Annali di Matematica pura ed applicata
(IV), Vol. CLXXVI (1999), pp. 57-72

Global Random Attractors are Uniquely Determined by Attracting Deterministic Compact Sets (*).

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Abstract. – *It is shown that for continuous dynamical systems an analogue of the Poincaré recurrence theorem holds for Ω-limit sets. A similar result is proved for Ω-limit sets of random dynamical systems* (RDS) *on Polish spaces. This is used to derive that a random set which attracts every (deterministic) compact set has full measure with respect to every invariant probability measure for the* RDS. *Then we show that a random attractor coincides with the Ω-limit set of a (nonrandom) compact set with probability arbitrarily close to one, and even almost surely in case the base flow is ergodic. This is used to derive uniqueness of attractors, even in case the base flow is not ergodic.*

1. – Introduction.

Given a measurable dynamical system with an invariant measure, the measure of the points which return infinitely often to a set is not smaller than the measure of the set itself. This is the Poincaré recurrence theorem. Given a continuous dynamical system, one can introduce the Ω-limit set of a set in the state space. This is the set of all points which are approached arbitrarily close by orbits starting from the initial set. It can be considered as «the opposite» of the set of recurrent points. We show first that for an invariant measure for a flow of continuous maps on a topological state space the measure of the Ω-limit set is not smaller than the measure of the set itself. Then we extend the result to random dynamical systems. This is then used to derive that for random dynamical systems on Polish spaces which have a (random) attractor every invariant measure is supported by the attractor. This already holds if the attractor attracts only compact sets (and not necessarily bounded sets, which makes a difference for infinite-dimensional systems).

We then address the question whether Ω-limit sets of deterministic bounded sets

(*) Entrata in Redazione il 10 marzo 1997.

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Mathematics Subject Classification: Primary 58F11 58F12, Secondary 58F39 60D05 60H10 60H15 93E03.

already give the whole random attractor. We obtain that there exist even compact sets whose Ω-limit sets almost surely give the whole attractor, provided the base flow is ergodic. Furthermore, on hand of a (one-dimensional) example it is shown that ergodicity of the base flow cannot be dispensed with. Still it is possible to find compact nonrandom sets such that the random attractor is contained in their Ω-limit set with probability arbitrarily close to one, even if the base flow is not ergodic. These results are interesting for numerical investigations. Usually there an ergodic base flow is assumed, so it suffices to find a compact set in the state space with the property that the random attractor is contained in this set with positive probability. Calculating the Ω-limit set of this deterministic compact set then gives the whole random attractor.

Another consequence is uniqueness of the attractor. Uniqueness even holds for the weaker notion of attractors for *compact sets*, and even in the case of a non-ergodic base flow.

The paper is organized as follows. In Section 2 we consider first the deterministic case. Here the result holds for general topological spaces. Then we recall the concept of a random dynamical system (RDS) and derive the result for this case in Section 3. Here we assume the state space to be a Polish space—in fact: Suslin suffices—in order to get measurability of Ω-limit sets. The result for RDS contains the deterministic one as a special case—but only for Suslin state spaces. Since, furthermore, the RDS case needs considerably more notation it seems justified to develop the idea for classical dynamical systems separately in order to exhibit the structure of the argument. In Section 4 we show that random attractors support invariant measures. Finally, in Section 5 it is proved that a random attractor coincides with the Ω-limit set of a compact deterministic set with probability arbitrarily close to one, and even almost surely provided the base flow is ergodic. This is then used to derive uniqueness of the global random attractor.

2. – Classical dynamical systems.

Suppose X is a topological space, and $\varphi_t: X \to X$, $t \in T$, is a family of continuous maps, where T is either $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$ or $\mathbb{Z}$, and $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all t, $s \in T$, and $\varphi_0 = \mathrm{id}$. Clearly, if T is two-sided ($\mathbb{R}$ or $\mathbb{Z}$) then φ_t is a homeomorphism with $\varphi_t^{-1} = \varphi_{-t}$. If T is discrete ($\mathbb{N}$ or $\mathbb{Z}$) then $\varphi_n = \varphi_1^n$.

2.1. DEFINITION. – *For any $B \subset X$ the Ω-limit set of B is*

$$\Omega_B = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi_\tau B},$$

where $\overline{D}$ stands for the closure of D in X.

For X a metric space another characterization of the Ω-limit set is

$$\Omega_B = \{x \in X: \text{ there exist } t_n \in T,\ b_n \in B,\ n \in \mathbb{N}, \text{ such that } x = \lim_{n \to \infty} \varphi_{t_n}(b_n)\}.$$

The Ω-limit set $\Omega_B \subset X$ is defined for any $B \subset X$; it is by definition closed.

We will need the following two elementary properties of maps in an essential way.

For an arbitrary map $f\colon Y \to Z$ from any set Y to any other set Z

$$A \subset f^{-1}(f(A)) \tag{1}$$

for any $A \subset Y$. If Y and Z are topological spaces then f is continuous (if and) only if

$$\overline{f^{-1}(A)} \subset f^{-1}(\overline{A}) \tag{2}$$

for every $A \subset Z$.

The following result is an analogue to the Poincaré recurrence theorem. Usually it is stated for discrete time only. But it is completely immediate that it holds for continuous time (semi-) flows. It suffices to note that the set of recurrent points along any (additive) sequence of discrete times is contained in the set of recurrent points along arbitrary real times. This holds just as well in the present case of Ω-limit sets. We nevertheless formulate everything without referring to a particular choice of discrete or continuous time.

2.2 THEOREM. – *Suppose μ is an invariant Borel probability measure for $(\varphi_t)_{t \in T}$. Then*

(i) $\mu(\Omega_B) \geq \mu(\overline{B})$ *for any* $B \subset X$,

(ii) $\mu(\Omega_B \cap B) = \mu(B)$ *for any Borel set* $B \subset X$.

