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Infinitely Many Radial Solutions to a Boundary Value Problem in a Ball (*)(**)

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Dedicated to Professor Tongren Ding on the occasion of his 70th birthday

Abstract. – *In this paper we are concerned with the existence and multiplicity of radial solutions to the BVP*

$$\begin{cases} \nabla\cdot(a(|\nabla u|)\,\nabla u)+f(|x|,u)=0 & \text{in } \mathcal{B} \\ u=0 & \text{on } \partial\mathcal{B}, \end{cases}$$

where $\mathcal{B}$ is an open ball in $\mathbb{R}^K$ and $u\mapsto\nabla\cdot(a(|\nabla u|)\nabla u)$ is a nonlinear differential operator (e.g. the p-laplacian or the mean curvature operator). The function f is defined in a neighborhood of $u=0$ and satisfies a «sublinear»-type growth condition for $u\to 0$. We use a degree approach combined with a time-map technique. Multiplicity results are obtained also for nonlinearities of concave-convex type.

1. – Introduction.

In this paper we study the existence and multiplicity of radial solutions, with prescribed nodal properties, to the boundary value problem

$$\begin{cases} \nabla\cdot(a(|\nabla u|)\,\nabla u)+f(|x|,u)=0 & \text{in } \mathcal{B} \\ u=0 & \text{on } \partial\mathcal{B}, \end{cases} \tag{1.1}$$

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where $\mathcal{B} = \{x \in \mathbb{R}^K, |x| < R\}$, $K > 1$ and $a: [0, \varepsilon_1] \to [0, +\infty)$, $(\varepsilon_1 > 0)$, and $f: [0, R] \times \times [-\varepsilon_2, \varepsilon_2] \to \mathbb{R}$, $(\varepsilon_2 > 0)$, are continuous functions.

The existence of radial solutions satisfying various boundary conditions has been investigated by many authors, starting from the classical situation where $a \equiv 1$: we quote, among others, the works of M. J. Esteban [16], E. W. C. Van Groesen [35], A. Castro-A. Kurepa [9], C. K. R. T. Jones [26], M. Grillakis [23], Z. Guo [24], A. El Hachimi-F. De Thelin [15], Y. Cheng [10], A. Ambrosetti-J. Garcia Azorero-I. Peral [3], F. I. Njoku-P. Omari-F. Zanolin [30].

In what follows, for $\phi(s) := sa(|s|)$ and $F(r, s) := \int_0^s f(r, u)\, du$, we shall assume:

(H_ϕ) $\phi: [-\varepsilon_1, \varepsilon_1] \to [-\phi(\varepsilon_1), \phi(\varepsilon_1)]$ is an odd increasing homeomorphism such that

$$\liminf_{s \to 0} \frac{\phi(\sigma s)}{\phi(s)} > 1, \qquad \forall \sigma > 1 \tag{1.2}$$

and

$$\limsup_{s \to 0} \frac{\phi(\sigma s)}{\phi(s)} < +\infty, \qquad \forall \sigma > 1; \tag{1.3}$$

these assumptions are clearly satisfied by the p-laplacian operator.

Moreover, we assume

(H_f) The function f is such that $f(r, 0) \equiv 0$ and

$$\lim_{s \to 0} \frac{f(r, s)}{\phi(s)} = +\infty, \qquad \text{uniformly in } r \in [0, R]; \tag{1.4}$$

(H_F) $F(r, s)$ is differentiable with respect to $r \in [0, R]$ and there exists a continuous function $\alpha: [0, R] \to \mathbb{R}^+$ such that

$$\left| \frac{\partial F}{\partial r}(r, s) \right| \leq \alpha(r) F(r, s), \qquad \forall r \in [0, R],\ \forall s \in [-\varepsilon_2, \varepsilon_2]. \tag{1.5}$$

The so-called «lower/upper» σ-conditions stated in (1.2)-(1.3) are typical for such kind of operators (see [18], [19], [20], [21]); they can also be found in [22], where it is assumed that the limits in (1.2)-(1.3) exist. Apart from the (classical) case of the p-laplacian, where $a(|s|) = |s|^{p-2}$, $p > 1$, these conditions are satisfied e.g. by the «mean curvature» operator, where $a(|s|) = (1 + s^2)^{-1/2}$, and by even more general ones like $a(|s|) = (1 + s^2)^{-a/2} s^{m-2}$, $a \geq 0$, $m > 1$, $m \geq a + 1$ (see [17]). We point out that a special feature of our approach is that we do not need, as it is frequently found in the literature, any homogeneity assumption on the nonlinear differential operator. In particular, we can achieve our results without passing through the study of associated eigenvalue problems.

An important consequence of assumption (H_ϕ) is stated in Proposition 2.2 and is applied in the proof of Proposition 3.7.

When dealing with radial solutions to (1.1) on a ball, one is led to study (setting $|x| = r$) the BVP

$$\begin{cases} (r^{(K-1)}\phi(u'))' + r^{(K-1)} f(r, u) = 0 \\ u'(0) = 0 = u(R) \end{cases} \tag{1.6}$$

and a singularity appears for $r = 0$. Beside this (intrinsic) aspect of (1.1), assumption (H_f) represents a singularity in the variable u which, for $a \equiv 1$, means that f has a sublinear growth at $u = 0$. Hence, in particular, when one tries to develop some shooting argument, global existence and uniqueness to initial value problems associated to the equation in (1.6) are not guaranteed. This is one of the reasons why a (relatively) small number of results for «sublinear» problems is available in the literature: we refer to the earlier works of G. J. Butler [6], H. Jacobowitz [25] (for the periodic case), M. A. Krasnosel'skii-A. I. Perov-A. I. Povolotskii-P. P. Zabreiko [27, Section 15], B. L. Shekhter [34, Section 15] for a more classical approach in the ODE's case. For other results, in the PDE's setting, we also refer to V. Moroz [29], P. Omari-F. Zanolin [31], E. W. C. Van Groesen [35], M. Willem [36], mainly for the case $a \equiv 1$.

