Annali di Matematica pura ed applicata
(IV), Vol. CLXXIX (2001), pp. 331-360

On the Dynamics of Deformable Ferromagnets I. Global Weak Solutions for Soft Ferromagnets at Rest (*).

M. BERTSCH - P. PODIO GUIDUGLI - V. VALENTE

Abstract. – *The paper begins with a fairly detailed presentation of a general mathematical model for the dynamics of ferromagnetic bodies undergoing arbitrarily large deformations. Next, a mathematical study is presented of the evolution problem for the magnetization field in a «soft» ferromagnetic body which is «mechanically at rest». No matter how special and simple this problem within the framework of the full theory, the governing equation is interesting: it is identical to the dynamic version of the harmonic-map equation usually referred to in the mathematical literature as the Gilbert form of the Landau-Lifshitz equation. Motivated by recent nonuniqueness results for the dynamic harmonic-map equation, we give a new proof of global existence of weak solutions to the Gilbert-Landau-Lifshitz equation.*

1. – Introduction.

The general object of the dynamics of ferromagnets is the formulation and the study of the evolution problem for ferromagnetic bodies undergoing arbitrarily large deformations. In this paper we begin our mathematical study of the model recently proposed by DeSimone and Podio-Guidugli [13, 14, 15].

1.1. *Plan of the paper. Results.*

A generalized account of the model of [13, 14, 15] is given in Section 2. This model is a nonlinear, dynamic version of *micromagnetics*, the variational model—essentially due to W.F. Brown [7, 8]—describing the statics of deformable ferromagnets; its most relevant feature is full consistency with the principles of rational continuum thermodynamics. The mathematical structure of the resulting evolution system is far too complex to be studied in complete generality. Hence we have chosen to consider here the special case of «soft» ferro-

(*) Entrata in Redazione il 18 dicembre 1999.
Indirizzo degli AA.: M. BERTSCH: Dipartimento di Matematica, Università di Roma «Tor Vergata», Via della Ricerca Scientifica, 00133 Roma, Italy (bertsch@mat.uniroma2.it); P. PODIO-GUIDUGLI: Dipartimento di Ingegneria Civile, Università di Roma «Tor Vergata», Via di Tor Vergata 110, 00133 Roma, Italy (ppg@uniroma2.it); V. VALENTE: Istituto per le Applicazioni del Calcolo, CNR, Viale del Policlinico 137, 00161 Roma, Italy (valente@iac.rm.cnr.it)

magnetic material bodies that are «mechanically at rest» (Section 3). Since this paper is meant as the first of a series, we have structured Sections 2 and 3 in such a way that they can be used in future papers as a reference for the basic concepts from continuum physics and their mathematical formulation.

For *soft ferromagnets at rest* the general evolution system reduces to a single equation for the magnetization $m(x, t)$ alone, namely,

$$\dot{m} = -\alpha m \times \dot{m} + m \times \Delta m, \qquad |m| = 1 . \tag{1.1}$$

Here α is a positive material parameter, the so-called Gilbert damping (in fact this equation is often referred to as the *Gilbert equation*), while other material parameters irrelevant to the mathematical analysis of the equation are lumped into a rescaling of the space variable (Subsection 3.3). Equation (1.1) is assumed to hold in the space-time cylinder $\mathcal{B} \times (0, T]$, with T a given positive number and $\mathcal{B}$ a nice three-dimensional region with boundary $\partial\mathcal{B}$ of outer normal $\mathbf{n}$, and is supplemented by the following initial and boundary conditions:

$$m(x, 0) = m_o(x) \quad \text{in } \mathcal{B}, \qquad |m_o| = 1 \, ; \tag{1.2$_1$}$$

$$\partial_{\mathbf{n}} m(x, t) = \mathbf{0} \quad \text{in } \partial\mathcal{B} \times (0, T] . \tag{1.2$_2$}$$

A physical interpretation of equation $(1.1)_1$ is that, at each fixed point $x \in \mathcal{B}$, the magnetization m progresses with an unforced, damped movement of precession with respect to the magnetic field Δm (the general model of Section 2 incorporates such features as precession and dissipation without any recourse to, respectively, analogy or *ad hoc* reasoning). As shown in Appendix A, three equivalent forms of $(1.1)_1$ are: the *Landau-Lifshitz equation* [23, 29, 30]

$$\dot{m} = m \times \Delta m - \alpha m \times (m \times \Delta m) \, ; \text{(1)} \tag{1.3}$$

the equation [1]

$$\dot{m} = m \times \Delta m + \alpha(\Delta m + |\nabla m|^2 m) \, ; \tag{1.4}$$

and the equation whose weak form we study, namely,

$$\alpha \dot{m} - m \times \dot{m} = \Delta m + |\nabla m|^2 m \, . \text{(2)} \tag{1.1$'$}$$

(1) In 1935 Landau and Lifshitz [24] derived the following equation to describe the evolution of spin fields in ferromagnets:

$$\dot{m} = m \times (\mathbf{h}_{\text{eff}} - \alpha m \times \mathbf{h}_{\text{eff}}) ,$$

where $\mathbf{h}_{\text{eff}}$, the effective magnetic field, has an expression more general than the one we here consider, that is, *modulo* the indicated rescalings, $\mathbf{h}_{\text{eff}} = \Delta m$.

(2) *Cf.*, respectively, equations (A.6)-(A.7), (A.8), and (A.10). In all of (1.3), (1.4), and (1.1)$'$ time has been rescaled by the factor $(1 + \alpha^2)$.

Existence of weak solutions to the Landau-Lifshitz equation coupled with Maxwell equations has been established by Visintin [29] in 1985. In 1992 Alouges and Soyeur [1] have proven that Problem (1.1)-(1.2) has a solution in a generalized sense (*cf.* also the work of Guo and Hong [17]), but that the solution is not always unique. In this paper we prove the following theorem of global existence for weak solutions (Section 4).

GLOBAL-WEAK EXISTENCE THEOREM. – Choose $\mathcal{B}$, an open, bounded region with smooth boundary, and choose a vector field $\boldsymbol{m}_0 \in H^1(\mathcal{B}; \mathbb{R}^3)$, with $|\boldsymbol{m}_0| = 1$ a.e. in $\mathcal{B}$, and with finite energy:

$$\int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}_0|^2 < \infty .$$

For each $T > 0$, there is a *global-weak solution* of Problem (1.1)-(1.2), *i.e.*, a vector field $\boldsymbol{m} \in H^1(\mathcal{B} \times (0, T]; \mathbb{R}^3)$ such that

(i) for each $\mathbf{z} \in C^\infty(\overline{\mathcal{B}} \times [0, T])$ vanishing at $t = 0$ and $t = T$,

$$-\int_0^T \int_{\mathcal{B}} \dot{\boldsymbol{m}} \cdot \mathbf{z} = \alpha \int_0^T \int_{\mathcal{B}} \mathbf{M} \dot{\boldsymbol{m}} \cdot \mathbf{z} + \int_0^T \int_{\mathcal{B}} \mathbf{M} \nabla \boldsymbol{m} \cdot \nabla \mathbf{z}$$

(here $\mathbf{M}$ is the skew matrix uniquely associated with the vector $\boldsymbol{m}$);

(ii) $|\boldsymbol{m}| = 1$ a.e. in $\mathcal{B} \times (0, T]$;

(iii) $\boldsymbol{m}(\cdot, t) \to \boldsymbol{m}_0(\cdot)$ in $L^2(\mathcal{B}; \mathbb{R}^3)$ as $t \to 0$.

The main idea of our proof is to modify equation (1.1)′ by introducing two positive small parameters, ε and τ:

$$\alpha \dot{\boldsymbol{m}} - \boldsymbol{m} \times \dot{\boldsymbol{m}} - \tau \Delta \dot{\boldsymbol{m}} = \Delta \boldsymbol{m} - \varepsilon^{-1}(|\boldsymbol{m}|^2 - 1)\, \boldsymbol{m} . \tag{1.5}$$

The τ-regularization allows us to treat (1.5) as an ODE in an appropriate function space, while the ε^{-1}-penalization replaces the magnitude constraint, $|\boldsymbol{m}| = 1$. A virtue of our model is that the τ-regularization can be assigned the physical meaning of an additional dissipation mechanism (although we know of no experiment supporting the inclusion of that term in the case of magnetostrictive materials). Taking the cross product of (1.5) by $\boldsymbol{m}$, we obtain a modified version of equation (1.1), for which we are able to pass to the limit up to subsequences for $(\varepsilon, \tau) \to (0, 0)$ along whatever smooth path. Our proof is flexible enough to treat slightly more complicated equations [28]. However, it is unclear to us whether different limiting procedures in (1.5) may lead to different solutions. We expand on the issue at the end of the next subsection.

1.2. *Singularities in statics and nonuniqueness in dynamics. Facts and conjectures.*

The fact that Problem (1.1)-(1.2) may have nonclassical solutions has been known since a few years. For example, if $\mathcal{B}$ is a ball centered at a point $x_0 \in \mathbb{R}^3$, then the field

$$m(x) = \frac{x - x_0}{|x - x_0|}, \tag{1.6}$$

which solves the *harmonic-map* equation

$$\Delta m + |\nabla m|^2 m = 0, \qquad |m| = 1, \tag{1.7}$$

is a static solution of Problem (1.1)-(1.2), with finite energy $\left(\int_{\mathcal{B}} \frac{1}{2} |\nabla m|^2 < +\infty\right)$ and with a *point singularity* at $x = x_0$. Equation (1.7) has been extensively studied (see, *e.g.*, Brezis, Coron, and Lieb [6]), not only because it is mathematically interesting *per se* but also because it offers a simplified and yet often qualitatively reliable model for the static theory of points and line defects in nematic *liquid crystals*. It is easy to see that (1.7) possesses infinitely many other solutions with a point singularity at a given point x_0, of less symmetric structure than (1.6). Alouges and Soyeur [1] have taken a singular solution of this less symmetric sort as the initial datum, and shown that there is then another, *not static* solution of Problem (1.1)-(1.2); Coron [11] and Bethuel, Coron, Ghidaglia, and Soyeur [5] have obtained similar results for the equation

$$\alpha \dot{m} = \Delta m + |\nabla m|^2 m, \qquad |m| = 1, \tag{1.8}$$

which is posited as the «heat flow» of the harmonic–map equation (as to this last equation, see the recent review article by Hardt [20] and the references listed therein, especially the papers by Struwe [26] and Chang, Ding, and Ye [9]).

Thus, in $\mathbb{R}^3$ nonuniqueness and point singularities turn out to be closely related. Very recently Bertsch, Dal Passo, and Van der Hout [4] have demonstrated a similar relation in $\mathbb{R}^2$ in the case of equation (1.8) with Dirichlet boundary conditions: if $\mathcal{B}$ is the two-dimensional disk, there are regular initial data for which equation (1.8) has more than one weak solution which is not smooth at the origin of the disk for all times. In addition, always in the case of a disk, preliminary results of Dal Passo and Vilucchi [12] for the equation

$$\alpha \dot{m} - \tau(\Delta \dot{m} - (m \cdot \Delta \dot{m})\, m) = \Delta m + |\nabla m|^2 m, \qquad |m| = 1, \tag{1.9}$$

would seem to indicate that different approximation methods lead to different solutions of (1.8). Since the precession term $m \times \dot{m}$ is powerless, we expect equation (1.1) to embody essentially the same mathematical phenomenology as (1.8). Our present, mostly conjectural description of this phenomenology is based on ideas exploited in [4] and [12].

Chang, Ding, and Ye [9] have shown that a finite amount of energy can concentrate on a line segment at a finite time, say, $t_0 > 0$. (³) Suppose now that one asks how, if ever, such a

(³) Actually, their construction—just as those in [4] and [12]—concerns a point singularity at the origin of a disk, a situation that cylindrical symmetry transforms into a line singularity at the axis of a cylinder in $\mathbb{R}^3$.

solution can be continued past t_0. We can think of two possibilities. Either the energy concentrated as $t \to t_0$ on the line segment is permanently trapped there, and for the continued solution the function $t \mapsto e(t) = \int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}(x, t)|^2$ is nonincreasing (curve (a) in Figure 1); (4) or the trapped energy is released at a later time $t_1 > t_0$, and the Dirichlet integral associated to the continued solution is nonmonotonic (curve (b) in Figure 1).

