$

such that

$$\mathcal{F}_\varepsilon(y) = \dot{y} - f(y) - \varepsilon g(\varepsilon t + \alpha, y, \varepsilon).$$

As we have just shown there exists a unique $y(t, \varepsilon) = y_\varepsilon$ where $\mathcal{F}_\varepsilon(y_\varepsilon) = 0$.

Let $\varepsilon_0 \neq 0$. Let us define $\tilde{\mathcal{F}}_\varepsilon = \mathcal{F}_\varepsilon(y) + \varepsilon \sigma L(y - y_{\varepsilon_0})$ such that $L: C_b^1 \rightarrow C_b^0$ is a linear operator with $\|L\| \leq 1$ and $|\sigma| < \sigma_0 \ll 1$.

Solving the following systems, as we did in the previous part of this theorem,

$$\begin{cases} \dot{y} - A(t)y - f(y) - \varepsilon g(\varepsilon t + \alpha, y, \varepsilon) + \varepsilon \sigma L(y - y_{\varepsilon_0}) + A(t)y = 0, \\ P_+ y(0) = \xi, \end{cases}$$

$$\begin{cases} \dot{y} - A(t)y - f(y) - \varepsilon g(\varepsilon t + \alpha, y, \varepsilon) + \varepsilon \sigma L(y - y_{\varepsilon_0}) + A(t)y = 0, \\ (I - P_-) y(0) = \eta, \end{cases}$$

we have two bounded solutions, $y_+(t)$ and $y_-(t)$ respectively.

The same method as above shows, then, that for any σ , $|\sigma| < \sigma_0 \ll 1$, $\tilde{\mathcal{F}}_\varepsilon(y) = 0$, $\varepsilon \neq 0$ has a unique solution $\tilde{y}_\varepsilon$ «near» to y_{ε_0} uniformly in σ .

However $\tilde{\mathcal{F}}_{\varepsilon_0}(y_{\varepsilon_0}) = \mathcal{F}_{\varepsilon_0}(y_{\varepsilon_0}) + \varepsilon_0 \sigma L(y_{\varepsilon_0} - y_{\varepsilon_0}) = 0$, so the unique solution of the equation $\tilde{\mathcal{F}}_{\varepsilon_0}(y) = 0$ is y_{ε_0} . As $D\mathcal{F}_\varepsilon(y_\varepsilon)$ is Fredholm with index zero we may apply the Theorem 2.2, concluding that $D\tilde{\mathcal{F}}_\varepsilon(y_{\varepsilon_0})$ is an invertible operator. Then for any $h \in C_b^0[\mathbb{R}, \mathbb{R}^n]$ the differential equation $\dot{y}[f_x(u(t, \varepsilon) + \gamma_0(t)) + \varepsilon g_x(\varepsilon t + \alpha, u(t, \varepsilon) + \gamma_0(t), \varepsilon)] \cdot y = h(t)$ has a unique bounded solution.

If $h(t) \equiv 0$ the system

$$(13) \quad \dot{y} - [f_x(u(t, \varepsilon) + \gamma_0(t)) + \varepsilon g_x(\varepsilon t + \alpha, u(t, \varepsilon) + \gamma_0(t), \varepsilon)] \cdot y = 0$$

has only the trivial solution as bounded solution. As

$$[f_x(u(t, \varepsilon) + \gamma_0(t)) + \varepsilon g_x(\varepsilon t + \alpha(\varepsilon), u(t, \varepsilon) + \gamma_0(t), \varepsilon)] \xrightarrow{\varepsilon \rightarrow 0^+} f_x(\gamma_0(t)),$$

it follows from the roughness of exponential dichotomies, that the system (13) has an exponential dichotomy on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with projection $\tilde{P}_+$, $\tilde{P}_-$ such that

$$\dim \mathcal{R}\tilde{P}_+ = \dim \mathcal{R}P_+, \quad \dim \mathcal{N}\tilde{P}_- = \dim \mathcal{N}P_-.$$

As a consequence $\dim \mathcal{R}\tilde{P}_+ + \dim \mathcal{N}\tilde{P}_- = n$ and $\mathcal{R}\tilde{P}_+ \cap \mathcal{N}\tilde{P}_- = \{0\}$ because of (13) has only the trivial bounded solution.

Then $\mathbb{R}^n = \mathcal{R}\tilde{P}_+ \oplus \mathcal{N}\tilde{P}_-$ and so

$$\dot{y} = [f_x(u(t, \varepsilon) + \gamma_0(t)) + \varepsilon g_x(\varepsilon t + \alpha(\varepsilon), u(t, \varepsilon) + \gamma_0(t), \varepsilon)] \cdot y$$

has an exponential dichotomy on $(-\infty, +\infty)$.