PROOF. – Choose $B \subset X$. For $t \geq 0$ put

$$C_t = \overline{\bigcup_{\tau \geq t} \varphi_\tau B}.$$

Then $C_s \supset C_t$ for any $0 \leq s \leq t$, so $C_t \downarrow C_\infty = \Omega_B$. Since μ is finite, this implies $\mu(C_t) \to \mu(\Omega_B)$ for $t \to \infty$ (it suffices to let t go through $\mathbb{N}$). We now claim that $C_t \subset \varphi_s^{-1} C_{t+s}$ for any $s \geq 0$. In fact,

$$C_t = \overline{\bigcup_{\tau \geq t} \varphi_\tau B} = \overline{\bigcup_{\tau \geq t+s} \varphi_{\tau - s} B} \subset$$

$$\subset \overline{\bigcup_{\tau \geq t+s} \varphi_s^{-1}(\varphi_s(\varphi_{\tau-s} B))} = \overline{\bigcup_{\tau \geq t+s} \varphi_s^{-1}(\varphi_\tau B)} = \overline{\left(\varphi_s^{-1} \bigcup_{\tau \geq t+s} \varphi_\tau B\right)} \subset$$

$$\subset \varphi_s^{-1}\left(\overline{\bigcup_{\tau \geq t+s} \varphi_\tau B}\right) = \varphi_s^{-1} C_{t+s},$$

where we used (1) and (2) for the first and the second inclusion, respectively. Now this implies

$$\mu(C_t) \leq \mu(\varphi_s^{-1} C_{t+s}) = \mu(C_{t+s})$$

for every $s \geq 0$, using invariance of μ for the identity. On the other hand $C_{t+s} \subset C_t$, hence

$\mu(C_{t+s}) \leq \mu(C_t)$ for any $t, s \geq 0$. We thus obtain, for any $0 \leq s \leq t$, $\mu(C_t) = \mu(C_s)$ and consequently

$$\mu(C_s \setminus C_t) = 0 . \tag{3}$$

To obtain (i) note that

$$\mu\left(\overline{\bigcup_{\tau \geq 0} \varphi_\tau B}\right) = \mu(C_0) \equiv \mu(C_t) \to \mu(\Omega_B)$$

for $t \to \infty$, hence $\mu(C_0) = \mu(\Omega_B)$. Since $B \subset C_0$, and consequently $\overline{B} \subset C_0$, we get $\mu(\overline{B}) \leq \mu(\Omega_B)$.

To obtain (ii) suppose $B \subset X$ is a Borel set. From (3) we obtain for any $0 \leq s \leq t$

$$\mu(C_s \cap B) = \mu(C_t \cap B) + \mu((C_s \setminus C_t) \cap B) = \mu(C_t \cap B),$$

hence $\mu(B) = \mu(C_0 \cap B) \equiv \mu(C_t \cap B) \equiv \mu(\Omega_B \cap B)$ for all $t \geq 0$. ■

Of course, for Borel sets B we immediately get $\mu(\Omega_B) \geq \mu(\overline{B}) \geq \mu(B)$ from (i). The argument of the proof in fact yields that for every Borel set $D \subset X$ we have $\mu(C_0 \cap D) = \mu(\Omega_B \cap D)$. Thus for every Borel set $D \subset C_0$ we get $\mu(\Omega_B \cap D) = \mu(D)$. Applying this, e.g., to $\overline{B}$, yields $\mu(\Omega_B \cap \overline{B}) = \mu(\overline{B})$ for any $B \subset X$. But note that possibly $\mu(\Omega_B \cap B) < \mu(\overline{B})$. Take, e.g., $\varphi = \mathrm{id}$ on $X = [0, 1]$, μ the Lebesgue measure, and $B = \mathbb{Q} \cap [0, 1]$, then $\Omega_B = [0, 1]$.

3. – Random dynamical systems.

Let $\{\vartheta_t : \Omega \to \Omega\}$, $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ such that $(t, \omega) \mapsto \vartheta_t \omega$ is measurable (with respect to the Borel-σ-algebra on $\mathbb{R}$), $\vartheta_0 = \mathrm{id}$, and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all t, s. Thus $(\vartheta_t)_{t \in T}$ is a flow, and $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t \in T})$ is a (measurable) dynamical system. Of course, the measurability condition is automatic for discrete time. Note also that time is a priori assumed to be two-sided here. In particular, ϑ_t is assumed to be measurably invertible, $\vartheta_t^{-1} = \vartheta_{-t}$.

Notation: we sometimes write (ϑ_t) instead of $(\vartheta_t)_{t \in T}$ when referring to properties of the whole family instead of a single ϑ_t.

For a topological space Y we denote by $\mathcal{B}(Y)$ or just $\mathcal{B}$ the σ-algebra of Borel sets of Y.

3.1. DEFINITION. – Suppose X is a topological space, and T is either $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$ or $\mathbb{Z}$, and suppose (ϑ_t) on $(\Omega, \mathcal{F}, P)$ is given as above with time $\mathbb{R}$ if $T = \mathbb{R}^+$ or $\mathbb{R}$ and with time $\mathbb{Z}$ if $T = \mathbb{N}$ or $\mathbb{Z}$. A *random dynamical system* (RDS) with time T on X over (ϑ_t) on $(\Omega, \mathcal{F}, P)$ is a map

$$\varphi : T \times X \times \Omega \to X ,$$

$$(t, x, \omega) \mapsto \varphi(t, \omega)\, x ,$$

such that

(i) $(t, \omega) \mapsto \varphi(t, \omega)\, x$ is measurable (with respect to $\mathcal{B}(T) \otimes \mathcal{F}$) for every $x \in$ $\in X$,

(ii) $x \mapsto \varphi(t, \omega)\, x$ is continuous for every $(t, \omega) \in T \times \Omega$,

(iii) $\varphi(0, \omega) = \mathrm{id}|_X$ (identity on X) and

(4) $$\varphi(t+s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega)$$

P-a.s. for every t, $s \in T$, where $\circ$ means composition.

We will often speak of an RDS φ, not mentioning the base flow (ϑ_t) on $(\Omega, \mathcal{F}, P)$.

3.2. REMARK. – (i) The time for the base flow (ϑ_t) is always assumed to be two-sided, even if φ is defined for nonnegative time only. Furthermore, the maps $\varphi(t, \omega)$: $X \to X$ are not assumed to be invertible a priori. The cocycle property implies that for two sided time ($T = \mathrm{R}$ or $T = \mathrm{Z}$) $\varphi(t, \omega)$ is automatically invertible P-a.s. for every $t \in T$. In fact, in this case $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t \omega)$ with probability one for every $t \in T$.

(ii) Definition 3.1 (i) and (ii) imply measurability of $(t, x, \omega) \mapsto \varphi(t, \omega)\, x$ as soon as X is separable and metrizable, see Crauel [5] Lemma 2.1. The definition of an RDS in the literature does not always assume continuity of $\varphi(t, \omega)$ on X, but only measurability of φ in all three components.

(iii) A family of maps $\varphi(t, \omega)$, $(t, \omega) \in T \times \Omega$, satisfying (4) is called a (*crude*) *cocycle* (with respect to (ϑ_t) on $(\Omega, \mathcal{F}, P)$), and (4) is the cocycle property. The family is said to be a *perfect* cocycle if (4) is satisfied for all t, $s \in T$ and ω from a (ϑ_t)-invariant set of full measure (or for all $\omega \in \Omega$). For details see Arnold and Scheutzow [2]. We do not need a perfect cocycle here.