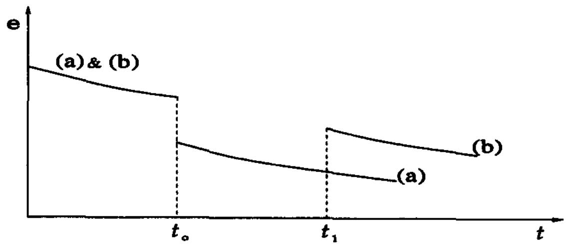

Figure 1

In the first instance the formation of singularities might somehow be regarded as a mechanism of instantaneous dissipation; in the second instance it would seem reasonable to incorporate a form of line energy into the model. Various questions come to mind, most of which we are unable to answer at the moment of this writing, some of which we list.

1) Should a model for the dynamics of ferromagnets allow for concentration of finite amounts of energy on sets of measure zero?

While our present model does not explicitly accomodate such a possibility, it does not exclude it as well. However, we believe that only experimental evidence could provide a definite answer. In this connection we observe that the counterexample to uniqueness constructed in [4] was strongly motivated by a different application (a flow problem for nematic liquid crystals, showing up in the process of fiber spinning [27]), an application in which one indeed expects energy concentration.

2) If we choose to allow for energies trapped on curves to be released, (i) can we formulate a uniqueness criterion? (ii) can we actually construct solutions along which energy is first trapped and then released?

When faced with lack of uniqueness one is driven to ask whether there is a criterion that selects the physically relevant solution(s) and whether the latter can be actually constructed. A precise mathematical formulation for such a criterion (something like, say, «trapped energies are to be released as soon as possible») is not obvious, not to speak about proving that it selects a solution uniquely. However, it would be useful to obtain at least good candidates

(4) Freire [16] has shown that, in the 2-D framework, this monotonicity criterion selects a unique solution.

for such solutions: our present paper offers a new existence proof for solutions of Problem (1.1)-(1.2) which are such candidates.

To state some of our conjectures more clearly, let us return to equation (1.5) with the help of Figure 2, where the path relevant to our existence proof is the generic path (c).

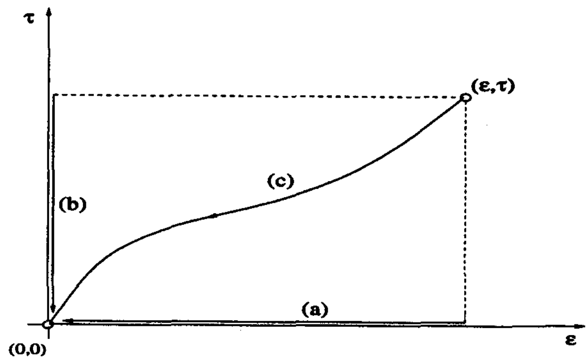

Figure 2

For the penalized equation

$$\alpha \dot{m} - m \times \dot{m} = \Delta m - \varepsilon^{-1}(|m|^2 - 1)\, m \tag{1.10}$$

we expect, for vanishing ε, solutions with monotonic Dirichlet integral (path (a) in Figure 2 and curve (a) in Figure 1). On the other hand, the results for equation (1.9) in [12] make us believe that solutions of the equation

$$\alpha \dot{m} - m \times \dot{m} - \tau(\Delta \dot{m} - (m \cdot \Delta \dot{m})\, m) = \Delta m + |\nabla m|^2 m \tag{1.11}$$

yield, in the limit of vanishing τ corresponding with path (b) in Figure 2, solutions associated with the nonmonotonic curve (b) in Figure 1. This latter case has special interest because, as already mentioned, our model allows for a physical interpretation of the dissipative term $\tau \Delta \dot{m}$.

2. – The dynamics of deformable ferromagnets in short.

This section serves the purpose of presenting a short, self-contained description of the mathematical model we study; we refer the reader to [13, 14, 15] for a complete discussion of the continuum mechanical development of the model, as well as for references to the technical literature on continuum theories of deformable ferromagnets.

Within the framework of a theory of continua with microstructure, a ferromagnet is pictured as the composition of two interacting continua, the one with a mechanical structure, the other with a magnetic structure. The model's construction consists of the following steps:

(i) The kinematics of the composite continuum is described by its *motion* with respect to a reference configuration and by the *magnetization*, a unit vector field over the current configuration.

(ii) The system of forces which are work-conjugate to such kinematics is split into forces peculiar to each constituent continuum and forces that define the interaction between the two. A distinctive feature of the magnetic continuum is the occurrence of interior forces of magnetic origin, that is, of mutual forces between body parts and of self-forces, the latter being the forces that a body part exerts on itself. *Interaction forces* and *interior forces* are associated with magnetostriction, the phenomenon that makes certain ferromagnets suitable for application as sensors and actuators.

(iii) Both for the composite continuum and for the constituents, *balance laws* for forces are posited.

(iv) The force response to histories of deformation and magnetization is prescribed in a manner consistent with a *dissipation principle* postulated for the composite continuum.

(v) Combination of balance laws and constitutive prescriptions yields a system of PDEs for the evolution of the (motion, magnetization) pair.

2.1. (*Motion, Magnetization*) *Pairs.*

Let the ambient space be a three-dimensional Euclidean space $\mathcal{E}$, with typical point x. We think of $\mathcal{E}$ as the image at time $t \in \mathbb{R}$ of a reference ambient space $\mathcal{E}_r$ under a smooth diffeomorphism $\tilde{f}(\cdot, t)$. Let B denote the *referential shape* of the material body under study, *i.e.*, the region the body occupies when placed in the space $\mathcal{E}_r$. The *deformation* $f(\cdot, t)$ of B at time t is the restriction to B of $\tilde{f}(\cdot, t)$; it is assumed that $f(\cdot, t)$ preserves the local orientation, in the sense that $\det(\partial_X f(X, t)) > 0$ for all $(X, t) \in B \times \mathbb{R}$. A typical point X of B occupies at time t the *place* $x = f(X, t)$; $\mathcal{B} = f(B, t)$ is the *current shape* of B.

A *motion* is a family of deformations $f(\cdot, t)$ of B. Given a motion and a referential mass-density field $\varrho_r(X) > 0$ for $X \in B$, we assume *conservation of mass*, in the form

$$(\det \mathbf{F})\, \varrho = \varrho_r, \tag{2.1}$$

where $\varrho(x, t)$ is the current mass density and $\mathbf{F}(X, t)$ stands for $\partial_X f(X, t)$. [5]

---

[5] More precisely, relation (2.1) should be written as

$$(\det \mathbf{F}(X, t))\, \varrho(x, t) = \varrho_r(X), \qquad x = f(X, t).$$

In the thermomechanics of continua undergoing finite deformations, where it is crucial to distinguish the reference configuration from the current configuration, one often finds formulae which combine fields over the current configuration with fields whose typical descriptions are instead over the refe-

For each given motion it makes sense to consider families of *magnetization* fields $m(\cdot, t)$ over the current shape of the body: for each time t fixed, the vector $m(x, t)$ has unit modulus for all $x \in \mathcal{B}$ (the *saturation condition*) and accounts for the current spatial distribution of magnetic dipoles per unit mass; it is convenient to introduce also the magnetization per unit current volume, the vector field defined by

$$\mathbf{m} = \varrho m . \tag{2.2}$$

2.2. *The magnetic field.*

Given a (motion, magnetization) pair for the body B, we define the *magnetic field* $\mathbf{h}_{\mathcal{B}}$ over $\mathcal{E}$ to be the unique square-integrable solution of *Maxwell equations* in the «quasistatic approximation»:

$$\operatorname{curl} \mathbf{h}_{\mathcal{B}} = \mathbf{0} , \qquad \operatorname{div} \mathbf{h}_{\mathcal{B}} = -\operatorname{div}(\chi_{\mathcal{B}} \mathbf{m}) \ \text{ in } \mathcal{E}, \tag{2.3}$$

where $\chi_{\mathcal{B}}$ is the characteristic function of $\mathcal{B}$, $\chi_{\mathcal{B}} \mathbf{m}$ is the extension of $\mathbf{m}$ to $\mathcal{E}$, and the equations are assumed to hold in the sense of distributions.

If $\mathbf{m}$ is smooth, the following representation formula for the solution of (2.3) holds at each point $x \in \mathcal{E} \backslash \partial \mathcal{B}$:

$$\mathbf{h}_{\mathcal{B}} = -\nabla \phi_{\mathcal{B}},$$

$$\phi_{\mathcal{B}}(x) = \int_{\mathcal{B}} |x-y|^{-1}(-\operatorname{div} \mathbf{m}(y))\, dv(y) + \int_{\partial \mathcal{B}} |x-y|^{-1} \mathbf{m}(y) \cdot \mathbf{n}(y)\, ds(y) \tag{2.4}$$

(recall from the preceding footnote that ∇ denotes the gradient operator with respect to current place, and note that here $\mathbf{n}(y)$ is the outer unit normal at a point $y \in \partial \mathcal{B}$, while time dependence is not displayed); moreover, at each point of $\partial \mathcal{B}$,

$$[\![\mathbf{h}_{\mathcal{B}}]\!] = (\mathbf{m} \cdot \mathbf{n})\, \mathbf{n} , \tag{2.5}$$

where $[\![\mathbf{h}_{\mathcal{B}}]\!] = \mathbf{h}_{\mathcal{B}}^{+} - \mathbf{h}_{\mathcal{B}}^{-}$ denotes the jump of $\mathbf{h}_{\mathcal{B}}$.

---

rence configuration, just like $\varrho(\cdot, t)$ and $\mathbf{F}(\cdot, t)$ in (2.1). Customarily, the independent variables are not displayed and the implicit adjustments are left to the reader. However, the gradient operators with respect to referential and current places are usually given different symbols; the same is done for those differential operators, like divergence and curl, that are defined starting from the one or the other gradient operator. In this paper we use ∂_X and ∇, respectively, and associate Div and Curl to the former, div, curl, and $\Delta = \operatorname{div} \nabla$ to the latter. We also denote by a superscript dot the *material time derivative*, that is, the time derivative of a field $\Phi(x, t)$ over $\mathcal{B}$:x

$$\dot{\Phi} = \partial_t \Phi + (\nabla \Phi)\, \mathbf{v} ,$$

where $\mathbf{v} = \partial_t f \circ (f(\cdot, t))^{-1}$ is the velocity field over $\mathcal{B}$ in the given motion.

It also follows from (2.3) that

$$\int_{\mathcal{E}} (\mathbf{h}_{\mathcal{B}} + \chi_{\mathcal{B}}\mathbf{m})\cdot\nabla\psi = 0 \tag{2.6}$$

for all test fields ψ that vanish sufficiently fast at infinity; in particular,

$$\int_{\mathcal{E}} |\mathbf{h}_{\mathcal{B}}|^2 = -\int_{\mathcal{B}} \mathbf{h}_{\mathcal{B}}\cdot\mathbf{m}\,.$$

By definition,

$$M(\mathcal{B}) = \frac{1}{2}\int_{\mathcal{E}} |\mathbf{h}_{\mathcal{B}}|^2 \tag{2.7}$$

is the *magnetostatic energy* of the body in its shape $\mathcal{B}$, *i.e.*, the energy of the magnetic field $\mathbf{h}_{\mathcal{B}}$ associated to the magnetization $\mathbf{m}$; a change in shape and/or in magnetization induces a change in magnetostatic energy.

At a typical point $x \in \mathcal{B}$ the total magnetic field $\mathbf{h}$ consists of $\mathbf{h}_{\mathcal{B}}$ and the *external magnetic field* $\mathbf{h}^e$:

$$\mathbf{h} = \mathbf{h}_{\mathcal{B}} + \mathbf{h}^e, \tag{2.8}$$

with $\mathbf{h}^e$ a field that we assume we can control, if needed.