We stress the fact that the «direct» methods of shooting type used by some of the authors quoted above require stronger regularity assumptions than (H_f)-(H_F), which turn out to be sufficient to treat (1.1) by means of an abstract continuation theorem. Assumption (H_F), indeed, is a well-known condition for the uniqueness of solutions to some Cauchy problems related to (1.6) (cf. the papers of Y. Cheng [10, p. 289], W. Reichel-W. Walter [33, Th. 4-(δ-ii)]). For more comments on (H_F) we refer to Remark 3.1 in Section 3.

Our main result (Th. 3.2) guarantees that under $(\dot{H}_\phi)$-(H_f)-(H_F) there exists a positive integer n_0 such that for all $n > n_0$ there are u_n and v_n, radial solutions to (1.1), with $u_n(0) > 0$ and $v_n(0) < 0$, both having exactly n zeros; moreover, we prove that $\lim_{n \to +\infty} |u_n(r)| + |u_n'(r)| = 0 = \lim_{n \to +\infty} |v_n(r)| + |v_n'(r)|$, uniformly in $r \in [0, R]$. We point out that no assumption on the behaviour of f at infinity is required; f can even be defined only in a neighborhood of $u = 0$.

The proof is developed through a (new) variant of the continuation theorem in [7, 8] (Th. 2.1), suitable for the application to nonlinearities satisfying only the local conditions (H_f)-(H_F). The needed estimates (2.2)-(2.3) in Theorem 2.1 are obtained via a time-map technique; some of the arguments are similar to those in [7] and to the ones developed in [9, 21] for the situation when f «grows faster than ϕ» at *infinity*. More precisely, Proposition 3.6 and Proposition 3.7 are upper and lower estimates (reminiscent of Sturmian theory) on the number of zeros of the solutions of a parameter-dependent problem associated to (1.6) (cf. the paper of E. Yanagida [37]). One might say that, on the lines of Remark 5.3, we show that the presence of the strongly nonlinear operator ϕ does not affect the properties of the number of zeros of solutions to *nonautonomous* problems when this number is considered as a function of the initial data.

Our continuation theorem needs also the degree condition (2.4), which is shown to be satisfied through the study of a suitable autonomous problem of the form

$$\begin{cases} (\phi(u'))' + g(u) = 0 \\ u'(0) = 0 = u(R), \end{cases} \tag{1.7}$$

where

$$\lim_{s\to 0} \frac{g(s)}{\phi(s)} = +\infty .$$

More precisely, the multiplicity result developed in [7] for the two-point BVP can be adapted with no difficulty to the boundary condition in (1.7); it is based on the notion of «generalized Fučik spectrum» introduced in [8] (cf. Proposition 5.1) and on the asymptotic behaviour near zero of the «time-maps» (cf. Proposition 5.2).

A combination of our main theorem with the results due to M. García-Huidobro-R. Manásevich-F. Zanolin in [21] enables us to consider a differential operator defined on the whole real line and satisfying conditions of the form (1.2)-(1.3) at infinity too and functions satisfying at the same time (H_f) and

$$\lim_{|s|\to +\infty} \frac{f(r,s)}{\phi(s)} = +\infty, \qquad \text{uniformly in } r\in[0,R]. \tag{1.8}$$

In the above situation, we can prove (Th. 4.1) (under suitable conditions of subcritical growth at infinity for f) the existence of four sequences of radial solutions u_n, v_n, w_n and z_n with $u_n(0)>0$, $v_n(0)<0$, $w_n(0)>0$ and $z_n(0)<0$, all having exactly n zeros in $[0,R)$. Moreover, we have

$$\lim_{n\to+\infty} |u_n(r)|+|u_n'(r)|=0=\lim_{n\to+\infty} |v_n(r)|+|v_n'(r)|, \quad \text{uniformly in } r\in[0,R] \tag{1.9}$$

and

$$\lim_{n\to+\infty} |w_n(r)|+|w_n'(r)|=+\infty=\lim_{n\to+\infty} |z_n(r)|+|z_n'(r)|, \text{ uniformly in } r\in[0,R]. \tag{1.10}$$

Combinations of conditions at zero and at infinity have been considered, among others, by H. Dang-R. Manásevich-K. Schmitt [13], H. Dang-K. Schmitt-R. Shivaji [14] when dealing with the existence of positive solutions.

In particular, as a consequence of Theorem 4.1, we can prove that the Dirichlet problem on the ball associated to

$$\Delta_p u + a_1 |u|^{\mu-2}u + a_2 |u|^{\nu-2}u = 0 \tag{1.11}$$

has, for each n sufficiently large, at least four radially symmetric solutions with n zeros in $[0,R)$ *for any* a_1, a_2 positive constants (cf. Remark 4.2), whenever

$$1<\mu<p<\nu<p^*.$$

If $p=2$, these are nonlinearities of «concave-convex» type; starting from the seminal paper by A. Ambrosetti-H. Brézis-G. Cerami [1], they have been widely studied (see also A. Ambrosetti-J. Garcia Azorero-I. Peral [2] for a general $p > 1$ and T. Bartsch-M. Willem [4]). In particular, as far as radial solutions which are positive in zero are concerned, for fixed a_1 we can obtain a diagram similar to Figure 4 in [1]. We also notice that, for the particular case of equation (1.11), the existence of infinitely many pairs of radial solutions with prescribed nodal properties follows also from the bifurcation result in [3, Th. 3.1].