(iv) Very often continuous time RDS are induced by stochastic differential equations (SDE). Then the map $\varphi(t, \omega)$ coincides with the stochastic flow $X_{0,t}(\omega)$ induced by the SDE. The map $\varphi(t, \vartheta_s \omega)$ then coincides with the stochastic flow evaluated in the times s and $s + t$, so $\varphi(t, \vartheta_s \omega) = X_{s, s+t}(\omega)$ for P-almost all ω. Here the exceptional set depends on s in general, so the stochastic flow generates a crude cocycle. The problem of perfecting crude cocycles induced by SDE is currently under investigation. For SDE which allow solutions for two-sided time this has been carried out by Arnold and Scheutzow [2]. In particular, for an RDS φ induced by an SDE $t \mapsto \varphi(t, \omega)$ is continuous (which is used as a short hand notation for continuity of $(t, x) \mapsto \varphi(t, \omega)\, x$).

(v) Under suitable conditions an SDE on a finite dimensional space generates a cocycle of homeomorphisms or even diffeomorphisms. In this case the (perfect) cocycle property implies that $t \mapsto \varphi(t, \vartheta_{-t}\omega) = \varphi^{-1}(-t, \omega)$ is continuous P-almost surely.

For a detailed account on the contemporary state of the art in the theory of RDS see Arnold [1].

We aim at a similar result as Theorem 2.2 for RDS instead of classical dynamical systems. We need the notions of invariant measures and of Ω-limit sets.

Denote by $Pr(X)$ the space of Borel probability measures on X, equipped with the smallest topology such that $\varrho \mapsto \int f d\varrho$ is continuous for every f: $X \to \mathrm{R}$ bounded and continuous (often referred to as «narrow» or «weak*» or «topology of weak convergence» on $Pr(X)$).

3.3. DEFINITION. – A *random measure* is a measurable map $\mu\colon \Omega \to Pr(X)$, $\omega \mapsto \mu_\omega$ (with respect to the Borel σ-algebra of the narrow topology on $Pr(X)$), where two such maps are identified if they coincide for P-almost all ω. An *invariant measure for* φ is a random measure μ such that

$$\varphi(t, \omega)\, \mu_\omega = \mu_{\vartheta_t \omega}$$

P-a.s. for every $t \geq 0$.

The above definition of φ-invariant measures only makes sense for a two-sided time flow (ϑ_t); see, e.g., Crauel [4] p. 160.

Any random measure $\omega \mapsto \mu_\omega$ induces a measure on the product space $(X \times \Omega, \mathcal{B} \otimes \mathcal{F})$ by

$$\mu(B) = \int_\Omega \mu_\omega(B(\omega))\, dP(\omega), \tag{5}$$

where $B \subset X \times \Omega$ is given by its sections $B(\omega) = \{x \in X\colon (x, \omega) \in B\}$, $\omega \in \Omega$. Furthermore, μ is uniquely determined by its values on product sets,

$$\mu(B \times F) = \int_F \mu_\omega(B)\, dP(\omega)$$

for $B \in \mathcal{B}$ and $F \in \mathcal{F}$. This only needs a measurable structure on X (and, of course, on Ω). In order to disintegrate a measure μ on the product space to obtain a random measure $\omega \mapsto \mu_\omega$ satisfying (5) needs more structure on X. It suffices, e.g., to have $\mathcal{B}$ countably generated and $\pi_X \mu$ compactly approximable (see Gänssler und Stute [8] Satz 5.3.21, p. 198); here $\pi_X\colon X \times \Omega \to X$ denotes the projection. In particular, for X Polish this is satisfied.

In order to introduce Ω-limit sets for RDS we need some notations. Given $B \subset X \times \Omega$, we denote by $\overline{B}$ the set with sections $\overline{B}(\omega) = \overline{B(\omega)}$. A map $B\colon \Omega \to 2^X$ is said to be a *random set*, if the associated $B = \{(x, \omega)\colon x \in B(\omega)\}$ is a measurable subset of the product space, i.e., if $B \in \mathcal{B} \otimes \mathcal{F}$. Clearly a random set can only take values in $\mathcal{B} \subset 2^X$.

From now on we assume X to be a Polish space, i.e., X is separable and there exists a complete metric inducing the topology. Denote by d such a metric. A map $C\colon \Omega \to 2^X$ is said to be a *closed random set* if $C(\omega)$ is closed for P-almost all ω, and if $\omega \mapsto d(x, C(\omega))$ is measurable for any $x \in X$, where $d(x, B) = \inf_{b \in B} d(x, b)$ for $x \in X$ and $B \subset X$ (so $d(x, B) = d(x, \overline{B})$). The notion of a closed random set does not depend on the choice of the metric d. Furthermore, every non-empty closed random set $\omega \mapsto C(\omega)$ admits a sequence of measurable selections $c_n\colon \Omega \to X$, $n \in \mathbb{N}$, such that $C(\omega) = \overline{\{c_n(\omega)\colon n \in \mathbb{N}\}}$, see Castaing and Valadier [3] Theorem III.8, p. 66. A closed random set is more than a random set which has closed sections. However, any random set with closed sections is a closed random set as soon as $\mathcal{F}$ is universally measurable, so, in particular, if $\mathcal{F}$ is complete with respect to P; see Castaing and Valadier [3] Theorem III.30, p. 80.

In general intersections of closed random sets need not give a closed random set again. But if $(C_t)_{t \in T}$ is a decreasing family of compact random sets (i.e., $C_t \subset C_s$ for $s \leq t$), then $C = \lim_t C_t = \bigcap_t C_t$ is a closed random set again. In fact, $\bigcap_t C_t = \bigcap_{n \in \mathbb{N}} C_n$, hence for $x \in X$

$$d(x, C) = \lim_{n \to \infty} d(x, C_n). \tag{6}$$

We need another elementary observation. If $\varphi: X \to X$ is any continuous map, then $\varphi \overline{B} \subset \overline{\varphi B}$ for any $B \subset X$, so $d(x, \varphi B) = d(x, \overline{\varphi B}) \geq d(x, \varphi \overline{B}) \geq d(x, \varphi B)$, hence $d(x, \varphi B) = d(x, \varphi \overline{B})$. In particular,

$$d(x, \varphi B) = d(x, \varphi D) \tag{7}$$

for any $B, D \subset X$ with $\overline{D} = \overline{B}$.

3.4. DEFINITION. – For any $B \subset X \times \Omega$ put

$$\Omega_B(\omega) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \varphi(s, \vartheta_{-s}\omega) B(\vartheta_{-s}\omega)},$$

$\omega \in \Omega$. Then $\omega \mapsto \Omega_B(\omega)$ is said to be the *Ω-limit set of B.*

The present notion of Ω-limit sets for RDS has been introduced for the study of random attractors by Crauel and Flandoli [7]. There it has been shown that the Ω-limit set of a bounded deterministic set is a closed random set at least with respect to the universal completion of $\mathcal{F}$. The following result extends this slightly for nondeterministic B such that $\overline{B}$ is a closed random set.