2.3. *Balance equations.*

Since we regard a ferromagnet as a composite continuum with two constituents, balance laws are needed for the compound and *one* of the constituents; interior forces appear in both the compound's balance and the constituent's balance, interaction forces only in the latter. We base our theory on a *balance law of forces for the composite continuum*:

$$\mathbf{b} + \operatorname{div}\mathbf{T} = \mathbf{0}\,, \tag{2.9}$$

and on a *balance law of torques for the magnetic continuum*:

$$\boldsymbol{m} \times (\boldsymbol{b} + \boldsymbol{k} + \operatorname{div}\boldsymbol{C}) = \mathbf{0}\,. \tag{2.10}$$

Here $\mathbf{T}$ is the *stress* in the composite continuum, $\boldsymbol{C}$ is the *couple stress* in the magnetic continuum, and $\boldsymbol{m}\times\boldsymbol{k}$ is the *interaction couple*; $\mathbf{T}$, $\boldsymbol{C}$, and $\boldsymbol{k}$ must each be constitutively specified (see the next subsection).

Two other balance laws should be posited in general, namely, balance of torques for the composite continuum and balance of forces for the magnetic continuum [14]. While certain customarily accepted constitutive choices allow one to dispense with the latter balance [13, 14], the restrictions posed by the former can be cast as an algebraic consistency condition for the constitutive choices of $\mathbf{T}$, $\boldsymbol{C}$, and $\boldsymbol{k}$, namely,

$$\operatorname{skw}[\mathbf{T} + \boldsymbol{C}(\nabla\boldsymbol{m})^T - \boldsymbol{k}\otimes\boldsymbol{m}] = \mathbf{0} \tag{2.11}$$

(here skw $[\cdot]$ evaluates the skew part of a tensor, and $\otimes$ denotes the usual dyadic product of vectors).

The tensor fields $\mathbf{T}$ and C are related to, respectively, the *contact force* in the composite continuum and the *contact couple* in the magnetic continuum. (6) The vector field $\mathbf{b}$, the *distance force* in the composite continuum, is thought of as split into three parts:

$$\mathbf{b} = \mathbf{b}^e + \mathbf{b}^i + \mathbf{b}^s, \tag{2.12}$$

with the superscripts standing, respectively, for *external*, *inertial*, and *self*. A completely similar splitting, following from

$$b = b^e + b^i + b^s, \tag{2.13}$$

is postulated for $m \times b$, the *distance couple* in the magnetic continuum. (7)

To specify $\mathbf{b}^i$ and b^i, as well as $\mathbf{b}^s$ and b^s, is a constitutive task, deferred to the next subsection. Instead, $\mathbf{b}^e$ and b^e are control fields that we can choose in order to generate one or another (motion, magnetization) process. Both $\mathbf{b}^e$ and b^e are split into mechanical and magnetical parts:

$$\mathbf{b}^e = \mathbf{b}^e_{me} + \mathbf{b}^e_{ma}, \qquad \mathbf{b}^e_{ma} = (\nabla \mathbf{h}^e)\,\mathbf{m}\,, \tag{2.14}$$

$$b^e = b^e_{me} + b^e_{ma}, \qquad b^e_{ma} = \varrho \mathbf{h}^e, \tag{2.15}$$

and we regard at our disposal each of the parts $\mathbf{b}^e_{me}$, $\mathbf{b}^e_{ma}$, and b^e_{me}.

2.4. *Constitutive prescriptions.*

We begin by specifying our choice of the inertial forces $\mathbf{b}^i$ and b^i. Two rules guide this choice [25, 14]: that the time rate of kinetic energy be balanced by the inertial power; and

(6) Formally, the contact force at (x, t) relative to the oriented plane of unit normal $\mathbf{n}$ is the vector

$$\mathbf{t}(x, t; \mathbf{n}) = \mathbf{T}(x, t)\mathbf{n} - \frac{1}{2}(\varrho m(x, t)\cdot\mathbf{n})^2\mathbf{n}\,; \tag{$\diamond$}$$

the contact couple is

$$\mathbf{c}(x, t; \mathbf{n}) = m(x, t) \times (C(x, t)\mathbf{n})\,. \tag{$\diamond\diamond$}$$

Both in $\mathbf{t}$ and in $\mathbf{c}$ one sees a manifestation of the interior forces of magnetic origin alluded to in the Introduction; these constructs account for the dipolar microstructure of a ferromagnet, at the macroscale typical of continuum mechanics and under the form of contact interactions. When it comes to specifying boundary conditions of the natural type, it is $\mathbf{t}$ and/or $\mathbf{c}$ that one assigns at a point of the current boundary of a ferromagnetic body (Subsection 2.5).

(7) The distance self-force $\mathbf{b}^s$ is another macroscopic manifestation of the interior forces of magnetic origin, and accompanies the contact self-force contributing to the macrostress, which appears in the right side of formula ($\diamond$) in the preceding footnote; similarly for the distance self-couple $m \times b^s$ (see [13], [14], and especially [15]).

that the power expended by each inertial force be linear in the conjugate velocity. As is standard in continuum theories for ferromagnetic solids, we choose:

$$\mathbf{b}^i = -\varrho \dot{\mathbf{v}}, \qquad b^i = \gamma^{-1} m \times \dot{m}, \qquad \gamma > 0, \tag{2.16}$$

where $\mathbf{v}$ is the velocity field over the current shape $\mathcal{B}$ and $\dot{m}$ is the material time derivative of the magnetization (*cf.* footnote 5). Thus, in the magnetic continuum the inertial power is everywhere null at each instant:

$$b^i \cdot \dot{m} = (m \times b^i) \cdot (m \times \dot{m}) = 0\,; \tag{2.17}$$

moreover,

$$\mathbf{b}^i \cdot \mathbf{v} = -\dot{\kappa}, \qquad \kappa = \frac{1}{2}\varrho |\mathbf{v}|^2, \tag{2.18}$$

where κ denotes the kinetic energy per unit current volume in the composite continuum.

Next, we stipulate that the distance self-forces $\mathbf{b}^s$ and b^s combine with, respectively, $\mathbf{b}^e_{ma}$ and b^e_{ma} as follows:

$$\mathbf{b}^s + \mathbf{b}^e_{ma} = (\nabla \mathbf{h})\, \mathbf{m}\,, \qquad b^s + b^e_{ma} = \varrho \mathbf{h} \tag{2.19}$$

(*cf.* (2.8), $(2.14)_2$, and $(2.15)_2$). With the use of (2.12)-(2.19) we find

$$\mathbf{b} + \varrho \dot{\mathbf{v}} = \mathbf{b}^e_{me} + (\nabla \mathbf{h})\, \mathbf{m}\,, \tag{2.20}$$

$$b - \gamma^{-1} \mu \times \dot{m} = b^e_{me} + \varrho \mathbf{h}\,, \tag{2.21}$$

where each of the terms on the right sides is controllable at each point of $\mathcal{B}$.

In the classical manner introduced by Coleman and Noll we regard the constitutive choices of the contact actions $\mathbf{T}$ and C, the interaction distance–force k, and the *free energy* ψ per unit mass, as collectively restricted by a *dissipation inequality* to be satisfied identically in all smooth (motion, magnetization) processes. This inequality we stipulate to have the form:

$$\mathbf{T} \cdot \nabla \mathbf{v} + C \cdot \nabla \dot{m} - k \cdot \dot{m} - \varrho \dot{\psi} \geqslant 0\,. \tag{2.22}$$

The left side of this inequality is interpreted as the *energy dissipated* in the composite continuum per unit current volume. (8)

We postulate such constitutive dependences of $\mathbf{T}$, C, k, and ψ on the process variables

(8) Given its physical interpretation, this quantity should be an *objective scalar, i.e.*, it should not change value under a change in observer; it is not difficult to check that this is indeed a consequence of the torque balance (2.11) for the composite continuum. A deduction of an equivalent form of (2.22) is found in Section 5 of [14].

in the list $(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m})$ (where we have denoted ∇m by G) as to accomodate for both *magnetoelastic equilibrium response* and *nonlinear viscosity* out of equilibrium. Precisely, we put

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m}), \tag{2.23}$$

$$C = \widehat{C}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m}), \tag{2.24}$$

$$k = \widehat{k}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m}), \tag{2.25}$$

$$\psi = \widehat{\psi}(\mathbf{F}, G, m) \tag{2.26}$$

(consistency with (2.22) rules out any dependence of ψ on $\dot{\mathbf{F}}$, $\dot{m}$, or $\dot{G}$; streching the standard terminology a bit, $\widehat{\psi}(\mathbf{F}, \cdot, m)$ can be called the *exchange energy* at $(\mathbf{F}, m)$, and $\widehat{\psi}(\mathbf{F}, G, \cdot)$ the *anisotropy energy* at $(\mathbf{F}, G)$). Since

$$\nabla \mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \qquad \nabla \dot{m} = \dot{G} + G(\nabla \mathbf{v}),$$

with the use of (2.26) the dissipation inequality can be given the form

$$\left((\mathbf{T} + G^T C)\mathbf{F}^{-T} - \varrho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}}\right)\cdot \dot{\mathbf{F}} + \left(C - \varrho \frac{\partial \widehat{\psi}}{\partial G}\right)\cdot \dot{G} - \left(k + \varrho \frac{\partial \widehat{\psi}}{\partial m}\right)\cdot \dot{m} \geq 0 .$$

At this point, an application of the algebraic lemma proved in Appendix B yields that the free-energy mapping $\widehat{\psi}$ determines the «equilibrium» part of each of the mappings $\widehat{\mathbf{T}}$, $\widehat{C}$, and $\widehat{k}$:

$$\mathbf{T}^{\text{eq}} = \widehat{\mathbf{T}}^{\text{eq}}(\mathbf{F}, G, m) = \varrho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, G, m)\, \mathbf{F}^T - \varrho G^T \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, m), \tag{2.27}$$

$$C^{\text{eq}} = \widehat{C}^{\text{eq}}(\mathbf{F}, G, m) = \varrho \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, m), \tag{2.28}$$

$$k^{\text{eq}} = \widehat{k}^{\text{eq}}(\mathbf{F}, G, m) = -\varrho \frac{\partial \widehat{\psi}}{\partial m}(\mathbf{F}, G, m), \tag{2.29}$$

where we have set

$$\widehat{\mathbf{T}}^{\text{eq}}(\mathbf{F}, G, m) = \widehat{\mathbf{T}}(\mathbf{F}, G, m; \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{2.30}$$

(and similarly for $\widehat{C}^{\text{eq}}$ and $\widehat{k}^{\text{eq}}$). Moreover, the «viscous» parts $\mathbf{T}^{\text{vs}}$, C^{vs}, and k^{vs} of, respectively, $\mathbf{T}$, C, and k are given by the constitutive mappings defined by

$$\widehat{\mathbf{T}}^{\text{vs}}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m}) = \widehat{\mathbf{T}}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m}) - \widehat{\mathbf{T}}(\mathbf{F}, G, m; \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{2.31}$$

etc., and satisfy the residual dissipation inequality

$$\mathbf{T}^{vs}\cdot\nabla\mathbf{v} + C^{vs}\cdot\nabla\dot{m} - k^{vs}\cdot\dot{m} \geqslant 0\,. \quad (2.32)$$ (9)

More information is extracted from (2.32) if the prescriptions (2.23)-(2.25), which are not invariant under change in observer, are replaced by properly invariant prescriptions such as

$$\widetilde{\mathbf{T}}(\mathbf{F}, G, m; \mathbf{D}, G^\circ, m^\circ) = \widehat{\mathbf{T}}(\mathbf{F}, G, m; \dot{\mathbf{F}}, \dot{G}, \dot{m})\,, \quad (2.33)$$

where the invariant rates $\mathbf{D}$, G°, and m° appear, defined by

$$\begin{aligned} \mathbf{D} &= \mathrm{sym}\,[\nabla\mathbf{v}]\,, \\ G^\circ &= \dot{G} - \mathbf{W}G + G\mathbf{W}\,, \\ m^\circ &= \dot{m} - \mathbf{W}m\,; \end{aligned} \quad (2.34)$$

here $\mathbf{D}$ is the *stretching* and $\mathbf{W} = \mathrm{skw}\,[\nabla\mathbf{v}]$ the *spin* in the composite continuum, while G° and m° are time rates relative to a local frame spinning together with the composite continuum. With (2.33) and (2.34), inequality (2.32) is easily shown to be equivalent to both

$$(\mathbf{T}^{vs} + G^T C^{vs})\cdot\mathbf{D} + C^{vs}\cdot G^\circ - k^{vs}\cdot m^\circ \geqslant 0 \quad (2.35)$$

and

$$\mathrm{skw}\,[\mathbf{T}^{vs} + C^{vs}G^T - k^{vs}\otimes m] = \mathbf{0}\,. \quad (2.36)$$

The latter relation, together with (2.11) and (2.27)-(2.31), imply that the assignment of the free-energy mapping satisfy

$$\mathrm{skw}\left[\frac{\partial\widehat{\psi}}{\partial\mathbf{F}}(\mathbf{F},G,m)\,\mathbf{F}^T - G^T\frac{\partial\widehat{\psi}}{\partial G}(\mathbf{F},G,m) + \frac{\partial\widehat{\psi}}{\partial G}(\mathbf{F},G,m)\,G^T + \frac{\partial\widehat{\psi}}{\partial m}(\mathbf{F},G,m)\otimes m\right] = \mathbf{0} \quad (2.37)$$

for each admissible $(\mathbf{F}, G, m)$. (10)

REMARK. – If we assume, as is done in Section 5 of [14], that none of $\mathbf{T}^{vs}$, C^{vs}, and k^{vs} de-

(9) Needless to say, this inequality is invariant under observer changes, just as (2.22).