The plan of the paper is the following.

In Section 2 we give an abstract continuation theorem and some preliminary results on uniqueness and continuous dependence of the solutions to Cauchy problems associated to the equation in (1.6).

In Section 3 we give our main result (Th. 3.2) under the (local) assumptions (H_ϕ)-(H_f)-(H_F).

In Section 4 we treat the case of «superlinear» nonlinearities, and give a multiplicity result under (H_f) and (1.8).

Section 5 (Appendix) is devoted to the study of the autonomous problem (1.7). We point out that ϕ need not be surjective on $\mathbb{R}$.

In the sequel, we will use the following notation: $\mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$, $C^1_\#([0, R]) = \{u \in C^1([0, R]): u'(0) = 0 = u(R)\}$ and $\|u\|_1 = \max_{r \in [0, R]} \sqrt{u(r)^2 + u'(r)^2}$ for every $u \in C^1([0, R])$. Moreover, $\deg_B$ and $\deg$ will denote the Brouwer and the Leray-Schauder degree, respectively.

2. – Preliminary results.

In this section we first study an abstract equation of the form

$$u = \mathcal{N}(u, \lambda) \tag{2.1}$$

where X is a Banach space and $\mathcal{N}$: $\operatorname{dom} \mathcal{N} \subset X \times [0, 1] \to X$ is a completely continuous operator. Moreover, we shall consider two open sets A and B such that $A \subset \overline{A} \subset B \subset \overline{B}$ and $(\overline{B} \setminus A) \subset \operatorname{dom} \mathcal{N}$.

Let Σ be the set of the solutions of (2.1), i.e.

$$\Sigma = \{(u, \lambda): u = \mathcal{N}(u, \lambda)\}$$

and, for any subset $D \subset X \times [0, 1]$, let us denote the section of D at $\lambda \in [0, 1]$ by $D_\lambda = \{x \in X: (x, \lambda) \in D\}$; we also set $\mathcal{N}_\lambda = \mathcal{N}(\cdot, \lambda)$. We have the following abstract theorem:

THEOREM 2.1. – *Let* $\boldsymbol{k}$: $\Sigma \cap (\overline{B} \setminus A) \to \mathbb{N}$ *be a continuous function; suppose that there exists a positive integer* n *satisfying the following conditions*:

$$n \notin \boldsymbol{k}(\partial((\overline{B} \setminus A) \cap \Sigma)) \tag{2.2}$$

and

$$\boldsymbol{k}^{-1}(n) \quad \textit{is bounded}\,. \tag{2.3}$$

Then, for an open bounded set U_0^n *such that* $(k^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset (\overline{B}\setminus A)_0$ *and* $\Sigma_0 \cap U_0^n = (k^{-1}(n))_0$, *the Leray-Schauder degree* $\deg(I - \mathcal{N}_0, U_0^n)$ *is defined. If*

$$\deg(I - \mathcal{N}_0, U_0^n) \neq 0\,, \tag{2.4}$$

then there is a continuum $C_n \subset \Sigma$ *with*

$$\{\lambda \in [0, 1]: \exists u \in X: (u, \lambda) \in C_n\} = [0, 1]$$

and such that

$$(u, \lambda) \in C_n \Rightarrow (u, \lambda) \in (B\setminus \overline{A}) \quad \text{and} \quad k(u, \lambda) = n\,.$$

In particular there is at least one solution $\tilde{u} \in (B\setminus\overline{A})_1$ *of the operator equation*

$$u = \mathcal{N}(u, 1)$$

with

$$k(\tilde{u}, 1) = n\,.$$

Theorem 2.1 is a variant of the continuation theorem in [7] (see also [21]). A version for a coincidence equation $Lu = M(u, \lambda)$, where L is a linear Fredholm operator of index zero and M is L-completely continuous [28], is valid as well.

Now, we introduce some notation and properties which will be useful in the sequel. More precisely, assume

(H_ϕ) For some $\varepsilon_1 > 0$, $\phi: [-\varepsilon_1, \varepsilon_1] \to [-\phi(\varepsilon_1), \phi(\varepsilon_1)]$ is an odd increasing homeomorphism such that

$$\liminf_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} > 1\,, \qquad \forall \sigma > 1 \tag{2.5}$$

and

$$\limsup_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} < +\infty\,, \qquad \forall \sigma > 1\,. \tag{2.6}$$

Then, let

$$\Phi(x) = \int_0^x \phi(s)\, ds\,, \qquad \forall |x| \leq \varepsilon_1, \tag{2.7}$$

$$\Phi_*(x) = \int_0^x \phi^{-1}(s)\, ds\,, \qquad \forall |x| \leq \varepsilon_1, \tag{2.8}$$

and

$$\mathcal{L}(x) = x\phi(x) - \Phi(x) = \Phi_*(\phi(x))\,, \qquad \forall |x| \leq \varepsilon_1\,. \tag{2.9}$$

As far as (2.9) is concerned, we observe that the functions Φ, Φ_* and $\mathcal{L}$ are even and, when restricted to $\mathbb{R}^+$, strictly increasing. Hence, the right inverse of $\mathcal{L}$ is defined and (without ambiguity) it will be denoted in what follows by $\mathcal{L}^{-1}$; the same meaning will be given to Φ_*^{-1}.