REMARK. – Suppose (3) satisfies the same conditions hold in Theorem 2.1, then there exists a unique T -periodic solution $x_0(t, \varepsilon)$ such that $\sup_{t \in \mathbb{R}} |x_0(t, \varepsilon) - p| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

The proof of this result follows the same procedure of the Theorem 2.1, and indeed, it is easier than it because of the invertibility of the linear part, and $\gamma(t, \varepsilon)$ tends to $x_0(t, \varepsilon)$ because of the saddle-point property.

3. - Chaos of the Poincaré map.

Let $f(t, x)$ be a vector function, defined, bounded and continuous in $\mathbb{R} \times \mathbb{R}^n$. Suppose also that the partial derivative $f_x(t, x)$ exists, $f_x \in C_b^0[\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n]$, f_x continuous in x , uniformly with respect to t . Let us recall the following result:

THEOREM 3.1 («Shadowing lemma»). - Suppose for each integer k the system

$$(14) \quad \dot{x} = f(t, x)$$

has a solution $w_k(t)$ defined on an interval $[t_{k-1}, t_k]$ such that

(i) the variational equation

$$(15) \quad \dot{x} = f_x(t, w_k(t)) \cdot x$$

has an exponential dichotomy on $[t_{k-1}, t_k]$, with projection matrix function $\mathcal{P}_k(t)$ and constants K, α ;

$$(ii) \quad |w_{k-1}(t_{k-1}) - w_k(t_{k-1})| \leq \delta;$$

$$(iii) \quad |\mathcal{P}_{k-1}(t_{k-1}) - \mathcal{P}_k(t_{k-1})| \leq \delta;$$

$$(iv) \quad t_k - t_{k-1} \geq \tau.$$

Then there exist positive constants μ_0, τ_0 and a function $\delta_0(\mu)$ such that if $\tau \geq \tau_0$, $0 \leq \mu \leq \mu_0$ and $\delta \leq \delta_0(\mu)$, system (14) has a unique solution $x(t)$, satisfying $|x(t) - w_k(t)| \leq \mu$ for $t_{k-1} \leq t \leq t_k$ and all k . ■

In Section 2 we showed that, given the singular system

$$(5) \quad \varepsilon \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

$f(x), g(t, x, \varepsilon)$, being C_b^r -function in their arguments ($r \geq 2$), and $g(t, x, \varepsilon)$ being T -periodic in t , with the same hypotheses of the Theorem 2.1, setting $y(t) = x(\varepsilon t + \alpha(\varepsilon))$, system

$$(11) \quad \dot{y} = f(y) + \varepsilon g(\varepsilon t + \alpha(\varepsilon), y, \varepsilon),$$

equivalent to the system (5) has a unique transversal homoclinic orbit $u(t)$ such that

$$\sup_{t \in \mathbb{R}} |u(t) - \gamma_0(t)| \leq C|\varepsilon|$$

where $u(t) = u(t, \varepsilon) = \gamma(\varepsilon t + \alpha(\varepsilon), \varepsilon)$.

Now we are able to show the following result:

THEOREM 3.2. – For each $a = \{a_k\}_{k \in \mathbb{Z}}$ system

$$\varepsilon \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

has a unique solution $x_a(t)$, $t \in \mathbb{R}$, satisfying

$$\sup_{-T/2 + \alpha(\varepsilon) < t < T/2 + \alpha(\varepsilon)} \left| x_a \left(t + (2k-1) \frac{T}{2} \right) - \gamma(t + a_k T, \varepsilon) \right| < \varrho$$

with ϱ sufficiently small.

PROOF. – Putting $y(t) = x(\varepsilon t + \alpha(\varepsilon))$ we may study system (11) and look for a solution $y_a(t)$ satisfying

$$\sup_{-T/2\varepsilon \leq t \leq T/2\varepsilon} \left| y_a \left(t + (2k-1) \frac{T}{2\varepsilon} \right) - u \left(t + a_k \frac{T}{\varepsilon} \right) \right| < \varrho.$$

For $u(t) = \gamma(\varepsilon t + \alpha(\varepsilon), \varepsilon)$ and for $k \in \mathbb{Z}$ we define

$$v_k(t) = u \left(t + a_k \frac{T}{\varepsilon} \right), \quad w_k(t) = v_k \left(t - (2k-1) \frac{T}{2\varepsilon} \right), \quad t_k = k \frac{T}{\varepsilon}.$$