(10) Note that, with no use of the dissipation inequality but simply as a consequence of constitutive splittings of type (2.30)-(2.31):

$$\mathbf{T} = \mathbf{T}^{eq} + \mathbf{T}^{vs}, \qquad \mathbf{T}^{vs} = \mathbf{T} - \mathbf{T}^{eq}, \quad etc.\,,$$

(2.11) is equivalent to both (2.36) and

$$\mathrm{skw}\,[\mathbf{T}^{eq} + C^{eq}G^T - k^{eq}\otimes m] = \mathbf{0}\,.$$

pend on $\mathbf{D}$ and G°, then (2.35) and (2.36) reduce to, respectively,

$$\operatorname{sym}[\mathbf{T}^{vs}] = \mathbf{0}\,, \qquad C^{vs} = \mathbf{0}\,, \qquad \boldsymbol{k}^{vs}\cdot \boldsymbol{m}^\circ \leqslant 0\,,$$

and

$$\operatorname{skw}[\mathbf{T}^{vs} - \boldsymbol{k}^{vs} \otimes \boldsymbol{m}] = \mathbf{0}$$

(*cf.* (5.30), (5.35), and (5.37) of [14]). If, in addition, as in [13], the free-energy mapping is assumed to have the simple form

$$\widehat{\psi}(\mathbf{F}, G, \boldsymbol{m}) = \frac{1}{2}\kappa_m |G|^2 + \overline{\psi}(\mathbf{F}, \boldsymbol{m})\,,$$

then (2.37) reduces to

$$\operatorname{skw}\left[\frac{\partial \overline{\psi}}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{m})\,\mathbf{F}^T + \frac{\partial \overline{\psi}}{\partial \boldsymbol{m}}(\mathbf{F}, \boldsymbol{m}) \otimes \boldsymbol{m}\right] = \mathbf{0}$$

(*cf.* (29) of [13]).

2.5. The General Evolution Problem.

We are now in a position to state formally a fairly general problem in the dynamics of ferromagnets.

Given a referential shape B and a referential mass-density $\varrho_r(X) > 0$ over B, find a (motion, magnetization) pair, *i.e.*, a pair of suitably smooth mappings $f(X, t)$ and $\boldsymbol{m}(x, t)$, with $(X, t) \in B \times [0, T)$, $T > 0$, $x = f(X, t)$, and $|\boldsymbol{m}(x, t)| \equiv 1$, consistent with the following system of evolution equations, initial conditions, and boundary conditions:

– the quasi-static Maxwell equations,

$$\operatorname{curl} \mathbf{h}_{\mathcal{B}} = \mathbf{0}\,, \qquad \operatorname{div} \mathbf{h}_{\mathcal{B}} = -\operatorname{div}(\varrho \chi_{\mathcal{B}} \boldsymbol{m}) \text{ in } \mathcal{E}\,, \tag{2.38}$$

with

$$\mathcal{B} = f(B, t)\,, \qquad \varrho = (\det \nabla f)^{-1} \varrho_r \tag{2.39}$$

(so that mass is conserved, *cf.* (2.1));

– the balance equations (2.9) and (2.10), combined with (2.20) and (2.21),

$$\varrho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \mathbf{b}^e_{me} + \varrho(\nabla \mathbf{h})\, \boldsymbol{m}\,, \tag{2.40}$$

$$\gamma^{-1} \dot{\boldsymbol{m}} = \boldsymbol{m} \times (\operatorname{div} C + \boldsymbol{k} + (\boldsymbol{b}^e_{me} + \varrho \mathbf{h}))\,, \text{(11)} \tag{2.41}$$

(11) Here we have used the fact that, due to the saturation condition, $-\boldsymbol{m} \times (\boldsymbol{m} \times \dot{\boldsymbol{m}}) = \dot{\boldsymbol{m}}$.

with

$$\mathbf{v} = \partial_t f \circ (f(\cdot, t))^{-1}, \qquad \mathbf{h} = \mathbf{h}_{\mathcal{B}} + \mathbf{h}^e; \tag{2.42}$$

– constitutive relations for the free energy of the form

$$\psi = \widehat{\psi}(\mathbf{F}, G, \boldsymbol{m}), \tag{2.26}$$

and for the stress, the couple stress, and the interaction distance-force of forms resulting by combining (2.27) with (2.33) *etc.*, namely,

$$\mathbf{T} = \varrho \frac{\partial \widehat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, G, \boldsymbol{m})\, \mathbf{F}^T - \varrho G^T \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, \boldsymbol{m}) + \widetilde{\mathbf{T}}^{\text{vs}}(\mathbf{F}, G, \boldsymbol{m}; \mathbf{D}, G^\circ, \boldsymbol{m}^\circ), \tag{2.43}$$

$$C = \varrho \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, \boldsymbol{m}) + \widetilde{C}^{\text{vs}}(\mathbf{F}, G, \boldsymbol{m}; \mathbf{D}, G^\circ, \boldsymbol{m}^\circ), \tag{2.44}$$

$$\boldsymbol{k} = -\varrho \frac{\partial \widehat{\psi}}{\partial \boldsymbol{m}}(\mathbf{F}, G, \boldsymbol{m}) + \widetilde{\boldsymbol{k}}^{\text{vs}}(\mathbf{F}, G, \boldsymbol{m}; \mathbf{D}, G^\circ, \boldsymbol{m}^\circ), \tag{2.45}$$

provided these constitutive relations be compatible with the dissipation inequality

$$(\mathbf{T}^{\text{vs}} + G^T C^{\text{vs}}) \cdot \mathbf{D} + C^{\text{vs}} \cdot G^\circ - \boldsymbol{k}^{\text{vs}} \cdot \boldsymbol{m}^\circ \geqslant 0 \tag{2.35}$$

and, in addition, with the consistency conditions

$$\operatorname{skw}[\mathbf{T}^{\text{vs}} + C^{\text{vs}} G^T - \boldsymbol{k}^{\text{vs}} \otimes \boldsymbol{m}] = \mathbf{0} \tag{2.36}$$

and

$$\operatorname{skw}\left[\frac{\partial \widehat{\psi}}{\partial \mathbf{F}}(\mathbf{F}, G, \boldsymbol{m})\, \mathbf{F}^T - G^T \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, \boldsymbol{m}) + \frac{\partial \widehat{\psi}}{\partial G}(\mathbf{F}, G, \boldsymbol{m})\, G^T + \frac{\partial \widehat{\psi}}{\partial \boldsymbol{m}}(\mathbf{F}, G, \boldsymbol{m}) \otimes \boldsymbol{m}\right] = \mathbf{0}; \tag{2.37}$$

– the compatibility conditions

$$\mathbf{F} = \partial_X f, \qquad G = \nabla \boldsymbol{m}, \tag{2.46}$$

$$\mathbf{D} = \operatorname{sym}[\nabla \mathbf{v}], \qquad \mathbf{W} = \operatorname{skw}[\nabla \mathbf{v}], \tag{2.47}$$

$$G^\circ = \dot{G} - \mathbf{W} G + G \mathbf{W}, \qquad \boldsymbol{m}^\circ = \dot{\boldsymbol{m}} - \mathbf{W} \boldsymbol{m}; \tag{2.48}$$

– the initial conditions

$$f(X, 0) = f_o(X), \qquad \partial_t f(X, 0) = \mathbf{v}_o(X) \quad \text{in } B, \tag{2.49}$$

$$\boldsymbol{m}(X, 0) = \boldsymbol{m}_o(X), \qquad |\boldsymbol{m}_o(X)| = 1 \quad \text{in } B; \tag{2.50}$$

– the boundary conditions

$$f(X, t) = X \quad \text{in } \partial_1 B \times (0, T], \tag{2.51$_1$}$$

$$\mathbf{T}(x, t)\,\mathbf{n}(x) - \frac{1}{2}(\varrho(x, t)\,\boldsymbol{m}(x, t)\cdot\mathbf{n}(x))^2\,\mathbf{n}(x) = \mathbf{t}_o(x, t) \quad \text{in } \partial_2 \mathcal{B} \times (0, T], \tag{2.51$_2$}$$

where

$$\partial_1 \mathcal{B} = f(\partial_1 B, t), \qquad \partial_1 B \cup \partial_2 B = \partial B, \qquad \partial_1 B \cap \partial_2 B = \emptyset, \tag{2.52}$$

and

$$\varrho(x, t)\,\boldsymbol{m}(x, t) \times (C(x, t)\,\mathbf{n}(x)) = \mathbf{c}_o(x, t) \quad \text{in } \partial\mathcal{B} \times (0, T]. \tag{2.53}$$ (12)

We have already remarked that, in this system, the body fields $\mathbf{b}^e_{me}$, b^e_{me}, and $\mathbf{h}^e$ can be regarded as distance controls; the contact controls are the surface fields $\mathbf{t}_o$ and $\mathbf{c}_o$.

3. – Soft ferromagnets at rest.

A ferromagnetic material is termed *soft* whenever its free-energy mapping does not depend on magnetization. If this is the case, it follows from (2.45) that the interaction distance-force $\boldsymbol{k}$ is null at equilibrium, a situation of weak mechanical/magnetical coupling, whence the terminology.

A ferromagnetic body B is *mechanically at rest* whenever it undergoes a magnetic process with no motion, *i.e.*, whenever for $\mathbf{v}_o(X) = \mathbf{0}$ in B

$$f(X, t) = f_o(X) \quad \text{in } B \times (0, T] \tag{3.1}$$

(*cf.* (2.49)), and yet the magnetization field evolves in time.

In view of the complexity of the general evolution problem formulated in Subsection 2.5, one may ask whether there is a set of reasonable assumptions under which purely magnetic processes would be possible, and composing a collection sufficiently rich to attract attention on theories of soft ferromagnets at rest; we introduce such a list of assumptions in the next subsection.

---

(12) Thus, while the part $\partial_1 B$ of the boundary is clamped, tractions $\mathbf{t}_o$ are applied on the current shape of the complementary and disjoint part $\partial_2 B$, and torques $\mathbf{c}_o$ are applied on the whole current boundary (*cf.* relations (◇) and (◇◇) in footnote 6). It would be easy to replace conditions (2.51)-(2.52) by more general ones. Instead, it does not seem physically sound to posit boundary conditions directly on the magnetization vector, at sharp variance with the case of nematic liquid crystals, where *anchoring conditions* on the director are believed realizable, and hence imposed mathematically; consequently, singularities in the magnetization field cannot be enforced by adjusting the boundary data [13, Section 6].