Now, we state some consequences of the lower/upper σ-conditions given in (H_ϕ); these are variants of similar results obtained in [20, 21] where, in the upper/lower σ-conditions (3.2)-(3.3), the limits are taken for $s \to \infty$ instead of $s \to 0$.

PROPOSITION 2.2. – (i) *There exist* $C > 1$ *and* $\bar{\varepsilon} > 0$, $\bar{\varepsilon} \leq \varepsilon_1$, *such that*

$$\xi\phi(\xi) \leq C\mathcal{L}(\xi), \qquad \forall |\xi| \leq \bar{\varepsilon}. \tag{2.10}$$

(ii) $\forall d_1 > 1\ \exists d_2 > 1 : \forall d > d_2\ \exists \varepsilon_d > 0$, $\varepsilon_d \leq \varepsilon_1$, *such that*

$$\phi^{-1}(d\xi) \geq d_1 \phi^{-1}(\xi), \qquad \forall |\xi| \leq \varepsilon_d. \tag{2.11}$$

(iii) $\forall c_1 > 1\ \exists c_2 > 1 : \forall c > c_2\ \exists \varepsilon_c > 0$, $\varepsilon_c \leq \varepsilon_1$, *such that*

$$\Phi_*^{-1}(c\xi) \geq c_1 \Phi_*^{-1}(\xi), \qquad \forall |\xi| \leq \varepsilon_c. \tag{2.12}$$

Now, for some $\varepsilon_2 > 0$ and for every $\lambda \in [0, 1]$, let $f_\lambda : [0, R] \times [-\varepsilon_2, \varepsilon_2] \to \mathbb{R}$ be a continuous function; we denote $F_\lambda(r, s) = \int_0^s f_\lambda(r, u)\, du$ for every $r \in [0, R]$, for every $s \in [-\varepsilon_2, \varepsilon_2]$ and for every $\lambda \in [0, 1]$. We assume:

(H_{F_λ}) $F_\lambda(r, s)$ is differentiable with respect to $r \in [0, R]$ and there exists a continuous function $\alpha : [0, R] \to \mathbb{R}^+$ such that

$$\left| \frac{\partial F_\lambda}{\partial r}(r, s) \right| \leq \alpha(r)\, F_\lambda(r, s), \qquad \forall r \in [0, R],\ \forall s \in [-\varepsilon_2, \varepsilon_2]. \tag{2.13}$$

We refer to the Introduction and to Remark 3.1 for comments on the verification of the above hypothesis. We only observe that, by (2.13), we deduce that

$$F_\lambda(r, s) \geq 0, \qquad \forall (r, s) \in [0, R] \times [-\varepsilon_2, \varepsilon_2],\ \forall \lambda \in [0, 1]. \tag{2.14}$$

Next, let $K \in \mathbb{N}$, $K > 1$, and let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ (so that (H_ϕ), (H_{F_λ}) and (2.14) hold with ε_0 instead of ε_1 and ε_2); consider the equation

$$(r^{(K-1)}\phi(u'))' + r^{(K-1)} f_\lambda(r, u) = 0\,, \qquad r \in (0, R)\,. \tag{2.15}$$

We prove some continuous dependence and uniqueness results for certain initial value problems associated to (2.15), under assumptions (H_ϕ) and (H_{F_λ}). They will enable us to define an operator k suitable for the validity of conditions (2.2), (2.3) and (2.4) in Theorem 2.1.

LEMMA 2.3. – *For every* $\varepsilon \leqslant \varepsilon_0$ *there exists* $d_\varepsilon \in (0, \varepsilon]$ *such that if* u *is a* (*local*) *solution of*

$$\begin{cases} (r^{\lambda(K-1)}\phi(u'))' + r^{\lambda(K-1)} f_\lambda(r, u) = 0 \\ u(0) = d, \qquad u'(0) = 0 \end{cases} \tag{2.16}$$

with $|d| \leqslant d_\varepsilon$, *then* u *can be defined on* $[0, R]$ *and* $\|u\|_1 \leqslant \varepsilon$.

REMARK 2.4. – *We observe that for every* $d \in [-d_\varepsilon, d_\varepsilon]$ *there exists at least one* (*local*) *solution of* (2.16). *For a proof of this result, see e.g.* [17,33] *and references therein.*

PROOF. – Let u be a solution of (2.16), and assume that, for some $\varepsilon > 0$,

$$|u(r)| \leqslant \varepsilon, \qquad |u'(r)| \leqslant \varepsilon, \qquad \forall r \in [0, \varrho], \tag{2.17}$$

with $0 < \varrho \leqslant R$. We introduce the functions

$$E_\lambda(r, x, y; d) = \mathcal{L}(y) + F_\lambda(r, x) \quad \forall r \in [0, \varrho], \quad \forall x, y \in [-\varepsilon, \varepsilon]^2, \quad \forall \lambda \in [0, 1] \tag{2.18}$$

and

$$v_\lambda(r) = E_\lambda(r, u(r), u'(r); d), \qquad \forall r \in [0, \varrho]. \tag{2.19}$$

From (2.14) we deduce that

$$\mathcal{L}(y) \leqslant E_\lambda(r, x, y; d), \qquad \forall r \in [0, \varrho] \ \forall x, y \in [-\varepsilon, \varepsilon]^2, \ \forall \lambda \in [0, 1]. \tag{2.20}$$