Then $w_k(t) = u(t - (2k-1)(T/2\varepsilon) + a_k(T/\varepsilon))$ satisfies

$$\begin{aligned} (16) \quad \dot{w} &= f(w) + \varepsilon g \left[\varepsilon \left(t - k \frac{T}{\varepsilon} + \frac{T}{2\varepsilon} + a_k \frac{T}{\varepsilon} \right) + \alpha(\varepsilon), w, \varepsilon \right] = \\ &= f(w) + \varepsilon g \left(\varepsilon t + \frac{T}{2} + \alpha(\varepsilon), w, \varepsilon \right) \end{aligned}$$

and the variational equation

$$\begin{aligned} \dot{w} &= \left[f'(w_k(t)) + \varepsilon g_x \left(\varepsilon t + \frac{T}{2} + \alpha(\varepsilon), w_k(t), \varepsilon \right) \right] w = \\ &= \left[f' \left(u \left(t - (2k-1) \frac{T}{2\varepsilon} + a_k \frac{T}{\varepsilon} \right) \right) + \varepsilon g_x \left(\varepsilon \left(t + \frac{T}{2\varepsilon} \right) + \right. \right. \\ &\quad \left. \left. + \alpha(\varepsilon), u \left(t - (2k-1) \frac{T}{2\varepsilon} + a_k \frac{T}{\varepsilon} \right), \varepsilon \right) \right] w \end{aligned}$$

has an exponential dichotomy on $\mathbb{R}$ and so on $[t_{k-1}, t_k]$ with projection matrix function $\mathcal{P}_k(t)$.

In fact, writing t in the place of $t - T/2\varepsilon + (k + a_k)(T/\varepsilon)$ and recalling that $g(t, x, \varepsilon)$ is T -periodic in t , we have

$$(17) \quad \dot{w}[f'(u(t)) + \varepsilon g_x(\varepsilon t + \alpha(\varepsilon), u(t), \varepsilon)],$$

which has an exponential dichotomy on $\mathbb{R}$.

So the first condition of the Theorem 3.1 is verified.

Now we prove (ii), (iii), (iv).

$$\begin{aligned} (ii) \quad & |w_{k-1}(t_{k-1}) - w_k(t_{k-1})| = \\ & = \left| v_{k-1} \left(t_{k-1} - (2k-3) \frac{T}{2\varepsilon} \right) - v_k \left(t_{k-1} - (2k-1) \frac{T}{2\varepsilon} \right) \right| = \\ & = \left| u \left(k \frac{T}{\varepsilon} - \frac{T}{\varepsilon} - k \frac{T}{\varepsilon} + 3 \frac{T}{2\varepsilon} + a_{k-1} \frac{T}{\varepsilon} \right) - u \left(k \frac{T}{\varepsilon} - \frac{T}{\varepsilon} - k \frac{T}{\varepsilon} + \frac{T}{2\varepsilon} + a_k \frac{T}{\varepsilon} \right) \right| = \\ & = \left| u \left(\left(\frac{1}{2} + a_{k-1} \right) \frac{T}{\varepsilon} \right) - u \left(\left(a_k - \frac{1}{2} \right) \frac{T}{\varepsilon} \right) \right| \leq \\ & \leq \left| u \left(\left(\frac{1}{2} + a_{k-1} \right) \frac{T}{\varepsilon} \right) - \gamma_0 \left(\left(\frac{1}{2} + a_{k-1} \right) \frac{T}{\varepsilon} \right) \right| + \\ & + \left| \gamma_0 \left(\left(\frac{1}{2} + a_{k-1} \right) \frac{T}{\varepsilon} \right) - \gamma_0 \left(\left(a_k - \frac{1}{2} \right) \frac{T}{\varepsilon} \right) \right| + \\ & + \left| u \left(\left(a_k - \frac{1}{2} \right) \frac{T}{\varepsilon} \right) - \gamma_0 \left(\left(a_k - \frac{1}{2} \right) \frac{T}{\varepsilon} \right) \right| \ll 1 \quad \text{if } \varepsilon \ll 1 \end{aligned}$$

in fact

$$\sup_{\mathbb{R}} |u(t) - \gamma_0(t)| \xrightarrow{\varepsilon \rightarrow 0^+} 0 \quad \text{and} \quad |\gamma_0(t) - p| \rightarrow 0 \quad \text{as } |t| \rightarrow \infty.$$

So if $|\varepsilon| < \varepsilon_0 \ll 1$, (ii) of Theorem 3.1 is satisfied.

To prove (iii), let $x_0(t)$ be the T -periodic solution of (5), whose existence is shown in the remark of Theorem 2.1, and put $u_0(t) = x_0(\varepsilon t + \alpha(\varepsilon), \varepsilon)$.