3.1. *Soft ferromagnets with linear magnetic response.*

We confine attention to soft ferromagnets with linear magnetic response, both equilibrium and viscous, by which we mean those ferromagnetic materials of type (2.43)-(2.45) such that

$$\varrho\widehat{\psi}(\mathbf{F}, G) = \varrho\overline{\psi}(\mathbf{F}) + \frac{1}{2}\kappa G\cdot G\,, \qquad \kappa > 0\,, \tag{3.2}$$

$$\widetilde{C}^{vs}(\mathbf{F}, G, m; \mathbf{D}, G^{\circ}, m^{\circ}) = \mu_C G^{\circ}\,, \qquad \mu_C \geqslant 0\,, \tag{3.3}$$

$$\widetilde{k}^{vs}(\mathbf{F}, G, m; \mathbf{D}, G^{\circ}, m^{\circ}) = -\mu_k m^{\circ}\,, \qquad \mu_k \geqslant 0\,. \tag{3.4}$$

Note that, with (3.2), condition (2.37) reduces to

$$\operatorname{skw}\left[\frac{\partial\overline{\psi}}{\partial\mathbf{F}}(\mathbf{F})\,\mathbf{F}^T\right] = \mathbf{0}\,; \tag{3.5}$$

we assume that the mapping $\overline{\psi}$ be such as to satisfy (3.5) identically. Note, in addition, that relations (2.43)-(2.45) become, respectively,

$$\mathbf{T} = \varrho\frac{\partial\overline{\psi}}{\partial\mathbf{F}}(\mathbf{F})\,\mathbf{F}^T - \kappa G^T G + \widetilde{\mathbf{T}}^{vs}(\mathbf{F}, G, m; \mathbf{D}, G^{\circ}, m^{\circ})\,, \tag{3.6}$$

$$C = \kappa G + \mu_C G^{\circ}\,, \tag{3.7}$$

$$k = -\mu_k m^{\circ}\,; \tag{3.8}$$

we assume that the mapping $\widetilde{\mathbf{T}}^{vs}$ be compatible with the disssipation inequality to which (2.35) reduces:

$$(\mathbf{T}^{vs} + \mu_C G^T G^{\circ})\cdot\mathbf{D} + \mu_C G^{\circ}\cdot G^{\circ} + \mu_k m^{\circ}\cdot m^{\circ} \geqslant 0\,, \tag{3.9}$$

and with (2.36) in its present form, namely,

$$\operatorname{skw}[\mathbf{T}^{vs} + \mu_C G^{\circ}G^T + \mu_k m^{\circ}\otimes m] = \mathbf{0}\,. \tag{3.10}$$ [13]

With (3.7), the boundary conditions (2.53) becomes

$$\varrho m\times(\kappa G + \mu_C G^{\circ})\,\mathbf{n} = \mathbf{c}_o \quad \text{in } \partial\mathcal{B}\times(0, T]\,. \tag{3.11}$$

We now stipulate the body to be initially at rest in its referential shape:

$$f_o(X) = X\,, \qquad \mathbf{v}_o(X) = \mathbf{0} \quad \text{in } B\,, \tag{3.12}$$

[13] While inequality (3.9) restricts only the symmetric part of $\mathbf{T}^{vs}$, relation (3.10) determines the skew part completely.

with constant mass density

$$\varrho_o(X) = \varrho_o \quad \text{in } B. \tag{3.13}$$

Moreover, as to boundary conditions, we take the applied surface torque $\mathbf{c}_o$ to be identically null, and we take $\partial_1 B$ to be either empty or all of ∂B (*cf.* (2.51); in the latter case, the boundary traction $\mathbf{t}_o$ has reactive nature, and hence drops off the list of available controls).

We choose to study only purely magnetic processes, for which, consistently with (3.12), the body remains at rest throughout the time interval of interest, *i.e.*,

$$f(X, t) = X \quad \text{in } B \times (0, T] \tag{3.14}$$

(*cf.* (3.1)). As a first consequence of this choice, we have that, during such processes,

$$\mathbf{F} \equiv \mathbf{1}, \qquad G = \partial_X m, \qquad \mathbf{D} = \mathbf{W} \equiv \mathbf{0}, \tag{3.15}$$

$$G^\circ \equiv \dot{G} = \partial_t(\partial_X m) = \partial_X(\partial_t m), \qquad m^\circ \equiv \dot{m} = \partial_t m\,; \tag{3.16}$$ (14)

hence, in particular, (2.39) and (2.2) yield, respectively,

$$\varrho(X, t) = \varrho_o \quad \text{and} \quad \mathbf{m}(X, t) = \varrho_o m(X, t) \quad \text{in } B \times (0, T], \tag{3.17}$$

while (3.9) is identically satisfied. Another consequence is that the Maxwell equations take the form

$$\operatorname{Curl} \mathbf{h}_B = \mathbf{0}, \quad \operatorname{Div} \mathbf{h}_B = -\operatorname{Div}(\varrho_o \chi_B m) \quad \text{in } \mathcal{E}_r; \tag{3.18}$$ (15)

we here for short write the functional dependence of the magnetic field on the magnetization field, whose form has been given in Subsection 2.2, as

$$\mathbf{h}_B = \mathfrak{h}(m). \tag{3.19}$$

Finally, the evolution equation (2.41) takes the form

$$\gamma^{-1}\dot{m} = m \times \left(\kappa \Delta m + \mu_C \Delta \dot{m} - \mu_k \dot{m} + b^e_{me} + \varrho_o(\mathbf{h}^e + \mathfrak{h}(m))\right) \quad \text{in } B \times (0, T]. \tag{3.20}$$

We supplement these equations with the initial condition

$$m(X, 0) = m_o(X), \qquad |m_o(X)| = 1 \quad \text{in } B, \tag{2.50}$$

and with the boundary condition to which (3.11) reduces under the current hypotheses, namely,

$$m \times (\kappa \partial_{\mathbf{n}} m + \mu_C \partial_{\mathbf{n}} \dot{m}) = \mathbf{0} \quad \text{in } \partial B \times (0, T]. \tag{3.21}$$

Once a solution $m(X, t)$ to (3.20), (2.50), and (3.21) is found, we are left with the task of

(14) Since, when there is no motion, material and partial differentiations with respect to time are the same operation, henceforth we choose to denote time differentiation by a superposed dot.

(15) *Cf.* equations (2.38), on recalling the notation introduced in footnote 5.

showing that, when combined with the constitutive relation (3.6) composed with $m(X, t)$, the evolution equation (2.40) has the solution (3.14) up to boundary of B for some $T > 0$. At the present level of generality in the choice of $\overline{\psi}$ and $\tilde{\mathbf{T}}^{vs}$ we cannot do any better than claiming that this is indeed the case. Our reason to do so is that we still have sufficient freedom in the selection of the available mechanical and magnetical controls. ([16])

3.2. *Two interesting cases.*

A first case of interest to study is the homogeneous version of equation (3.20):

$$\gamma^{-1}\dot{m} = m \times (\kappa \Delta m + \mu_C \Delta \dot{m} - \mu_k \dot{m}) \quad \text{in } B \times (0, T], \tag{3.22}$$

again supplemented by the initial and boundary conditions (2.50) and (3.21). Once a solution to this problem is found, the Maxwell equations can be solved for the corresponding $\mathbf{h}_B$, and the body controls partly disposed of by assuming that

$$b^e_{me} + \varrho_0(\mathbf{h}^e + \mathbf{h}_B) = \mathbf{0}, \tag{3.23}$$

just as needed to justify *a posteriori* the study of (3.22).

A second interesting case is when, instead, the body control vanishes:

$$b^e_{me} + \varrho_0 \mathbf{h}^e = \mathbf{0}, \tag{3.24}$$

so that the equation to solve is

$$\gamma^{-1}\dot{m} = m \times (\kappa \Delta m + \mu_C \Delta \dot{m} - \mu_k \dot{m} + \mathfrak{h}(m)) \quad \text{in } B \times (0, T]. \tag{3.25}$$

This problem has been considered by Vilucchi [28] under the additional hypotheses that the viscosity is solely that provided by the interaction force k, and hence

$$\mu_k > 0, \quad \mu_C = 0 \tag{3.26}$$

(see the remark closing the next section).

3.3. *The case studied here.*

In the next section we establish global existence of weak solutions to problem (3.22)-(2.50)-(3.21) under assumption (3.26). With this assumption, the evolution equation (3.22) for the magnetization vector becomes

$$\gamma^{-1}\dot{m} = m \times (\kappa \Delta m - \mu_k \dot{m}) \quad \text{in } B \times (0, T]. \tag{3.27}$$

In (3.27), κ, μ_k, and γ are given material constants, with κ the material's stiffness with respect to spatial changes of magnetization, μ_k measuring dissipation, and γ measuring the

([16]) Precisely, both the body control ($\mathbf{b}^e_{me} + \varrho_0(\nabla \mathbf{h}^e)\, m$) and the surface control $\mathbf{t}_0$ when the whole boundary is free ($\partial_1 B = \emptyset$), the body control only when the whole boundary is clamped ($\partial_1 B = \partial B$).

nondissipative inertial force associated with time changes of magnetization. We find it convenient to scale the space variable with a factor $(\gamma\kappa)^{1/2}$ and, with slight abuse of notation, we continue to denote by $\mathcal{B}$ the scaled region occupied by the body and by x the typical point of $\mathcal{B}$. Moreover, we set $\alpha = \gamma\mu_k$, so that equation (3.27) takes the form displayed in the Introduction, namely,

$$(1.1)_1 \qquad \dot{\boldsymbol{m}} = -\alpha \boldsymbol{m} \times \dot{\boldsymbol{m}} + \boldsymbol{m} \times \Delta \boldsymbol{m} .$$

While the initial condition (2.50) needs no changes, as a consequence of (3.26) the boundary condition (3.21) does: to the evolution equation (1.1) we append

$$(1.2)_1 \qquad \boldsymbol{m}(x, 0) = \boldsymbol{m}_0(x) \quad \text{in } \mathcal{B}, \ |\boldsymbol{m}_0| = 1 ;$$

$$(1.2)_2 \qquad \partial_{\mathbf{n}} \boldsymbol{m}(x, t) = \mathbf{0} \quad \text{in } \partial\mathcal{B} \times (0, T] ,$$

where $\boldsymbol{m}_0$ is the initial magnetization field over $\mathcal{B}$, and where $\partial_{\mathbf{n}} \boldsymbol{m} = (\nabla \boldsymbol{m})\,\mathbf{n}$.([17])

We call Problem $\mathcal{P}$ the initial-boundary value problem of finding, for each datum $\boldsymbol{m}_0$ of finite energy $\int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}_0|^2$ and for each fixed positive time T, a field $\boldsymbol{m}(x, t)$ over $\overline{\mathcal{B}} \times [0, T]$ that satisfies (1.1)-(1.2).

4. – Global existence of weak solutions.

As a premise to our proof of the title result for Problem $\mathcal{P}$, we observe that solutions of (1.1)-(1.2) satisfy formally the energy estimates

$$(4.1) \qquad \frac{d}{dt} \int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}|^2 dx + \alpha \int_{\mathcal{B}} |\dot{\boldsymbol{m}}|^2 dx = 0 ;$$

$$(4.2) \qquad \alpha \int_0^T \int_{\mathcal{B}} |\dot{\boldsymbol{m}}|^2 dx dt \leq \int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}_0|^2 dx , \qquad \int_{\mathcal{B}} |\nabla \boldsymbol{m}|^2 dx \leq \int_{\mathcal{B}} |\nabla \boldsymbol{m}_0|^2 dx .$$

Relation (4.1) obtains by taking the inner product of (1.1) and $(\Delta \boldsymbol{m} - \alpha \dot{\boldsymbol{m}})$ and making use of the boundary condition $(1.2)_2$; (4.2) follow from (4.1) and the initial condition $(1.2)_1$. The physical interpretation is that, along orbits, the time rate of the exchange energy $\int_{\mathcal{B}} \frac{1}{2} |\nabla \boldsymbol{m}|^2$ compensates for the dissipation, while the initial exchange energy separately bounds both energy and dissipation at any later time.