Moreover, from (2.15) we infer:

$$\frac{d}{dr} v_\lambda(r) = \Big(\phi(u'(r))' u'(r) + \frac{\partial F_\lambda}{\partial r}(r, u(r)) + f_\lambda(r, u(r))\, u'(r) =$$

$$= -\frac{\lambda(K-1)}{r}\phi(u'(r))\, u'(r) + \frac{\partial F_\lambda}{\partial r}(r, u(r))\Big).$$

From (2.13) we obtain

$$v_\lambda'(r) \leqslant \alpha(r)\, F_\lambda(r, u(r)) \leqslant \alpha(r)\, v_\lambda(r), \qquad \forall r \in [0, \varrho]$$

and, integrating on $(0, r)$, we get

$$v_\lambda(r) \leqslant v_\lambda(0)\, e^{\int_0^r \alpha(s)\, ds} \leqslant H F_\lambda(0, d), \tag{2.21}$$

where $H = e^{\int_0^R \alpha(s)\, ds}$.

Now, let us consider $(a_1, a_2) \in (0, 1)^2$ such that

$$\text{(2.22)} \qquad \begin{cases} a_1 + Ra_2 \leqslant \dfrac{1}{2} \\[2ex] a_2 \leqslant \dfrac{1}{2} \end{cases}$$

(observe that, for every $R > 0$, a similar choice of a_1 and a_2 is always possible). Then, for every $\varepsilon \leqslant \varepsilon_0$, let $d_\varepsilon > 0$ be such that

$$d_\varepsilon \leqslant a_1 \varepsilon$$

and

$$\mathcal{L}^{-1}(HF_\lambda(0, d)) \leqslant a_2 \varepsilon, \qquad \forall 0 < |d| \leqslant d_\varepsilon, \quad \forall \lambda \in [0, 1]$$

(this is possible since $\lim_{d \to 0} \mathcal{L}^{-1}(HF_\lambda(0, d)) = 0$ uniformly in $\lambda \in [0, 1]$).

Now, for $|d| \leqslant d_\varepsilon$, from (2.20) and (2.21), we deduce:

$$\mathcal{L}(u'(r)) \leqslant v_\lambda(r) \leqslant HF_\lambda(0, d)$$

and

$$\text{(2.23)} \qquad |u'(r)| \leqslant \mathcal{L}^{-1}(HF_\lambda(0, d)) \leqslant a_2 \varepsilon \leqslant \frac{1}{2}\varepsilon, \qquad \forall r \in [0, \varrho].$$

Hence, from (2.23), we have:

$$\text{(2.24)} \qquad |u(r)| \leqslant d + \int_0^r |u'(s)|\,ds \leqslant d + R\mathcal{L}^{-1}(HF_\lambda(0, d)) \leqslant$$

$$\leqslant a_1 \varepsilon + Ra_2 \varepsilon = (a_1 + Ra_2)\varepsilon \leqslant \frac{1}{2}\varepsilon, \qquad \forall r \in [0, \varrho].$$

Since (2.23) and (2.24) hold independently on ϱ, we can extend u on $[0, R]$ as a C^1-function.

Finally, (2.23) and (2.24) imply that:

$$\|u\|_1 = \max_{r \in [0, R]} \sqrt{|u(r)|^2 + |u'(r)|^2} \leqslant \frac{1}{2}\varepsilon \leqslant \varepsilon. \qquad \blacksquare$$

LEMMA 2.5. – *Let u be a solution of*

$$\text{(2.25)} \qquad \begin{cases} (r^{\lambda(K-1)}\phi(u'))' + r^{\lambda(K-1)} f_\lambda(r, u) = 0 \\ u(r_0) = 0 = u'(r_0), \qquad r_0 \in (0, R]. \end{cases}$$

Then $u \equiv 0$ in $[0, R]$.

PROOF. – Let $C > 1$ and $\bar{\varepsilon} > 0$ be as in Proposition 2.2; let u be a solution of (2.25) such that

$$|u(r)| \leq \bar{\varepsilon}, \quad |u'(r)| \leq \bar{\varepsilon}, \quad \forall r \in (0, R] \cap [r_0 - \eta, r_0 + \eta] := I_\eta, \tag{2.26}$$

for some $\eta > 0$, with $r_0 - \eta > 0$. Let us consider the function v_λ defined in (2.19). We have already proved that

$$|v_\lambda'(r)| \leq \left| \frac{\partial F_\lambda}{\partial r} \right| + \frac{\lambda(K-1)}{r} |\phi(u'(r))| \, |u'(r)|;$$

using (2.13) and (2.10), we obtain:

$$|v_\lambda'(r)| \leq \alpha(r) F_\lambda(r, u(r)) + C\frac{K-1}{r} \mathcal{L}(u'(r)) \leq \xi(r) v_\lambda(r), \qquad \forall r \in I_\eta, \tag{2.27}$$

where $\xi(r) = \max\left\{\alpha(r), C\frac{K-1}{r}\right\}$. We observe that $\xi \in L^1_{\mathrm{loc}}((0, R])$.

Now, integrating (2.27) on (r_0, r), for $r \in I_\eta$, we obtain

$$v_\lambda(r) \leq v_\lambda(r_0) + \int_{r_0}^{r} \xi(s) v_\lambda(s) \, ds$$

and, as a consequence,

$$v_\lambda(r) \leq v_\lambda(r_0) \, e^{\int_{r_0}^{r} \xi(s) \, ds}, \qquad \forall r \in I_\eta,$$

i.e., since $v_\lambda(r_0) = 0$,

$$v_\lambda(r) \leq 0, \qquad \forall r \in I_\eta.$$

We deduce that $u \equiv 0$ in I_η. ■

3. – The main result.