For an RDS φ put

$$\mathcal{F}_{\leq 0} = \sigma\{\omega \mapsto \varphi(t, \vartheta_{-s}\omega)\, x \colon 0 \leq t \leq s,\ x \in X\} \subset \mathcal{F}.$$

The σ-algebra $\mathcal{F}_{\leq 0}$ is said to be the *past of the system* or *the past of φ.*

3.5. LEMMA. – *Suppose φ is an* RDS *on a Polish space X. For every $B: \Omega \to 2^X$ such that $\omega \mapsto \overline{B}(\omega)$ is a closed random set with respect to the universal completion $\overline{\mathcal{F}}$ of the past $\mathcal{F}_{\leq 0}$ of φ the Ω-limit set $\omega \mapsto \Omega_B(\omega)$ is a closed random set with respect to $\overline{\mathcal{F}}$ (so, in particular, Ω_B is a closed random set with respect to $\mathcal{F}_{\leq 0}$ as soon as $\mathcal{F}_{\leq 0}$ is complete with respect to P).*

PROOF. – Suppose B is a random set such that $\overline{B}$ is a closed random set with respect to $\mathcal{F}_{\leq 0}$ or $\overline{\mathcal{F}}$, respectively. Let $\{b_n: n \in \mathbb{N}\}$ be a sequence of measurable selections with respect to $\mathcal{F}_{\leq 0}$ or $\overline{\mathcal{F}}$, resp., with $\overline{B}(\omega) = \overline{\{b_n(\omega): n \in \mathbb{N}\}}$ for every $\omega \in \Omega$. From (7) we get for any $x \in X$ and for any $t \in T$

$$d(x, \varphi(t, \vartheta_{-t}\omega) B(\vartheta_{-t}\omega)) = d(x, \varphi(t, \vartheta_{-t}\omega)\{b_n(\vartheta_{-t}\omega): n \in \mathbb{N}\}) =$$

$$= \inf_{n \in \mathbb{N}} d(x, \varphi(t, \vartheta_{-t}\omega)\, b_n(\vartheta_{-t}\omega)).$$

Thus $(t, \omega) \mapsto d(x, \varphi(t, \vartheta_{-t}\omega) B(\vartheta_{-t}\omega))$ is measurable with respect to $\mathcal{B}(T) \otimes \overline{\mathcal{F}}$ or $\mathcal{B}(T) \otimes \mathcal{F}_{\leq 0}$, respectively. (For T discrete measurability in t is, of course, trivial.) Now for any $t \in T$

$$d\Big(x, \overline{\bigcup_{s \geq t} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)}\Big) = d\Big(x, \bigcup_{s \geq t} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)\Big) =$$

$$= \inf_{s \geq t} d(x, \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)) \,.$$

In the particular cases where T is discrete and $\overline{B}$ is a closed random set with respect to $\mathcal{F}_{\leq 0}$, or where $t \mapsto \varphi(t, \vartheta_{-t}\omega)$ is continuous and B is nonrandom, it is thus immediate that $\omega \mapsto \bigcup_{s \geq t} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)$ is a closed random set with respect to $\mathcal{F}_{\leq 0}$.

In the general case, joint measurability of $(t, \omega) \mapsto d(x, \varphi(t, \vartheta_{-t}\omega)\, B(\vartheta_{-t}\omega))$ implies

$$D = \{(t, \omega) \colon d(x, \vartheta(t, \vartheta_{-t}\omega)\, B(\vartheta_{-t}\omega)) < a\} \in \mathcal{B}(T) \otimes \overline{\mathcal{F}}$$

for any $a > 0$, so the projection theorem (see Castaing and Valadier [3] Theorem III.23, p. 75) yields

$$\{\omega \colon \inf_{s \geq t} d(x, \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)) < a\} = \pi_{\Omega}(D \cap ([t, \infty) \times \Omega)) \in \overline{\mathcal{F}}$$

for every $a > 0$, where $\pi_{\Omega} \colon X \times \Omega \to \Omega$ is the projection. Thus $\omega \mapsto d(x, C_t(\omega))$ is measurable (with respect to $\overline{\mathcal{F}}$) for every $x \in X$, where $C_t(\omega) = \overline{\bigcup_{s \geq t} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)}$. Thus C_t is a closed random set with respect to $\overline{\mathcal{F}}$ for any $t \geq 0$. Since the intersection $\Omega_B = \bigcap_{t \geq 0} C_t$ of the decreasing sequence of closed random sets C_t is a closed random set again, the proof is complete. ■

3.6. REMARK. – (i) Often also Ω-limit sets of B with $\overline{B}$ not measurable will be closed random sets. But for $\varphi(t, \omega) \equiv \mathrm{id}$ one has $\Omega_B = \overline{B}$ for every B, so in general measurability of $\overline{B}$ cannot be discarded with.

(ii) For a discussion of continuity of $t \mapsto \varphi(t, \vartheta_{-t}\omega)$ see Remark 3.2 (v). The assumption of a nonrandom B can be replaced by assuming suitable regularity of $s \mapsto B(\vartheta_{-s}\omega)$.

(iii) The arguments of the previous Lemma go through for the original σ-algebra $\mathcal{F}$ instead of the past $\mathcal{F}_{\leq 0}$ when starting with a general random set B. But then, of course, the Ω-limit set need not be a closed random set with respect to the past, but just with respect to the whole σ-algebra.

(iv) Essentially the method of Lemma 3.5 gives measurability of Ω-limit sets in Suslin instead of Polish spaces. However, in this case also the discrete time case needs completion.

3.7. REMARK. – Given an RDS φ, for any two deterministic sets $C \subset D$ it is immediate that P-almost surely $\Omega_C \subset \Omega_D$. But note that for two random sets C and D one does not get existence of a set of full P-measure in which $C(\omega) \subset D(\omega)$ implies $\Omega_C(\omega) \subset \Omega_D(\omega)$.

In fact, on non-compact X one easily gets C and D with $0 < P(C = D) \leqslant P(C \subset D)$, but $P(\Omega_C \subset \Omega_D) = 0$. Just let φ be a (possibly even deterministic) RDS with two different fixed points c and d, and a transient set, i.e., a set with empty Ω-limit set. Put $C(\omega) = \{c\}$ and $D(\omega) = \{d\}$ for $\omega \in F \subset \Omega$ with $0 < P(F) < 1$. On F^c let $C(\omega) = D(\omega)$ equal the transient set. If (ϑ_t) is ergodic, then, P-almost surely, $\Omega_C(\omega) = \{c\}$, whereas $\Omega_D(\omega) = \{d\}$, whence, in the present case,

$$P(\Omega_C = \Omega_D) \leqslant P(\Omega_C \subset \Omega_D) = P(\Omega_D \subset \Omega_C) \leqslant P(\Omega_C \cap \Omega_D \neq \emptyset) = 0 < P(C = D).$$

One can even have $P(C = D) \geqslant 1 - \delta$ for $\delta > 0$ arbitrarily small, and still $\Omega_C \cap \Omega_D = \emptyset$ with probability one.