We also note that, for smooth functions $\boldsymbol{m}$, equation (1.1) can be given various alterna-

---

([17]) The equivalence of $(1.2)_2$ and

$$\boldsymbol{m} \times \partial_{\mathbf{n}} \boldsymbol{m} = \mathbf{0} \quad \text{in } \partial\mathcal{B} \times (0, T]$$

is an easy consequence of the saturation condition.

tive forms, derived in Appendix A; one of these forms,

$$\alpha \dot{\boldsymbol{m}} - \boldsymbol{m} \times \dot{\boldsymbol{m}} = \Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m}\,, \qquad |\boldsymbol{m}| = 1\,, \tag{1.1$'$}$$

has been used in the Introduction to make evident the kinship of (1.1) with the harmonic-map equation (1.7) and its «heat flow» (1.8). Equation (1.1)′ can be interpreted as the requirement that, in the intrinsic orthonormal triad $\{\boldsymbol{m}, \dot{\boldsymbol{m}}, \boldsymbol{m} \times \dot{\boldsymbol{m}}\}$, the laplacian of the magnetization has components $(-|\nabla \boldsymbol{m}|^2, \alpha, -1)$; in particular, the stationary solutions of our evolution problem are characterized by the condition that the magnetization vector and its laplacian are antiparallel, with $|\Delta \boldsymbol{m}|$ proportional to the exchange energy.

4.1. *Existence of global-strong solutions to problem* $\mathcal{P}^{\varepsilon,\tau}$.

For each fixed positive value of the parameter ε and τ, consider the following

PROBLEM $\mathcal{P}^{\varepsilon,\tau}$. – For each initial datum $\boldsymbol{m}_0$ of finite energy, and for each fixed positive time T, find a field $\boldsymbol{m}^{\varepsilon,\tau}(x, t)$ over $\overline{\mathcal{B}} \times [0, T]$ that satisfies the evolution equation

$$\alpha \dot{\boldsymbol{m}} - \boldsymbol{m} \times \dot{\boldsymbol{m}} - \tau \Delta \dot{\boldsymbol{m}} = \Delta \boldsymbol{m} - \varepsilon^{-1}(|\boldsymbol{m}|^2 - 1)\, \boldsymbol{m}\,, \tag{1.5}$$

subject to the initial and boundary conditions

$$\boldsymbol{m}(x, 0) = \boldsymbol{m}_0^\varepsilon(x) \quad \text{in } \overline{\mathcal{B}}, \tag{4.3$_1$}$$

$$\partial_{\mathbf{n}} \boldsymbol{m} = \mathbf{0} \quad \text{in } \partial\mathcal{B} \times (0, T]\,, \tag{4.3$_2$}$$

where $\boldsymbol{m}_0^\varepsilon(x) \in C^\infty(\overline{\mathcal{B}})$, $\boldsymbol{m}_0^\varepsilon(x) \to \boldsymbol{m}_0(x)$ in $L^2(\mathcal{B})$, $\partial_{\mathbf{n}} \boldsymbol{m}_0^\varepsilon = \mathbf{0}$ in $\partial\mathcal{B}$, $\boldsymbol{m}_0^\varepsilon(x)$ is uniformly bounded in $H^1(\mathcal{B})$, and $\left(\varepsilon^{-1} \int_{\mathcal{B}} (|\boldsymbol{m}_0^\varepsilon|^2 - 1)^2 dx\right) \to 0$ as $\varepsilon \to 0$.

With a view to proving global existence and uniqueness of a strong solution $\boldsymbol{m}^{\varepsilon,\tau}$ to Problem $\mathcal{P}^{\varepsilon,\tau}$, we now establish some *a priori* estimates. Multiplying (1.5) by $\dot{\boldsymbol{m}}^{\varepsilon,\tau}$, integrating by parts, and considering the boundary condition $(4.3)_2$, we obtain the energy identity

$$\alpha \int_{\mathcal{B}} |\dot{\boldsymbol{m}}^{\varepsilon,\tau}|^2 dx + \tau \int_{\mathcal{B}} |\nabla \dot{\boldsymbol{m}}^{\varepsilon,\tau}|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{B}} |\nabla \boldsymbol{m}^{\varepsilon,\tau}|^2 dx + \frac{1}{4\varepsilon}\frac{d}{dt}\int_{\mathcal{B}} (|\boldsymbol{m}^{\varepsilon,\tau}|^2 - 1)^2 dx = 0\,, \tag{4.4}$$

from which we deduce the following integral estimates:

$$\alpha \int_0^T \int_{\mathcal{B}} |\dot{\boldsymbol{m}}^{\varepsilon,\tau}|^2 dx\, dt \leq \frac{1}{2} \int_{\mathcal{B}} |\nabla \boldsymbol{m}_0^\varepsilon|^2 dx + \frac{1}{4\varepsilon} \int_{\mathcal{B}} (|\boldsymbol{m}_0^\varepsilon|^2 - 1)^2 dx\,, \tag{4.5}$$

$$\tau \int_0^T \int_{\mathcal{B}} |\nabla \dot{\boldsymbol{m}}^{\varepsilon,\tau}|^2 dx\, dt \leq \frac{1}{2} \int_{\mathcal{B}} |\nabla \boldsymbol{m}_0^\varepsilon|^2 dx + \frac{1}{4\varepsilon} \int_{\mathcal{B}} (|\boldsymbol{m}_0^\varepsilon|^2 - 1)^2 dx\,, \tag{4.6}$$

$$\int_{\mathcal{B}} |\nabla m^{\varepsilon,\tau}|^2 dx \leqslant \int_{\mathcal{B}} |\nabla m_0^\varepsilon|^2 dx + \frac{1}{2\varepsilon} \int_{\mathcal{B}} (|m_0^\varepsilon|^2 - 1)^2 dx , \tag{4.7}$$

$$\frac{1}{\varepsilon} \int_{\mathcal{B}} (|m^{\varepsilon,\tau}|^2 - 1)^2 dx \leqslant 2 \int_{\mathcal{B}} |\nabla m_0^\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_{\mathcal{B}} (|m_0^\varepsilon|^2 - 1)^2 dx . \tag{4.8}$$

Moreover, from (4.7) and (4.6), respectively, and from Sobolev embedding theorem, we have that, for $p \leqslant 6$, there is a constant K, independent of ε, τ, t, and T, such that, for all fixed ε, $\tau > 0$,

$$\int_{\mathcal{B}} |m^{\varepsilon,\tau}|^p dx \leqslant K , \qquad 0 < t \leqslant T , \tag{4.9}$$

and

$$\int_0^T \int_{\mathcal{B}} |\dot{m}^{\varepsilon,\tau}|^p dx\, dt \leqslant K\tau^{-1}. \tag{4.10}$$

THEOREM. – For each ε and $\tau > 0$ fixed, there is a unique *global-strong solution* of Problem $\mathcal{P}^{\varepsilon,\tau}$, *i.e.*, a vector field $m^{\varepsilon,\tau}$ such that, for some $\lambda \in (0, 1)$, both $m^{\varepsilon,\tau}(x, t) \in C^{2,\lambda}(\overline{\mathcal{B}}; \mathbb{R}^3)$ for all $t \in [0, T]$ and $\|m^{\varepsilon,\tau}(x, t)\|_{C^{2,\lambda}(\overline{\mathcal{B}}; \mathbb{R}^3)} \in W^{1,\infty}((0, T))$.

PROOF. – We introduce the space

$$V = \{\mathbf{v} \mid \mathbf{v} \in C^{2,\lambda}(\overline{\mathcal{B}}; \mathbb{R}^3),\ \partial_{\mathbf{n}} \mathbf{v} = \mathbf{0} \text{ in } \partial\mathcal{B}\},$$

equipped with the norm of $C^{2,\lambda}(\overline{\mathcal{B}}; \mathbb{R}^3)$. We write Problem $\mathcal{P}^{\varepsilon,\tau}$ as an ODE in V:

$$\dot{m}^{\varepsilon,\tau} = F_{\varepsilon,\tau}(m^{\varepsilon,\tau}) \quad \text{for } t > 0 , \qquad m^{\varepsilon,\tau}(0) = m_0^\varepsilon. \tag{4.11}$$

The definition of the operator $F_{\varepsilon,\tau}$ requires some care.

For $\mathbf{w} \in C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)$ fixed, we consider the linear elliptic operator

$$G_\tau^{\mathbf{w}}: V \to C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)$$

defined by

$$G_\tau^{\mathbf{w}}(\mathbf{u}) = \alpha\mathbf{u} - \tau\Delta\mathbf{u} - \mathbf{w} \times \mathbf{u} . \tag{4.12}$$

It follows from classical Schauder estimates for the elliptic system

$$G_\tau^{\mathbf{w}}(\mathbf{u}) = \mathbf{y} \quad \text{in } \mathcal{B}, \qquad \partial_{\mathbf{n}} \mathbf{u} = \mathbf{0} \quad \text{in } \partial\mathcal{B},$$

where $\mathbf{y} \in C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)$, that

$$\|\mathbf{u}\|_{C^{2,\lambda}(\overline{\mathcal{B}}; \mathbb{R}^3)} \leqslant K \|\mathbf{y}\|_{C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)}, \tag{4.13}$$

for some constant K which depends on τ and $\|\mathbf{w}\|_{C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)}$, but not on $\mathbf{y}$. Hence, $G_\tau^{\mathbf{w}}$ is invertible and, since it is linear, $(G_\tau^{\mathbf{w}})^{-1}$ is Lipschitz continuous for any $\mathbf{w} \in C^\lambda(\overline{\mathcal{B}}; \mathbb{R}^3)$. Defining the operator $F_{\varepsilon,\tau}: V \to V$ by

$$F_{\varepsilon,\tau}(m) = (G_\tau^m)^{-1}(\Delta m - \varepsilon^{-1}(|m|^2 - 1)\, m)\,,$$

we find that (4.11) and (1.5) are equivalent.

For the existence and uniqueness of a *local* solution of the ODE (4.11), it is enough to show that $F_{\varepsilon,\tau}$ is locally Lipschitz continuous in V. Let $m_1, m_2 \in V$, and let

$$\mathbf{u}_1 = F_{\varepsilon,\tau}(m_1) \quad \textit{and} \quad \mathbf{u}_2 = F_{\varepsilon,\tau}(m_2)\,.$$

Then,

$$G_\tau^{m_i}(\mathbf{u}_i) = \Delta m_i - \varepsilon^{-1}(|m_i|^2 - 1)\, m_i,\ i = 1, 2\,, \tag{4.14}$$

and hence

$$\begin{aligned} G_\tau^{m_1}(\mathbf{u}_1 - \mathbf{u}_2) &= G_\tau^{m_1}(\mathbf{u}_1) - G_\tau^{m_2}(\mathbf{u}_2) - G_\tau^{m_1}(\mathbf{u}_2) + G_\tau^{m_2}(\mathbf{u}_2) = \\ &= \Delta(m_1 - m_2) - \varepsilon^{-1}(|m_1|^2 - 1)\, m_1 + \varepsilon^{-1}(|m_2|^2 - 1)\, m_2 + ((m_1 - m_2) \times \mathbf{u}_2)\,. \end{aligned} \tag{4.15}$$

Since, by (4.13) and (4.14),

$$\|\mathbf{u}_2\|_{C^{2,\lambda}(\overline{\mathcal{B}};\, \mathbb{R}^3)} \leq K \|m_2\|_{C^{2,\lambda}(\overline{\mathcal{B}};\, \mathbb{R}^3)} \tag{4.16}$$

for some constant K depending on $\|m_2\|_{C^\lambda(\overline{\mathcal{B}};\, \mathbb{R}^3)}$, it follows from (4.15) and (4.16) that

$$\|F_{\varepsilon,\tau}(m_1) - F_{\varepsilon,\tau}(m_2)\|_V \leq C \|m_1 - m_2\|_V$$

for some constant C depending on $\|m_i\|_{C^\lambda(\overline{\mathcal{B}};\, \mathbb{R}^3)}$ and $\|m_2\|_{C^{2,\lambda}(\overline{\mathcal{B}};\, \mathbb{R}^3)}$. In particular $F_{\varepsilon,\tau}$ is locally Lipschitz continuous in V, and equation (4.11) possesses a uniquely determined local solution $m^{\varepsilon,\tau}$.