Let us consider the following boundary value problem:

$$\begin{cases} \nabla \cdot (a(|\nabla u|) \nabla u) + f(|x|, u) = 0 & \text{in } \mathcal{B} \\ u = 0 & \text{on } \partial \mathcal{B} \end{cases} \tag{3.1}$$

where $\mathcal{B}$ is the open ball of center 0 and radius $R > 0$ in $\mathbb{R}^K$ $(K > 1)$ and $a: [0, \varepsilon_1] \to [0, +\infty)$ for some $\varepsilon_1 > 0$; we also set $\phi(s) = sa(|s|)$, for $s \in [-\varepsilon_1, \varepsilon_1]$, and $F(r, s) = \int_0^s f(r, u) \, du$.

Let us assume the following hypotheses:

(H_ϕ) $\phi\colon[-\varepsilon_1, \varepsilon_1]\to[-\phi(\varepsilon_1), \phi(\varepsilon_1)]$ is an odd increasing homeomorphism such that

$$\liminf_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} > 1\,, \qquad \forall\sigma > 1 \tag{3.2}$$

and

$$\limsup_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} < +\infty\,, \qquad \forall\sigma > 1\,. \tag{3.3}$$

(H_f) For some $\varepsilon_2 > 0$, $f\colon[0, R]\times[-\varepsilon_2, \varepsilon_2]\to\mathbb{R}$ is a continuous function such that $f(r, 0)\equiv 0$ and

$$\lim_{s\to 0} \frac{f(r, s)}{\phi(s)} = +\infty\,, \qquad \text{uniformly in } r\in[0, R]\,. \tag{3.4}$$

(H_F) $F(r, s)$ is differentiable with respect to $r\in[0, R]$ and there exists a continuous function $\alpha\colon[0, R]\to\mathbb{R}^+$ such that

$$\left|\frac{\partial F}{\partial r}(r, s)\right| \leq \alpha(r)F(r, s)\,, \qquad \forall r\in[0, R],\ \ \forall s\in[-\varepsilon_2, \varepsilon_2]\,, \tag{3.5}$$

REMARK 3.1. – *As it was already mentioned in the Introduction, condition* (H_F) *can be found also in* [10,33]. *We observe that it is satisfied by functions F of the form* $F(r, u) = p(r)\int_0^u g(t)\,dt$, *being* $p(\cdot)$ *positive and continuously differentiable and g a continuous function satisfying the sign condition* $g(u)\,u > 0$ *for every u in a neighbourhood of the origin,* $u\neq 0$. *We point out that some regularity for p is crucial in order to avoid the difficulties which may arise (as it is shown, even for the case* $a\equiv 1$, *in* [5,11]) *when continuability of solutions to the ODE in* (1.6) *is studied.*

We set $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$; moreover, ε_0 will be now taken such that, according to (H_f), $f(r, s)\,s > 0$, for every $0 < |s| \leq \varepsilon_0$ and for every $r\in[0, R]$. Finally, let us define the function $\hat{f}\colon[-\varepsilon_0, \varepsilon_0]\to\mathbb{R}$ by:

$$\hat{f}(s) = \begin{cases} \sup\{f(r, u)\colon r\in[0, R],\ u\in[0, s]\} & \text{if } 0 < s \leq \varepsilon_0 \\ \inf\{f(r, u)\colon r\in[0, R],\ u\in[s, 0]\} & \text{if } -\varepsilon_0 \leq s < 0\,. \end{cases} \tag{3.6}$$

We are now in position to state our main result:

THEOREM 3.2. – *Assume* (H_ϕ), (H_f) *and* (H_F). *Then, there exists* $n_0\in\mathbb{N}$ *such that for every* $n > n_0$ *problem* (3.45) *has at least two radial solutions* u_n *and* v_n *with* $u_n(0) > 0$ *and*

$v_n(0) < 0$, *both having exactly n zeros in* $[0, R)$. *Moreover, we have*:

$$\lim_{n \to +\infty} |u_n(r)| + |u_n'(r)| = 0 = \lim_{n \to +\infty} |v_n(r)| + |v_n'(r)|, \quad \textit{uniformly in } r \in [0,R]. \tag{3.7}$$

We will study problem (3.1) by means of a degree approach; to this end, we first introduce a continuous nondecreasing function $g: [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ such that

$$\lim_{s \to 0} \frac{g(s)}{\phi(s)} = +\infty; \tag{H_g}$$

we shall also assume (without loss of generality) that $g(s)\, s > 0$, for every $s \in [-\varepsilon_0, \varepsilon_0]$, $s \neq 0$. We also denote $G(s) = \int_0^s g(u)\, du$; a possible choice for g is the function $\hat{f}$ given in (3.6).

Moreover, for $\lambda \in [0, 1]$, we define the functions $f_\lambda: [0, R] \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ by $f_\lambda(r, s) = \lambda f(r, s) + (1 - \lambda) g(s)$ and $F_\lambda(r, s) = \int_0^s f_\lambda(r, u)\, du$. It is straightforward to check that, by (H_f) and (H_g), we have:

$$\lim_{s \to 0} \frac{f_\lambda(r, s)}{\phi(s)} = +\infty, \qquad \text{uniformly in } r \in [0, R] \text{ and } \lambda \in [0, 1]. \tag{3.8}$$