The argument of the proof of the following result goes very much along the lines of the arguments used to obtain Theorem 2.2. Nevertheless we go through the complete argument in order to point out the places where the argument has to be modified.

3.8. THEOREM. – *Suppose μ is an invariant measure for an RDS φ. Then*

(i) *$\mu(\Omega_B) \geqslant \mu(\overline{B})$ for any $B \subset X \times \Omega$ with $\overline{B} \in \mathcal{B} \otimes \mathcal{F}$,*

(ii) *$\mu(\Omega_B \cap B) = \mu(B)$ for any $B \in \mathcal{B} \otimes \mathcal{F}$.*

PROOF. – For $t \geqslant 0$ put

$$C_t(\omega) = \overline{\bigcup_{\tau \geqslant t} \varphi(\tau, \vartheta_{-\tau}\omega) B(\vartheta_{-\tau}\omega)}.$$

Then $C_s(\omega) \supset C_t(\omega)$ for any $0 \leqslant s \leqslant t$. Consequently, denoting the associated subsets of the product space $X \times \Omega$ by $C_t = \{(x, \omega): x \in C_t(\omega)\}$, also $C_s \supset C_t$, and $C_t \downarrow \Omega_B$ for $t \to \infty$. Since μ is finite, this implies $\mu(C_t) \to \mu(\Omega_B)$ for $t \to \infty$. We now claim that

$$C_t(\omega) \subset \varphi(s, \omega)^{-1} C_{t+s}(\vartheta_s \omega)$$

for any $s \geqslant 0$. In fact,

$$C_t(\omega) = \overline{\bigcup_{\tau \geqslant t} \varphi(\tau, \vartheta_{-\tau}\omega) B(\vartheta_{-\tau}\omega)} = \overline{\bigcup_{\tau \geqslant t+s} \varphi(\tau - s, \vartheta_{s-\tau}\omega) B(\vartheta_{s-\tau}\omega)} \subset$$

$$\subset \overline{\bigcup_{\tau \geqslant t+s} \varphi(s, \omega)^{-1}(\varphi(s, \omega)\varphi(\tau - s, \vartheta_{s-\tau}\omega) B(\vartheta_{s-\tau}\omega))} =$$

$$= \overline{\bigcup_{\tau \geqslant t+s} \varphi(s, \omega)^{-1}(\varphi(\tau, \vartheta_{s-\tau}\omega) B(\vartheta_{s-\tau}\omega))} = \overline{\left(\varphi(s, \omega)^{-1} \bigcup_{\tau \geqslant t+s} \varphi(\tau, \vartheta_{s-\tau}\omega) B(\vartheta_{s-\tau}\omega)\right)} \subset$$

$$\subset \varphi(s, \omega)^{-1}\left(\overline{\bigcup_{\tau \geqslant t+s} \varphi(\tau, \vartheta_{-\tau} \circ \vartheta_s \omega) B(\vartheta_{-\tau} \circ \vartheta_s \omega)}\right) = \varphi(s, \omega)^{-1} C_{t+s}(\vartheta_s \omega),$$

where we used (1) and (2) for the first and the second inclusion, respectively. Now this implies

$$\mu_\omega(C_t(\omega)) \leqslant \mu_\omega(\varphi(s, \omega)^{-1} C_{t+s}(\vartheta_s \omega)) = \varphi(s, \omega)\mu_\omega(C_{t+s}(\vartheta_s \omega)) = \mu_{\vartheta_s \omega}(C_{t+s}(\vartheta_s \omega)).$$

Integrating with respect to P yields $\mu(C_t) \leqslant \mu(C_{t+s})$. On the other hand $C_{t+s} \subset C_t$, hence

$\mu(C_{t+s}) \leq \mu(C_t)$ for any $s, t \geq 0$. We thus get, for $0 \leq s \leq t$ arbitrary, $\mu(C_t) = \mu(C_s)$ and consequently

$$\mu(C_s \backslash C_t) = 0 . \tag{8}$$

To obtain (i) note that

$$\mu(C_0) \equiv \mu(C_t) \to \mu(\Omega_B)$$

for $t \to \infty$, hence $\mu(C_0) = \mu(\Omega_B)$. Since

$$B(\omega) \subset C_0(\omega) = \overline{\bigcup_{\tau \geq 0} \varphi(\tau, \vartheta_{-\tau}\omega)\, B(\vartheta_{-\tau}\omega)},$$

and consequently $\overline{B}(\omega) \subset C_0(\omega)$, we get $\mu(\overline{B}) \leq \mu(\Omega_B)$.

To obtain (ii) suppose $B \in \mathcal{B} \otimes \mathcal{F}$. From (8) we obtain for any $0 \leq s \leq t$

$$\mu(C_s \cap B) = \mu(C_t \cap B) + \mu((C_s \backslash C_t) \cap B) = \mu(C_t \cap B),$$

hence $\mu(B) = \mu(C_0 \cap B) \equiv \mu(C_t \cap B) \equiv \mu(\Omega_B \cap B)$ for all $t \geq 0$. ∎

Of course, for $B \in \mathcal{B} \otimes \mathcal{F}$ we immediately get $\mu(\Omega_B) \geq \mu(\overline{B}) \geq \mu(B)$ from (i). The argument of the proof in fact yields that for every $D \in \mathcal{B} \otimes \mathcal{F}$ we have $\mu(C_0 \cap D) = \mu(\Omega_B \cap D)$. Thus for every $D \in \mathcal{B} \otimes \mathcal{F}$ with $D \subset C_0$ we get $\mu(\Omega_B \cap D) = \mu(D)$. Applying this, e.g., to $\overline{B}$, yields $\mu(\Omega_B \cap \overline{B}) = \mu(\overline{B})$ for any B with $\overline{B} \in \mathcal{B} \otimes \mathcal{F}$.

3.9. COROLLARY. – *Suppose φ is an* RDS *on a Polish space X, and suppose there exists a random set $\omega \mapsto A(\omega)$ such that P-almost surely $\Omega_K \subset A$ for every compact $K \subset \subset X$. Then every invariant measure for φ is supported by A.*

PROOF. – Suppose μ is invariant for φ. Since X is Polish, μ is tight, i.e., for every $\varepsilon > 0$ there exists $K = K_\varepsilon \subset X$ compact such that $\mu(K \times \Omega) \geq 1 - \varepsilon$ (Crauel [5] Lemma 4.27, p. 29). For any such K Theorem 3.8 (i) yields

$$\mu(A) \geq \mu(\Omega_K) \geq \mu(K) \geq 1 - \varepsilon .$$

Since ε is arbitrary we have $\mu(A) = 1$, which gives the assertion. ∎

4. – Invariant measures are supported by random attractors.