To complete the proof we have to show that $m^{\varepsilon,\tau}$ can be extended for all t. Since

$$G_\tau^m(\dot{m}^{\varepsilon,\tau}) = \Delta m^{\varepsilon,\tau} - \varepsilon^{-1}(|m^{\varepsilon,\tau}|^2 - 1)\, m^{\varepsilon,\tau},$$

it follows from (4.12) that, as long as the solution exists,

$$\|\dot{m}^{\varepsilon,\tau}\|_V \leq C \|m^{\varepsilon,\tau}\|_V$$

for some constant C which depends on $\|m^{\varepsilon,\tau}\|_{C^\lambda(\overline{\mathcal{B}};\, \mathbb{R}^3)}$. Hence the existence of a *global* solution is garanteed if we find an *a priori* bound in $C^\lambda(\overline{\mathcal{B}};\, \mathbb{R}^3)$ for $m^{\varepsilon,\tau}$, for all time and for some $\lambda \in (0, 1)$. Since $W^{2,2}(\mathcal{B}) \subseteq C^{1/2}(\overline{\mathcal{B}})$ if $\mathcal{B} \subseteq R^3$, global existence is a consequence of the following estimate. We multiply equation (1.5) by $\Delta m^{\varepsilon,\tau}$, and integrate by parts with the use of the boundary condition $(4.3)_2$. Omitting superscripts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{B}}(\alpha|\nabla m|^2+\tau|\Delta m|^2)\,dx=-\int_{\mathcal{B}}(m\times\dot m)\,\Delta m\,dx-$$

$$-\int_{\mathcal{B}}|\Delta m|^2\,dx-\frac{1}{\varepsilon}\int_{\mathcal{B}}\frac{1}{2}|\nabla(|m|^2-1)|^2\,dx-\varepsilon^{-1}\int_{\mathcal{B}}(|m|^2-1)|\nabla m|^2\,dx,$$

whence

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{B}}(\alpha|\nabla m|^2+\tau|\Delta m|^2)\,dx+\int_{\mathcal{B}}|\Delta m|^2\,dx-\varepsilon^{-1}\int_{\mathcal{B}}|\nabla m|^2\,dx\leqslant$$

$$\leqslant\int_{\mathcal{B}}|m\times\dot m|^2\,dx+\frac{1}{4}\int_{\mathcal{B}}|\Delta m|^2\,dx.$$

In view of (4.9) and (4.10), it follows from Gronwall Lemma that, for all $T>0$, $|\Delta m^{\varepsilon,\tau}|$ and $|\nabla m^{\varepsilon,\tau}|$ are bounded in $L^2(\mathcal{B})$ for all $t\in[0,T]$. ■

4.2. *Existence of global-weak solutions to problem $\mathcal{P}$.*

We are now in a position to prove our theorem of global existence of weak solutions to Problem $\mathcal{P}$; for the reader's convenience, we begin by reproducing the statement given in the Introduction.

GLOBAL-WEAK EXISTENCE THEOREM. – Choose $\mathcal{B}$, an open, bounded region with smooth boundary, and choose a vector field $m_0\in H^1(\mathcal{B};\mathbb{R}^3)$, with $|m_0|=1$ a.e. in $\mathcal{B}$, and with finite energy. For each $T>0$, there is a *global-weak solution* of Problem $\mathcal{P}$, *i.e.*, a vector field $m\in H^1(\mathcal{B}\times(0,T];\mathbb{R}^3)$ such that

(i) for each $\mathbf{z}\in C^\infty(\overline{\mathcal{B}}\times[0,T])$ vanishing at $t=0$ and $t=T$,

$$-\int_0^T\int_{\mathcal{B}}\dot m\cdot\mathbf{z}=\alpha\int_0^T\int_{\mathcal{B}}\mathbf{M}\dot m\cdot\mathbf{z}+\int_0^T\int_{\mathcal{B}}\mathbf{M}\nabla m\cdot\nabla\mathbf{z}\tag{4.17}$$

(here $\mathbf{M}$ is the skew matrix uniquely associated with the vector m);

(ii) $|m|=1$ a.e. in $\mathcal{B}\times(0,T]$;

(iii) $m(\cdot,t)\to m_0(\cdot)$ in $L^2(\mathcal{B};\mathbb{R}^3)$ as $t\to 0$.

PROOF. – For each $\varepsilon,\tau>0$ we have a unique global-strong solution $m^{\varepsilon,\tau}$ of Problem $\mathcal{P}^{\varepsilon,\tau}$, satisfying the estimates (4.5)-(4.8). Then, for $\varepsilon_k,\tau_k\to 0$ as $k\to\infty$, there is a subsequence

$m^{\varepsilon_k, \tau_k}$, denoted for short by m^k, such that

$$m^k \to m \quad \text{in } L^2(\mathcal{B} \times [0, T]; \mathbb{R}^3),$$

$$m^k \rightharpoonup m \quad \text{in } H^1(\mathcal{B} \times [0, T]; \mathbb{R}^3),$$

and

$$\lim_{k \to \infty} \int_{\mathcal{B}} (|m^k|^2 - 1)^2 dx = 0 \quad (\text{or rather, } |m| = 1 \text{ a.e.}).$$

The typical element of this subsequence satisfies the evolution equation

$$\alpha \dot{m}^k - m^k \times \dot{m}^k - \tau_k \Delta \dot{m}^k = \Delta m^k - \varepsilon_k^{-1}(|m^k|^2 - 1)\, m^k \tag{4.18}$$

(*cf.* (1.5)). Let $\mathbf{M}^k$ be the skew matrix uniquely associated to the vector m^k by relation (A.1) in Appendix A, and let $\mathbf{z} \in C^\infty(\mathcal{B} \times [0, T]; \mathbb{R}^3)$ be a test function. In order to show that the limit element m of the subsequence satisfies equation (1.1), we take the scalar product of (4.18) by $\mathbf{M}^k\mathbf{z}$ and obtain

$$-\alpha \mathbf{M}^k \dot{m}^k \cdot \mathbf{z} - |m^k|^2 \dot{m}^k \cdot \mathbf{z} + (m^k \cdot \dot{m}^k)\, m^k \cdot \mathbf{z} + \tau_k \mathbf{M}^k \Delta \dot{m}^k \cdot \mathbf{z} = -\mathbf{M}^k \Delta m^k \cdot \mathbf{z}\,. \tag{4.19}$$

Note now the following differential identities:

$$\mathbf{M}^k \Delta \dot{m}^k \cdot \mathbf{z} = \operatorname{div}((\mathbf{M}^k \nabla \dot{m}^k)^T \mathbf{z}) - \mathbf{M}^k \nabla \dot{m}^k \cdot \nabla \mathbf{z} + \frac{1}{2} \mathbf{Z} \nabla \dot{m}^k \cdot \nabla m\,,$$

$$\mathbf{M}^k \Delta m^k \cdot \mathbf{z} = \operatorname{div}((\mathbf{M}^k \nabla m^k)^T \mathbf{z}) - \mathbf{M}^k \nabla m^k \cdot \nabla \mathbf{z}\,,$$

where $\mathbf{Z}$ is the skew matrix associated to the vector $\mathbf{z}$. With these identities, since the boundary conditions $(4.3)_2$ implies that both $\partial_{\mathbf{n}} m^k$ and $\partial_{\mathbf{n}} \dot{m}^k$ vanish over the boundary, integration of (4.19) yields, after some rearrangement of terms,

$$-\int_0^T \int_{\mathcal{B}} |m^k|^2 \dot{m}^k \cdot \mathbf{z}\, dx\, dt = \alpha \int_0^T \int_{\mathcal{B}} \mathbf{M}^k \dot{m}^k \cdot \mathbf{z}\, dx\, dt + \int_0^T \int_{\mathcal{B}} \mathbf{M}^k \nabla m^k \cdot \nabla \mathbf{z}\, dx\, dt$$

$$-\int_0^T \int_{\mathcal{B}} (\dot{m}^k \cdot m^k)\, m^k \cdot \mathbf{z} + \tau_k \int_0^T \int_{\mathcal{B}} \left(\mathbf{M}^k \nabla \dot{m}^k \cdot \nabla \mathbf{z} - \frac{1}{2} \mathbf{Z} \nabla \dot{m}^k \cdot \nabla m \right) dx\, dt\,.$$

With the use of the estimates (4.5)-(4.8), we see that, in the limit as $k \to \infty$,

$$|m^k|^2 \dot{m}^k \to \dot{m}, \quad \mathbf{M}^k \dot{m}^k \to \mathbf{M} \dot{m}, \quad \mathbf{M}^k \nabla m^k \to \mathbf{M} \nabla m \quad \text{weakly in } L^2(\mathcal{B} \times [0, T]).$$

This takes care of the first three integrals in the last relation; since the remaining terms all vanish as $k \to \infty$, we obtain the weak form (4.17) of (1.1). ∎

REMARK. – As mentioned in Subsection 3.2, Vilucchi [28] has studied the equation

$$\gamma^{-1}\dot{m} = m \times (\kappa \Delta m - \mu_k \dot{m} + \mathfrak{h}(m)) \quad \text{in } B \times (0, T],$$

that is, the generalization of equation (3.27) that obtains when the magnetic field $\mathfrak{h}(m)$ is included. Vilucchi's existence proof has the same structure as ours. The starting point is to consider the following approximate problem:

$$\alpha \dot{m}^{\varepsilon,\tau,\delta} - m^{\varepsilon,\tau\delta} \times \dot{m}^{\varepsilon,\tau,\delta} - \tau \Delta \dot{m}^{\varepsilon,\tau,\delta} =$$

$$\Delta m^{\varepsilon,\tau,\delta} - \varepsilon^{-1}(|m^{\varepsilon,\tau,\delta}|^2 - 1)\, m^{\varepsilon,\tau,\delta} + \alpha_1 \mathbf{h}^{\varepsilon,\tau,\delta} \quad \text{in } B \times (0, T],$$

$$m^{\varepsilon,\tau,\delta}(x, 0) = m_0^{\varepsilon}(x) \quad \text{in } \overline{\mathcal{B}}, \qquad \partial_{\mathbf{n}} m^{\varepsilon,\tau,\delta} = \mathbf{0} \quad \text{in } \partial\mathcal{B} \times (0, T],$$

with the material moduli α, α_1 positive and the approximation parameters $\varepsilon, \tau > 0$, $0 < \delta \leq 1$. Here $\mathbf{h}^{\varepsilon,\tau,\delta} = \mathfrak{h}(\widetilde{m}^{\varepsilon,\tau,\delta})$ is the solution of the approximate Maxwell equations:

$$\operatorname{Curl} \mathbf{h}^{\varepsilon,\tau,\delta} = \mathbf{0}, \qquad \operatorname{Div} \mathbf{h}^{\varepsilon,\tau,\delta} = -\operatorname{Div}(g_\delta \widetilde{m}^{\varepsilon,\tau,\delta}) \quad \text{in } \mathbb{R}^3,$$

where the function $g_\delta \in C_0^\infty(\mathbb{R}^3)$ mollifies the characteristic function of the set $\mathcal{B}_\delta = \{x + y,\ x \in \mathcal{B},\ |y| < \delta\}$:

$$g_\delta(x) = 1 \quad \text{for } x \in \overline{\mathcal{B}}, \qquad 0 \leq g_\delta(x) < 1 \quad \text{for } x \in \mathcal{B}_\delta / \overline{\mathcal{B}}, \qquad g_\delta(x) = 0 \quad \text{for } x \in \mathbb{R}^3 / \overline{\mathcal{B}}_\delta,$$

and where $\widetilde{m}^{\varepsilon,\tau,\delta}$ is the $C^{1,\lambda}$ extension of $m^{\varepsilon,\tau,\delta}$ to $\overline{\mathcal{B}}_\delta$. The global existence of a classical solution for this approximate problem is established along lines completely analogous to the ones we have used just above. The limit process as ε, τ, and $\delta \to 0$ makes use also of some convergence results to be found in [3] and [22].