Moreover, we observe that, by (H_F) and the choice of g, for every $\lambda \in [0, 1]$, $F_\lambda(r, s)$ is differentiable with respect to $r \in [0, R]$ and for $\alpha: [0, R] \to \mathbb{R}^+$ in (H_F) one has:

$$\left| \frac{\partial F_\lambda}{\partial r}(r, s) \right| \leq \alpha(r) F_\lambda(r, s), \qquad \forall r \in [0, R], \ \forall s \in [-\varepsilon_0, \varepsilon_0], \ \forall \lambda \in [0, 1]; \tag{3.9}$$

we also observe that (3.9) guarantees that

$$F_\lambda(r, s) > 0, \qquad \forall r \in [0, R], \qquad \forall s \in [-\varepsilon_0, \varepsilon_0], \qquad s \neq 0, \qquad \forall \lambda \in [0, 1]. \tag{3.10}$$

We then consider the parameter dependent boundary value problem:

$$\begin{cases} (r^{\lambda(K-1)} \phi(u'))' + r^{\lambda(K-1)} f_\lambda(r, u) = 0 \\ u'(0) = 0 = u(R) = 0. \end{cases} \tag{3.11}$$

We already observed that, by our hypotheses, we are in the setting of Section 2. Hence we are allowed to use Lemma 2.3 and Lemma 2.5, together with the results of this Section, in order to show that the assumptions of the abstract continuation theorem 2.1 are satisfied.

Now, we prove two results which give some estimates on the «energy» function introduced in (2.18). To this aim, let us consider the number d_{ε_0} given in Lemma 2.3 and let us set

$d_0 = d_{\varepsilon_0}$; for every $d \in \mathbb{R}$, $0 < |d| \leq d_0$, let $u(\cdot\,; d)$ be a solution of (2.16). Integrating, we obtain

$$-\phi(u') = r^{-\alpha}\int_0^r s^\alpha f_\lambda(s, u(s))\,ds > 0\,, \qquad \alpha = \lambda(K-1)\,, \tag{3.12}$$

for every r in a (sufficiently small) neighbourhood of zero; hence, being u decreasing, arguing like in [9], for every $\theta \in (0, 1)$ we can consider the first point $r_0(d; \theta)$ such that

$$u(r_0(d; \theta); d) = \theta d\,. \tag{3.13}$$

Moreover, we denote by $r_0(d)$ the first zero of $u(\cdot\,; d)$.

The following lemma gives an estimate on $r_0(d; \theta)$ as a function of d:

LEMMA 3.3. – *For every $d \in (0, d_0]$ and for every $\theta \in (0, 1)$ there exists $A > 0$ such that*

$$r_0(d; \theta) \geq A\frac{(1-\theta)}{\hat{f}(d) + g(d)}\,\Phi_*^{-1}(d(\hat{f}(d) + g(d)))\,; \tag{3.14}$$

an analogous result holds for $d \in [-d_0, 0)$.

PROOF. – For $d \in (0, d_0]$ and by the definition of $r_0(d)$, inequality (3.12) is valid for every $r \in (0, r_0(d))$; on the interval $(0, r_0(d))$ we have (being u decreasing)

$$f_\lambda(s, u(s)) \leq \lambda \hat{f}(d) + (1-\lambda)\,g(d) \leq \hat{f}(d) + g(d)\,;$$

then, we obtain

$$u'(r) \geq -\phi^{-1}\left(r^{-\alpha}(\hat{f}(d) + g(d))\int_0^r s^\alpha\,ds\right) = -\phi^{-1}\left(\frac{(\hat{f}(d) + g(d))}{1+\alpha}r\right), \quad \forall r \in (0, r_0(d))\,;$$

integrating this relation on $(0, r_0(d; \theta))$ we get

$$u(r_0(d; \theta)) - u(0) \geq -\int_0^{r_0(d;\theta)} \phi^{-1}\left(\frac{(\hat{f}(d) + g(d))}{1+\alpha}r\right)dr$$

and, using the definitions of Φ_* and $r_0(d; \theta)$,

$$(1-\theta)\,d \leq \frac{\alpha+1}{(\hat{f}(d) + g(d))}\,\Phi_*\left(\frac{(\hat{f}(d) + g(d))\,r_0(d; \theta)}{1+\alpha}\right),$$

i.e.

$$r_0(d;\,\theta) \geq \frac{\alpha+1}{(\hat{f}(d)+g(d))}\,\Phi_*^{-1}\left(\frac{(1-\theta)\,d(\hat{f}(d)+g(d))}{1+\alpha}\right);$$

now, we observe that Φ_*^{-1} is concave: we thus deduce that

$$r_0(d;\,\theta) \geq \frac{(1-\theta)}{(\hat{f}(d)+g(d))}\,\Phi_*^{-1}(d(\hat{f}(d)+g(d)))$$

and the result is proved. ■

We now give an estimate on the function E_λ previously defined:

LEMMA 3.4. – *There exist $\bar{d} \leq d_0$ and $\delta_0 > 0$ such that for any solution u of* (3.11) *with* $|u(0)| = \bar{d}$ *we have* $E_\lambda(r, u(r), u'(r); \bar{d}) \geq \delta_0$ *for every* $r \in [0, R]$.