The notion of a *random attractor* has found interest in several applications. See Crauel and Flandoli [7] and Crauel, Debussche and Flandoli [6]. We will show that random attractors as introduced in [7] and [6] satisfy the condition of Corollary 3.9.

We say that a random set A *attracts* another random set B if

$$\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega)\, B(\vartheta_{-t}\omega), A(\omega)) = 0 , \qquad P\text{-a.s.}, \tag{9}$$

where $d(B, A) = \sup\{d(b, A) : b \in B\}$ is the Hausdorff semi-distance, defined for $A, B \subset X$ arbitrary. Note that $d(B, A) = d(\overline{B}, A)$, and for A closed $d(B, A) = 0$ if and only if $B \subset A$.

4.1. LEMMA. – *If a random set A attracts another random set B then*

$$\Omega_B \subset A\,, \qquad P\text{-a.s.}$$

PROOF. – Since A attracts B, (9) gives that for any $\varepsilon > 0$ there is τ $(=\tau(\omega))$ with $d(\varphi(t, \vartheta_{-t}\omega)\, B(\vartheta_{-t}\omega), A(\omega)) < \varepsilon$ for all $t \geq \tau$. Thus

$$d\Big(\overline{\bigcup_{s\geq\tau} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)}, A(\omega)\Big) = d\Big(\bigcup_{s\geq\tau} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega), A(\omega)\Big) =$$

$$= \sup_{s\geq\tau} d(\varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega), A(\omega)) \leq \varepsilon\,.$$

Since $\Omega_B(\omega) \subset \overline{\bigcup_{s\geq t} \varphi(s, \vartheta_{-s}\omega)\, B(\vartheta_{-s}\omega)}$ for every $t \geq 0$, we have $d(\Omega_B(\omega), A(\omega)) < \varepsilon$ for $\varepsilon > 0$ arbitrary, hence $\Omega_B \subset A$. ■

Note that $\Omega_B \subset A$ does not imply that A attracts B. It does as soon B is absorbed by a bounded set, where a (random) set D is said to *absorb* B if $\varphi(t, \vartheta_{-t}\omega)\, B(\vartheta_{-t}\omega) \subset D(\omega)$ for all t sufficiently big.

4.2. DEFINITION. – A closed random set A is said to be a *(global random) attractor (for bounded deterministic sets)* or just a *global attractor* for an RDS φ on a Polish space, equipped with a complete metric, if

- $A(\omega)$ is compact for P-almost all $\omega \in \Omega$,
- A is strictly invariant, i.e., $\varphi(t, \omega)\, A(\omega) = A(\vartheta_t\omega)$ for P-almost all ω, for every $t \geq 0$,
- $\omega \mapsto A(\omega)$ attracts every bounded deterministic set, i.e., $d(\varphi(t, \vartheta_{-t}\omega)\, B, A(\omega)) \to 0$ for $t \to \infty$ for P-almost all ω, for every bounded deterministic $B \subset X$.

We speak of a (global) attractor for *compact* deterministic sets if A is compact and strictly invariant, and if it attracts every compact (instead of bounded) set. In finite dimensions this does not make a difference. In infinite dimensional Banach or Hilbert spaces attraction of all compact sets is a considerably weaker property than attraction of all bounded sets. Speaking of bounded sets refers to the choice of a metric, whereas speaking of compact sets refers to the topology only.

Note that a random set can attract all other sets without being an attractor in the sense of Definition 4.2. It need neither be compact nor invariant. Theorem 3.11 of Crauel and Flandoli [7] together with Proposition 2.3 of Crauel, Debussche and Flandoli [6] give

4.3. THEOREM. – *Suppose φ is an* RDS *on a metric complete separable space X. If there exists a compact random set K which absorbs every bounded deterministic set then there exists a global random attractor $\omega \mapsto A(\omega)$, and $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}_{\leq 0}$, the past of the system. If (ϑ_t) is ergodic, there exists a bounded deterministic set B such that P-almost surely $\Omega_B = A$. In particular, in this case the attractor is P-a.s. unique.*

The proof of the above result proceeds by showing that, provided bounded sets are compactly absorbed, the union of all Ω_B with B bounded (non-random) gives an attractor. So it goes the other direction of Lemma 4.1.

Existence of attractors can be proved for several classes of infinite dimensional stochastic differential equations coming from mathematical physics, see [6] and [7].

We formulate the consequences for random attractors as a corollary.

4.4. COROLLARY. – *Suppose φ is an* RDS *which has a global random attractor $\omega \mapsto A(\omega)$ for compact deterministic sets. Then for every invariant probability measure μ for φ*

$$\mu(A) = \int_\Omega \mu_\omega(A(\omega))\, dP(\omega) = 1\,,$$

so μ is supported by A.

PROOF. – Immediate from Corollary 3.9 in view of Lemma 4.1. ∎

Of course, this also applies for deterministic φ, saying that a global attractor supports every invariant probability measure. But this can be derived more directly, without mentioning RDS, by using Theorem 2.2. It seems that this has not been published previously.

5. – Random attractors are unique.

Suppose φ is an RDS on a Polish space X with a random attractor A. We will show that for every $\varepsilon > 0$ there exists a deterministic compact $K \subset X$ such that A coincides with the Ω-limit set of K with probability not smaller than $1-\varepsilon$. Furthermore, as soon as (ϑ_t) is ergodic, there exist compact $K \subset X$ such that $A = \Omega_K$ with probability one.

We will need the fact that Ω-limit sets are invariant (they need not be strictly invariant in general). This already appears in Crauel and Flandoli [7] Lemma 3.2, p. 368. We give a different formulation of the proof.

5.1. LEMMA. – *The Ω-limit set of any $B \in \mathcal{B} \otimes \mathcal{F}$ is invariant, i.e.,*

$$\varphi(t, \omega)\, \Omega_B(\omega) \subset \Omega_B(\vartheta_t \omega)$$

for all $t \geq 0$.