Then $u_0(t)$ is a (T/ε) -periodic solution of system (11).

Denote $F(t, x) = f(x) + \varepsilon g(\varepsilon t + \alpha(\varepsilon), x, \varepsilon)$.

Then $|F_x(t, v_k(t)) - F_x(t, u_0(t))| \rightarrow 0$ as $|t| \rightarrow +\infty$ uniformly with respect to k , because of

$$|v_k(t) - u_0(t)| = |\gamma(\varepsilon t + a_k T + \alpha(\varepsilon), \varepsilon) - x_0(\varepsilon t + a_k T + \alpha(\varepsilon), \varepsilon)| \rightarrow 0 \quad \text{as } |t| \rightarrow +\infty.$$

It follows from the roughness theorem that the variational equation

$$(18) \quad \dot{x} = F_x(t, u_0(t))x$$

has an exponential dichotomy on $[0, +\infty)$, and being $F_x(t, u_0(t))$ (T/ε) -periodic, (18) has an exponential dichotomy on $\mathbb{R}$ and the corresponding to projection matrix function $Q(t)$ is (T/ε) -periodic.

If we put

$$A_k(t) = F_x(t, v_k(t)), \quad B_k(t) = F_x(t, u_0(t)) - F_x(t, v_k(t))$$

then we may apply the roughness theorem to the following system

$$\dot{x} = [A_k(t) + B_k(t)]x = F_x(t, u_0(t))x.$$

So, if $\tilde{\mathcal{P}}_k(t)$ is the projection matrix function corresponding to the exponential dichotomy for

$$\dot{x} = A_k(t)x,$$

we have $|\tilde{\mathcal{P}}_k(t) - Q(t)| \rightarrow 0$ as $|t| \rightarrow +\infty$ uniformly with respect to k because of $|B_k(t)| \rightarrow 0$ uniformly with respect to k .

We can notice that $\mathcal{P}_k(t) = \mathcal{P}_k(t - (2k-1)(T/2\varepsilon))$, and that $\mathcal{P}_k(t)$, $\tilde{\mathcal{P}}_k(t)$ and $Q(t)$ are uniquely determined, because the corresponding linear systems have an exponential dichotomy on $\mathbb{R}$.

Now

$$\begin{aligned} & |\mathcal{P}_{k-1}(t_{k-1}) - \mathcal{P}_k(t_{k-1})| = \\ & = \left| \tilde{\mathcal{P}}_{k-1} \left(k \frac{T}{\varepsilon} - \frac{T}{\varepsilon} - 2k \frac{T}{2\varepsilon} + 3 \frac{T}{2\varepsilon} \right) - \tilde{\mathcal{P}}_k \left(k \frac{T}{\varepsilon} - \frac{T}{\varepsilon} - 2k \frac{T}{2\varepsilon} + \frac{T}{2\varepsilon} \right) \right| = \\ & = \left| \tilde{\mathcal{P}}_{k-1} \left(\frac{T}{2\varepsilon} \right) - \tilde{\mathcal{P}}_k \left(-\frac{T}{2\varepsilon} \right) \right| \leq \left| \tilde{\mathcal{P}}_{k-1} \left(\frac{T}{2\varepsilon} \right) - Q \left(\frac{T}{2\varepsilon} \right) \right| + \left| Q \left(\frac{T}{2\varepsilon} \right) - Q \left(-\frac{T}{2\varepsilon} \right) \right| + \\ & \quad + \left| Q \left(-\frac{T}{2\varepsilon} \right) - \tilde{\mathcal{P}}_k \left(-\frac{T}{2\varepsilon} \right) \right| \xrightarrow{\varepsilon \rightarrow 0^+} 0 \text{ uniformly with respect to } k \in \mathbb{Z}. \end{aligned}$$

$$(iv) \quad t_k - t_{k-1} = \frac{T}{\varepsilon} \geq \tau \quad \text{if } \varepsilon \ll 1.$$

All the conditions of theorem 3.1 are verified, hence there exists a unique solution