Appendix A.

Let $\mathbf{M}$ denote the skew-symmetric matrix uniquely associated to the unit vector m by the relation

$$\mathbf{M}\mathbf{v} = m \times \mathbf{v}, \qquad \forall \mathbf{v} \in \mathcal{V}. \tag{A.1}$$

As is well known, for $\mathbf{1}$ the unit matrix,

$$-\mathbf{M}^2 = \mathbf{1} - m \otimes m, \tag{A.2}$$

and hence $-\mathbf{M}^2$ can be viewed as the orthogonal projector onto the plane perpendicular to m; moreover,

$$\mathbf{M}^2 \dot{m} = -\dot{m}, \qquad -\mathbf{M}^3 = \mathbf{M}. \tag{A.3}$$

With this notation, equation

$$\dot{m} = -\alpha m \times \dot{m} + m \times \Delta m, \tag*{(1.1)$_1$}$$

the Gilbert form of the Landau-Lifshitz equation (A.8), can be written as

$$\dot{m} = -\alpha \mathbf{M} \dot{m} + \mathbf{M} \Delta m . \tag{A.4}$$

Since the matrix $(\mathbf{1} + \alpha \mathbf{M})$ is invertible, ([18]) with inverse

$$(\mathbf{1} + \alpha \mathbf{M})^{-1} = \frac{1}{1+\alpha^2}(\mathbf{1} - \alpha \mathbf{M} + \alpha^2 m \otimes m), \tag{A.5}$$

another form of (A.4) is

$$\dot{m} = \mathbf{A}(\alpha, m) \Delta m , \tag{A.6}$$

with

$$\mathbf{A}(\alpha, m) = (\mathbf{1} + \alpha \mathbf{M})^{-1} \mathbf{M} = \frac{1}{1+\alpha^2}(\mathbf{M} - \alpha \mathbf{M}^2). \text{([19])} \tag{A.7}$$

For (A.6) to take the form of the Landau-Lifshitz equation

$$\dot{m} = m \times \Delta m - \alpha m \times (m \times \Delta m), \tag{1.3}$$

it is enough to rescale the time variable by the factor $(1 + \alpha^2)$, and use of the last of (A.7) and (A.2).

A third equivalent form of equation (A.4) is

$$(1 + \alpha^2)\, \dot{m} = \mathbf{M} \Delta m + \alpha(\Delta m + |\nabla m|^2 m), \tag{A.8}$$

(this follows from (A.6), (A.2), and the following identity over the sphere of unit vectors:

$$\mathbf{v} \cdot \Delta \mathbf{v} = -|\nabla \mathbf{v}|^2, \tag{A.9}$$

where $|\nabla \mathbf{v}|^2 = tr[\nabla \mathbf{v} (\nabla \mathbf{v})^T]$). *Modulo* the same time rescaling as above, equation (A.8) takes

([18]) One finds that

$$\det(\mathbf{1} + \alpha \mathbf{M}) = 1 + \alpha^2 .$$

([19]) The nonnull orthogonal invariants of $\mathbf{A}$ are

$$\iota_1 = tr\,\mathbf{A} = \frac{2\alpha}{1+\alpha^2}, \qquad \iota_2 = \frac{1}{2}[(tr\,\mathbf{A})^2 - tr\,\mathbf{A}^2] = \frac{\alpha^2}{1+\alpha^2};$$

the nonnull proper values of $\mathbf{A}$ are the complex conjugate numbers

$$\frac{\alpha}{1+\alpha^2}(\alpha \pm i).$$

the form given in the Introduction, namely,

$$\dot{m} = m \times \Delta m + \alpha(\Delta m + |\nabla m|^2 m). \tag{1.4}$$

An interesting consequence of (A.4) and (A.8) is that

$$\Delta m = -|\nabla m|^2 m + \alpha \dot{m} - \mathbf{M}\dot{m}; \tag{A.10}$$

this relation shows how the vector Δm decomposes in the orthogonal basis consisting of m, $\dot{m}$, and $\mathbf{M}\dot{m}$. With the use of (A.3), equation (A.10) can given the form (A.4). In addition, (A.10) implies that

$$\alpha|\dot{m}|^2 = \dot{m}\cdot\Delta m = div((\nabla m)^T\dot{m}) - \nabla m\cdot\nabla\dot{m} = div((\nabla m)^T\dot{m}) - \frac{1}{2}\frac{d}{dt}|\nabla m|^2. \tag{A.11}$$

Appendix B.

LEMMA. – Let V be a finite-dimensional vector space, let $f_o \in V$, and let f be a mapping of class C^1 from an open neighborhood $\mathcal{O}$ of the origin in V into V, such that

$$v\cdot(f(v) - f_o) \geqslant 0, \qquad \forall v \in \mathcal{O}. \tag{B.1}$$

Then, $f(v)$ has the representation

$$f(v) = f_o + F(v)\,v, \tag*{(B.2)$_1$}$$

with

$$f(0) = f_o, \qquad F(v) = \int_0^1 Df(\varepsilon v)\,d\varepsilon, \qquad v\cdot F(v)\,v \geqslant 0 \qquad \forall v \in \mathcal{O} \tag*{(B.2)$_2$}$$

(here D denotes differentiation).

PROOF (*Cf.* [19], [18, Theorems 6A and 16B], and [2, Section XII.14]). – Choose $\varepsilon > 0$, and write (B.1) for $w = \varepsilon v$. Then,

$$v\cdot(f(\varepsilon v) - f_o) \geqslant 0 \qquad \forall v \in \mathcal{O},$$

and hence, passing to the limit for $\varepsilon \to 0$,

$$f(0) - f_o = 0,$$

which is the first of relations $(B.\,2)_2$. With this, the remain of $(B.\,2)_2$ follows by calling upon the fundamental theorem of integral calculus. ■

Acknowledgments. The work of Bertsch and Podio-Guidugli was partly supported by TMR Contract FMRX-CT98-0229 «Phase Transitions in Crystalline Solids». The work of Podio-Guidugli was also supported by M.U.R.S.T., Progetto Nazionale «Termomeccanica dei Continui Classici e dei Materiali Nuovi» and Progetto Cofin 98 «Modelli Matematici per la Scienza dei Materiali».

We gratefully acknowledge valuable discussions with A. De Simone and M. E. Gurtin. After the completion of this work, Stefan Müller informed us that, in the 2D-framework, the τ-regularization has also been used for wave maps [MS,Z].

REFERENCES

[1] F. ALOUGES - A. SOYEUR, *On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness*, Nonlinear Anal. Theory, Meth. Appl., **18** (1992), pp. 1071-1084.

[2] S. S. ANTMAN, *Nonlinear Problems of Elasticity*, Springer-Verlag (1995).

[3] G. ANZELLOTTI - S. BALDO - A. VISINTIN, *Asymptotic behavior of the Landau-Lifshitz model of ferromagnetism*, Appl. Math. Optim., **23** (1991), pp. 171-192.

[4] M. BERTSCH - R. DAL PASSO - R. VAN DER HOUT, *Nonuniqueness for the heat flow of harmonic maps on the disk*, preprint (1999).

[5] F. BETHUEL - J. M. CORON - J. M. GHIDAGLIA - A. SOYEUR, *Heat flows and relaxed energies for harmonic maps*, in: Nonlinear Diffusion Equations and their Equilibrium States (N. G. Lloyd, W. N. Ni, L. A. Peletier, J. Serrin, Ed.), pp. 99-109, Gregynog, Birkhaüser (1990).

[6] H. BREZIS - J. M. CORON - E. H. LIEB, *Harmonic maps with defects*, Commun. Math. Phys, **107** (1986), pp. 649-705.

[7] W. F. BROWN, *Magnetoelastic Interactions*, Springer-Verlag (1966).

[8] W. F. BROWN, *Micromagnetics*, Krieger, 1978.

[9] K. C. CHANG - W. Y. DING - R. YE, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geometry, **36** (1992), pp. 507-515.

[10] Y. CHEN - M. STRUWE, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z., **201** (1989), pp. 83-103.

[11] J. M. CORON, *Nonuniqueness for the heat flow of harmonic maps*, Ann. Inst. Henri Poincaré, **7** (1990), pp. 335-344.

[12] R. DAL PASSO - E. VILUCCHI, in preparation (1999).

[13] A. DE SIMONE - P. PODIO GUIDUGLI, *Inertial and self interactions in structured continua: liquid crystals and magnetostrictive solids*, MECCANICA, **30** (1995), pp. 629-640.

[14] A. DE SIMONE - P. PODIO GUIDUGLI, *On the continuum theory of deformable ferromagnetic solids*, Arch. Rational Mech. Anal, **136** (1996), pp. 201-233.

[15] A. DE SIMONE - P. PODIO GUIDUGLI, *Pointwise balances and the construction of stress fields in dielectrics*, Math. Models Methods Appl. Sci., **7** (1997), pp. 447- 485.

[16] A. FREIRE, *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment. Math. Helv., **70** (1995), pp. 310-338.

[17] B. GUO - M. C. HONG, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Cal. Var., **1** (1993), pp. 311-334.

[18] M. E. GURTIN, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Clarendon Press, 1993.

[19] M. E. GURTIN - P. W. VOORHEES, *The continuum mechanics of two-phase elastic solids with mass transport*, Proc. Roy. Soc. London A, **440** (1993), pp. 323-343.

[20] R. M. HARDT, *Singularities of harmonic maps*, Bull. Amer. Math. Soc., **34** (1997), pp. 15-34.

[21] M. C. HONG, *Some new examples for nonuniqueness of the evolution problem of harmonic maps*, Comm. Anal. Geom., **6** (1998), pp. 779-787.

[22] R. D. JAMES - D. KINDERLEHRER, *Frustration in ferromagnetic materials*, Cont. Mech. Therm., **2** (1990), pp. 215-239.

[23] M. LAKSHMANAN - K. NAKAMURA, *Landau-Lifshitz equation of ferromagnetism: exact treatment of the Gilbert Damping*, Phys. Rev. Letters, **53** (1984), pp. 2497-2499.

[24] L. LANDAU - E. LISHITZ, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies*, Phys. Z. Sowjet., **8** (1935), pp. 153, Reprinted on; pp. 101-114 of Collected Papers of L. D. Landau (D. ter Haar, Ed.), Pergamon Press (1965).

[25] P. PODIO GUIDUGLI, *Inertia and invariance*, Ann. Mat. Pura Appl., **172** (1997), pp. 103-124.

[26] M. STRUWE, *Geometric evolution problems* of Nonlinear Partial Differential Equations in Differential Geometry, IAS Park City Math. Ser., Vol. **2**, Am. Math. Soc. (1996), pp. 257-339.

[27] R. VAN DER HOUT, *Flow alignment in nematic liquid crystals in flows with cylindrical symmetry* (1997), to appear.

[28] E. VILUCCHI, *An existence theorem for soft ferromagnets subject to magnetic actions*, IAC/C.N.R., Quad. 16/1997.

[29] A. VISINTIN, *On Landau-Lifshitz' equations for ferromagnetism*, Japan J. Appl. Math., **2** (1985), pp. 69-84.

[30] A. VISINTIN, *Modified Landau-Lifshitz equation for ferromagnetism*, Physica B, **233** (1997), pp. 365-369.

[MS] S. MÜLLER - M. STRUWE, *Global existence of wave maps in* 1+2 *dimensions with finite energy data*, Topol. Meth. Nonlinear Analysis, **7** (1996), pp. 245-259.

[Z] Y. ZHOU, *Global weak solutions for* 1+2 *dimensional wave maps into homogeneous spaces*, preprint, 1995.