PROOF. – For any $B \subset X$, $t \geq 0$ and ω

$$\varphi(t, \omega)\, \Omega_B(\omega) = \varphi(t, \omega) \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s} \varphi(\tau, \vartheta_{-\tau}\omega)\, B(\vartheta_{-\tau}\omega)} \subset$$

$$\subset \bigcap_{s \geq 0} \varphi(t, \omega) \overline{\bigcup_{\tau \geq s} \varphi(\tau, \vartheta_{-\tau}\omega)\, B(\vartheta_{-\tau}\omega)} \subset \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s} \varphi(t, \omega)\, \varphi(\tau, \vartheta_{-\tau}\omega)\, B(\vartheta_{-\tau}\omega)} =$$

$$= \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s} \varphi(t+\tau, \vartheta_{-\tau}\omega) B(\vartheta_{-\tau}\omega)} = \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s} \varphi(t+\tau, \vartheta_{(-t-\tau)} \circ \vartheta_t \omega) B(\vartheta_{(-t-\tau)} \circ \vartheta_t \omega)} =$$

$$= \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s+t} \varphi(\tau, \vartheta_{-\tau} \circ \vartheta_t \omega) B(\vartheta_{-\tau} \circ \vartheta_t \omega)} =$$

$$= \bigcap_{s \geq t} \overline{\bigcup_{\tau \geq s} \varphi(\tau, \vartheta_{-\tau} \circ \vartheta_t \omega) B(\vartheta_{-\tau} \circ \vartheta_t \omega)} = \Omega_B(\vartheta_t \omega),$$

where we have used $f\left(\bigcap_\alpha A_\alpha\right) \subset \bigcap_\alpha f(A_\alpha)$ for arbitrary f, and $f(\overline{A}) \subset \overline{f(A)}$ for f continuous to obtain the two inclusions. ■

An argument similar to that of the following result has been used to obtain Proposition 2.3, p. 11-12, of Crauel, Debussche and Flandoli [6].

5.2. Proposition. – *Suppose $\omega \mapsto I(\omega)$ is a strictly invariant random set, i.e.,*

$$\varphi(t, \omega) I(\omega) = I(\vartheta_t \omega)$$

P-almost surely for every $t \geq 0$. Then

$$P\{\omega : I(\omega) \subset D(\omega)\} \leq P\{\omega : I(\omega) \subset \Omega_D(\omega)\},$$

or, for short,

$$P(I \subset D) \leq P(I \subset \Omega_D)$$

for any random set $D \in \mathcal{B} \otimes \mathcal{F}$.

PROOF. – Put $F = \{\omega : I(\omega) \subset D(\omega)\}$. Then the classical Poincaré recurrence theorem, applied to the discrete time family $\{\vartheta_{-n} : n \in \mathbb{N}\} = \{\vartheta_{-1}^n : n \in \mathbb{N}\}$ (which is, of course, induced by the P-preserving $\vartheta_{-1} : \Omega \to \Omega$), yields that

$$F_\infty = \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \vartheta_n F = \{\omega : \vartheta_{-n}\omega \in F \text{ infinitely often}\}$$

satisfies $P(F_\infty) \geq P(F)$. Now for $\omega \in F_\infty$ we know that $I(\vartheta_{-n}\omega) \subset D(\vartheta_{-n}\omega)$ for infinitely many $n \in \mathbb{N}$, hence, by strict invariance of I,

$$I(\omega) = \varphi(n, \vartheta_{-n}\omega) I(\vartheta_{-n}\omega) \subset \varphi(n, \vartheta_{-n}\omega) D(\vartheta_{-n}\omega)$$

for infinitely many $n \in \mathbb{N}$. Consequently,

$$I(\omega) \subset \bigcup_{n \geq N} \varphi(n, \vartheta_{-n}\omega) D(\vartheta_{-n}\omega)$$

for every $N \in \mathbb{N}$, hence

$$I(\omega) \subset \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \varphi(n, \vartheta_{-n}\omega) D(\vartheta_{-n}\omega) \subset$$

$$\subset \bigcap_{N \in \mathbb{N}} \bigcup_{s \geq N, s \in \mathbb{R}} \varphi(s, \vartheta_{-s}\omega) D(\vartheta_{-s}\omega) \subset \bigcap_{N \in \mathbb{N}} \overline{\bigcup_{s \geq N} \varphi(s, \vartheta_{-s}\omega) D(\vartheta_{-s}\omega)} = \Omega_D(\omega).$$

This means $F_\infty \subset \{\omega\colon I(\omega) \subset \Omega_D(\omega)\}$, hence

$$P(I \subset \Omega_D) \geq P(F_\infty) \geq P(F) = P(I \subset D),$$

which is the assertion. ■

Compare Remark 3.7 to see that invariance of I is essential for Proposition 5.2.

5.3. Corollary. – *Suppose φ is an* RDS *over an ergodic base flow (ϑ_t). If I is a strictly invariant random set for φ, then*

$$P(I \subset \Omega_D) = 1$$

for every $D \in \mathcal{B} \otimes \mathcal{F}$ with $P(I \subset D) > 0$.

PROOF. – Suppose D is a random set with $P(I \subset D) > 0$. Put $F = \{\omega\colon I(\omega) \subset \Omega_D(\omega)\}$. For $\omega \in F$ we get, using strict invariance of I and invariance of Ω_D (see Lemma 5.1),

$$I(\vartheta_t \omega) = \varphi(t, \omega) I(\omega) \subset \varphi(t, \omega) \Omega_D(\omega) \subset \Omega_D(\vartheta_t \omega).$$

Consequently, $F \subset \vartheta_t^{-1} F$ for every $t \geq 0$, hence $P(F \Delta \vartheta_t^{-1} F) = P(\vartheta_t^{-1} F \backslash F) = 0$ for every $t \geq 0$. This means that F is in the σ-algebra of invariant sets with respect to (ϑ_t). Proposition 5.2 gives $P(F) = P(I \subset \Omega_D) \geq P(I \subset D) > 0$. By ergodicity of (ϑ_t) any invariant set with positive measure must have full measure, so $P(I \subset \Omega_D) = 1$. ■

5.4 COROLLARY. – *Suppose φ is an* RDS, *and I is a strictly invariant compact random set. Then for every $\varepsilon > 0$ there exists a compact nonrandom $K \subset X$ with*

$$P(I \subset \Omega_K) \geq 1 - \varepsilon.$$

In particular, if (ϑ_t) is ergodic then there exist compact nonrandom $K \subset X$ with $P(I \subset \Omega_K) = 1$, and this holds already for every compact nonrandom K provided $P(I \subset K) \neq 0$.

PROOF. – Since $\omega \mapsto I(\omega)$ is a compact random set, for every $\varepsilon > 0$ there exists a compact nonrandom $K = K_\varepsilon \subset X$ with $P\{\omega\colon I(\omega) \subset K\} \geq 1 - \varepsilon$, see Crauel [5] Proposition 3.15, p. 15-16. Consequently, Proposition 5.2 gives

$$P(I \subset \Omega_K) \geq P(I \subset K) \geq 1 - \varepsilon.$$

In case (ϑ_t) is ergodic, $P(I \subset \Omega_K) = 1$ for any K with $P(I \subset K) \neq 0$ is immediate from Corollary 5.3. ■

5.5. COROLLARY. – *Suppose φ is an RDS, and suppose that $\omega \mapsto A(\omega)$ is a global attractor for bounded or for compact sets, respectively. Then*

(i) *if the base flow (ϑ_t) is ergodic, then*

$$P(A = \Omega_D) = 1$$

for every (nonrandom) bounded or compact, resp., $D \subset X$ with $P(A \subset D) > 0$.