$y(t)$ of

$$\dot{y} = f(y) + \varepsilon g\left(\varepsilon t + \alpha(\varepsilon) + \frac{T}{2}, y, \varepsilon\right)$$

satisfying

$$\sup_{t \in [t_{k-1}, t_k]} |y(t) - w_k(t)| < \varrho.$$

Changing t with $t + (2k - 1) T/2\varepsilon$ and writing y_a instead of y we have

$$\dot{y}_a(t) = f(y_a(t)) + \varepsilon g(\varepsilon t + \alpha(\varepsilon), y_a(t), \varepsilon)$$

and

$$\sup_{-T/2\varepsilon \leq t \leq T/2\varepsilon} \left| y_a\left(t + (2k - 1) \frac{T}{2\varepsilon}\right) - u\left(t + a_k \frac{T}{\varepsilon}\right) \right| < \varrho$$

or putting $x_a(t) = y_a(\varepsilon^{-1}(t - \alpha(\varepsilon)))$ we have

$$\varepsilon \dot{x}_a(t) = f(x_a(t)) + \varepsilon g(t, x_a(t), \varepsilon)$$

and

$$(19) \quad \sup_{-T/2\varepsilon + \alpha(\varepsilon) \leq t \leq T/2\varepsilon + \alpha(\varepsilon)} \left| x_a\left(t + (2k - 1) \frac{T}{2}\right) - \gamma(t + a_k T, \varepsilon) \right| < \varrho.$$

Define now the function $\phi: \Sigma \rightarrow \mathbb{R}^n$, where Σ is the set of the sequences $\{s_k\}_{k \in \mathbb{Z}}$, putting $\phi(a) = x_a(0)$.

PROPOSITION 3.1. – The function $\phi: \Sigma \rightarrow \mathbb{R}^n$, defined above, is a homeomorphism onto a compact subset of $\mathbb{R}^n$.

PROOF. – The proof follows closely the one of Corollary 3.6 in [8], so we omit it. ■

Now we are able to prove the main theorem of this section:

THEOREM 3.3. – Let $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$ the Poincaré map of system (3). Then the following diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{R}^n & \xrightarrow{P} & \mathbb{R}^n \end{array}$$

commutes. In other words for all $a \in \Sigma$ we have

$$(P^0 \phi)(a) = (\phi^0 \sigma)(a).$$

Hence P is isomorphic to the Bernoulli shift and then behaves chaotically.

PROOF. — Let $x_0 \in \mathbb{R}^n$, and let $x(t, x_0)$ be the solution of

$$\begin{cases} \varepsilon \dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \\ x(0) = x_0. \end{cases}$$

The Poincaré map P is defined by

$$Px_0 = x(T, x_0).$$

We want to prove that

$$(P^0 \phi)(a) = (\phi^0 \sigma)(a)$$

or that

$$\phi(\sigma(a)) = x(T, \phi(a)) = x_a(T).$$

Put $x(t) = x_a(t + T)$. We find

$$\varepsilon \dot{x}(t) = f(x(t)) + \varepsilon g(t, x(t), \varepsilon)$$

and

$$\begin{aligned} \left| x\left(t + (2k-1)\frac{T}{2}\right) - \gamma(t + a_{k+1}T, \varepsilon) \right| &= \left| x_a\left(t + (2k+1)\frac{T}{2}\right) - \gamma(t + a_{k+1}T, \varepsilon) \right| = \\ &= \left| x_a\left(t + [2(k+1)-1]\frac{T}{2}\right) - \gamma(t + a_{k+1}T, \varepsilon) \right| < \varrho \\ &\text{for } t \in \left[-\frac{T}{2} + \alpha(\varepsilon), \frac{T}{2} + \alpha(\varepsilon)\right]. \end{aligned}$$

By the uniqueness of $x_a(t)$ we have $x(t) = x_{\sigma(a)}(t)$ and so $x_a(T) = x(0) = x_{\sigma(a)}(0) = \phi(\sigma(a))$. ■

REFERENCES

- [1] F. BATTELLI - M. FECKAN, *Subharmonic solutions in singular systems*, to appear in J. Differential Equations.
- [2] F. BATTELLI - K. J. PALMER, *Chaos in the Duffing equation*, J. Differential Equations, **102** (2) (1993), pp. 276-301.

- [3] S.-N. CHOW - J. K. HALE - J. MALLET-PARET, *An example of bifurcation to homoclinic orbits*, J. Differential Equations, **37** (1980), pp. 351-373.
 - [4] W. A. COPPEL, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, **629**, Springer-Verlag, Berlin (1978).
 - [5] M. FECKAN, *Note on a local invertibility*, to appear on Mathematica Slovaca.
 - [6] J. GUCKENHEIMER - P. HOLMES, *Non Linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York (1983).
 - [7] U. KIRCHGRABER - D. STOFFER, *Chaotic behavior in simple dynamical systems*, SIAM Review, **32** (3) (1990), pp. 424-452.
 - [8] K. J. PALMER, *Exponential dichotomie and transversal homoclinic points*, J. Differential Equations, **55** (1984), pp. 225-256.
 - [9] S. SMALE, *Differentiable dynamical systems*, Bull. Amer. Math. Soc., **73** (1967), pp. 747-817.
-