(ii) *In particular, if* (ϑ_t) *is ergodic then there exists a compact nonrandom* $K \subset X$ *such that* $A(\omega) = \Omega_K(\omega)$ *for P-almost all* $\omega \in \Omega$.

(iii) *Even if* (ϑ_t) *is not ergodic, still for every* $\varepsilon > 0$ *there exists a compact nonrandom* $K = K_\varepsilon \subset X$ *with*

$$P(A = \Omega_K) \geqslant 1 - \varepsilon .$$

PROOF. – First note that A, by assumption, attracts every bounded or compact set, respectively. Thus

$$P(\Omega_B \subset A) = 1 \tag{10}$$

for every bounded or compact, resp., (nonrandom) $B \subset X$ by Lemma 4.1. On the other hand, A is strictly invariant, hence Corollary 5.3 yields $P(A \subset \Omega_D) = 1$ for every $D \subset X$ with $P(A \subset D) > 0$ as soon as (ϑ_t) is ergodic. Thus $P(A = \Omega_D) = 1$ for every D bounded or compact, resp., with $P(A \subset D) > 0$, which gives (i).

Furthermore, $\omega \mapsto A(\omega)$ being compact P-a.s., Corollary 5.4 gives for every $\varepsilon > 0$ existence of a compact nonrandom $K \subset X$ with $P(A \subset \Omega_K) \geqslant 1 - \varepsilon$, and $P(A \subset \Omega_K) = 1$ in case (ϑ_t) is ergodic. Together with (10) this gives $P(A = \Omega_K) \geqslant 1 - \varepsilon$ in general and $P(A = \Omega_K) = 1$ in case (ϑ_t) is ergodic. ■

5.6. REMARK. – If (ϑ_t) is not ergodic, then in general there is no bounded $B \subset X$ with $A = \Omega_B$ almost surely. Take, e.g., $\Omega = \mathbb{R}$ with the Borel sets, $\vartheta_t \equiv \mathrm{id}$, and P the standard normal distribution (or any other probability measure with noncompact support). Let $\varphi(t, \omega)$ be the semiflow induced by the differential equation

$$\dot{x} = (x - \omega) - (x - \omega)^3$$

on the state space $X = \mathbb{R}$. Then $A(\omega) = [\omega - 1, \omega + 1]$, and $\Omega_B(\omega) \neq A(\omega)$ for all ω with $\sup B < \omega$ or $\inf B < \omega$, which have positive probability for every bounded nonrandom B.

5.7. COROLLARY. – *Suppose* φ *is an* RDS *on a Polish space X which has a global attractor* $\omega \mapsto A(\omega)$ *for compact sets. Then every strictly invariant compact random set* $\omega \mapsto I(\omega)$ *for* φ *satisfies*

$$P(I \subset A) = 1$$

(*also in case* (ϑ_t) *is not ergodic*).

PROOF. – By Corollary 5.4, for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \subset X$ compact such that $P(I \subset \Omega_K) \geqslant 1 - \varepsilon$. Now A attracts K, so by Lemma 4.1 $P(\Omega_K \subset A) = 1$, whence

$$P(I \subset A) \geqslant 1 - \varepsilon .$$

This holds for $\varepsilon > 0$ arbitrary, hence P-almost surely $I \subset A$. ■

Compactness of I is needed to get $P(I \subset A) = 1$. For instance, often the whole state space X is strictly invariant for φ.

5.8. COROLLARY. – *Suppose φ is an* RDS *on a Polish space X. If A_1 and A_2 are random attractors for compact nonrandom sets, then P-almost surely $A_1 = A_2$. Thus a random attractor is almost surely unique.*

PROOF. – Corollary 5.7 gives $P(A_1 \subset A_2) = 1$ as well as $P(A_2 \subset A_1) = 1$. ∎

5.9. REMARK. – (i) The previous result strengthens Theorem 4.3 insofar uniqueness of the attractor is obtained even if (ϑ_t) is not ergodic. Furthermore, the random attractor A is uniquely determined already by the property of attracting compact deterministic sets (and, of course, being strictly invariant and compact).

(ii) Corollary 5.8 also shows that a random attractor for *bounded* sets does not depend on the choice of a metric on X. Choosing a metric on X, then either there exists an attractor or not. In case there exist attractors for two different metrics (both inducing the topology of the Polish space, of course), then the two attractors must coincide already.

(iii) By Corollary 5.5 a random attractor is always measurable with respect to the past, i.e., it is a closed random set with respect to (the universal completion of) $\mathcal{F}_{\leq 0}$. For (ϑ_t) ergodic this is, of course, immediate from $A = \Omega_K$ almost surely for a suitable compact $K \subset X$ by Corollary 5.5, since Ω_K is measurable with respect to the past by Lemma 3.5. If (ϑ_t) is not ergodic, choose an increasing sequence K_n, $n \in \mathbb{N}$, of compact sets with $P(A = \Omega_{K_n}) \geq 1 - 1/n$ (again by Corollary 5.5). Then Ω_{K_n} is measurable with respect to the past for every $n \in \mathbb{N}$, again by Lemma 3.5, and the sequence Ω_{K_n}, $n \in \mathbb{N}$, is increasing (the K_n are nonrandom; see Remark 3.7). Furthermore, $A = \bigcup_n \Omega_{K_n}$ almost surely. Thus, for any $x \in X$,

$$\omega \mapsto d(x, A(\omega)) = \inf_{n \in \mathbb{N}} d(x, \Omega_{K_n}(\omega)) = \lim_{n \to \infty} d(x, \Omega_{K_n}(\omega))$$

is measurable with respect to the past, hence A is a closed random set with respect to (the universal completion of) $\mathcal{F}_{\leq 0}$.

REFERENCES

[1] L. ARNOLD, *Random Dynamical Systems*, Springer, Berlin (1998).

[2] L. ARNOLD - M. SCHEUTZOW, *Perfect cocycles through stochastic differential equations*, Probab. Theory Relat. Fields, **101** (1995), pp. 65-88.

[3] C. CASTAING - M. VALADIER, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, **580**, Springer, Berlin (1977).

[4] H. CRAUEL, *Markov measures for random dynamical systems*, Stochastics and Stochastics Rep., **37** (1991), pp. 153-173.

[5] H. CRAUEL, *Random Probability Measures on Polish Spaces*, Habilitationsschrift, Bremen (1995).

[6] H. CRAUEL - A. DEBUSSCHE - F. FLANDOLI, *Random attractors*, Journal of Dynamics and Differential Equations, **9** (1997), pp. 307-341.

[7] H. CRAUEL - F. FLANDOLI, *Attractors for random dynamical systems*, Probab. Theory Relat. Fields, **100** (1994) pp. 365-393.

[8] P. GÄNSSLER - W. STUTE, *Wahrscheinlichkeitstheorie*, Springer, Berlin (1977).