PROOF. – As above, let us denote $\alpha = \lambda(K-1)$; consider the numbers C and $\bar{\varepsilon}$ given in Proposition 2.2 and $d_{\bar{\varepsilon}} := \bar{d}$ given in Lemma 2.3. If u is a solution of (3.11) with $|u(0)| = \bar{d}$, then, by Lemma 2.3 $\|u\|_1 \leq \varepsilon_C$ and, by (2.10),

$$u'(r)\,\phi(u'(r)) \leq C\mathcal{L}(u'(r)), \qquad \forall r \in [0, R]. \tag{3.15}$$

From (3.5) we can find a constant $\gamma \geq C(K-1)$ such that

$$\frac{\partial F_\lambda}{\partial r}(r, s) + \frac{\gamma}{r}F_\lambda(r, s) \geq 0, \qquad \forall r \in (0, R],\ \ \forall s \in [-\varepsilon_0, \varepsilon_0],\ \ \forall \lambda \in [0, 1]. \tag{3.16}$$

Then we deduce that:

$$\frac{d}{dr}E_\lambda(r, u(r), u'(r); \bar{d}) + \frac{\gamma}{r}E_\lambda(r, u(r), u'(r); \bar{d}) = -\frac{\alpha}{r}u'(r)\,\phi(u'(r)) +$$

$$+\frac{\partial F_\lambda}{\partial r} + \frac{\gamma}{r}\mathcal{L}(u'(r)) + \frac{\gamma}{r}F_\lambda(r, u(r)) \geq -\frac{K-1}{r}C\mathcal{L}(u'(r)) + \frac{\gamma}{r}\mathcal{L}(u'(r)) \geq 0\,;$$

by multiplying the last relation by r^γ and integrating from $r_0(d;\theta)$ (which has been defined in (3.13)) to r, we obtain:

$$E_\lambda(r, u(r), u'(r); \bar{d})\,r^\gamma - E_\lambda(r_0, u(r_0), u'(r_0); \bar{d})\,r_0(d;\,\theta)^\gamma \geq 0$$

and

$$E_\lambda(r, u(r), u'(r); \bar{d}) \geq E_\lambda(r_0, u(r_0), u'(r_0); \bar{d})\,r_0^\gamma R^{-\gamma} =$$

$$= R^{-\gamma}(\mathcal{L}(u'(r_0)) + F_\lambda(r_0, u(r_0)))\,r_0^\gamma \geq R^{-\gamma}F^0(\theta\bar{d})\,r_0^\gamma,$$

where (recall (3.10)) $F^0(\theta\bar{d}) = \min\{F_\lambda(r, \theta\bar{d}) : r \in [0, R], \lambda \in [0, 1]\} > 0$. Finally, using

(3.14), we find

$$E_\lambda(r, u(r), u'(r); \bar{d}) \geq K_0 \left(\frac{1}{\hat{f}(\bar{d}) + g(\bar{d})} \right)^\gamma \left(\Phi_*^{-1}(\bar{d}(\hat{f}(\bar{d}) + g(\bar{d}))) \right)^\gamma F^0(\theta \bar{d});$$

taking $\delta_0 = K_0(\hat{f}(\bar{d}) + g(\bar{d}))^{-\gamma}(\Phi_*^{-1}(\bar{d}(\hat{f}(\bar{d}) + g(\bar{d}))))^\gamma F^0(\theta\bar{d})$, the result is proved. ∎

It is easy to deduce the following consequence of Lemma 3.4:

LEMMA 3.5. – *For $\bar{d}$ given in Lemma 3.4, there exists $\bar{\delta} > 0$ such that if u is a solution of* (3.11) *with $|u(0)| = \bar{d}$ then*

$$|u(r)|^2 + |u'(r)|^2 \geq \bar{\delta}, \qquad \forall r \in [0, R].$$

Now, for every $d \in \mathbb{R}$, $d \neq 0$, let us define

(3.17) $\Sigma_d = \{(u, \lambda): (u, \lambda)$ is a solution of (3.11) and $u(0) > d$ if $d > 0$, $u(0) < d$ if $d < 0\}$.

From Lemma 2.5 we deduce that the function given by

$$n : \Sigma_d \to \mathbb{N} : (u, \lambda) \mapsto n(u),$$

where $n(u)$ is the number of zeros of u in $[0, R)$, is well defined; as in [21], it can be proved that n is a continuous map.

Moreover, from Lemma 3.1 of [21], using Lemma 2.3 and Lemma 3.5, we have the following estimate from above on n:

PROPOSITION 3.6. – *Consider the number $\bar{d}$ given in Lemma 3.4. Then there exists $n^* \in \mathbb{N}$ such that for any solution u of* (3.11) *we have*:

$$|u(0)| = \bar{d} \Rightarrow n(u) < n^*.$$

Now we prove an estimate from below on n; the argument is developed through some techniques of [21], where functions f rapidly growing at infinity are treated. However, since we are concerned with the dual condition near zero, we give the details.

PROPOSITION 3.7. – *For every $N > 0$ there exists $d_N > 0$, $d_N < \bar{d}$, such that for any solution $(u, \lambda) \in \Sigma_d$ (for some d) we have*

$$|u(0)| \leq d_N \Rightarrow n(u) > N.$$

PROOF. – Consider $(u, \lambda) \in \Sigma_d$. First of all, we observe that for every $N > 0$ there is $M(N) > 0$ such that

$$\frac{1}{M(N)} \int_0^s \frac{du}{\phi^{-1}(\Phi_*^{-1}(\Phi(s) - \Phi(u)))} < \frac{1}{N}, \qquad \forall |s| \leq \varepsilon_0 \tag{3.18}$$

(cf. (5.7) in the Appendix).

Now, using assumption (3.8), we deduce that there is $\varepsilon_{M(N)} > 0$ such that

$$|f_\lambda(r, s)| > M(N) |\phi(s)|, \qquad \forall r \in [0, R], \ \forall 0 < |s| \leq \varepsilon_{M(N)}, \ \ \forall \lambda \in [0, 1]. \tag{3.19}$$

Moreover, from Lemma 2.3, we can consider $d_{\varepsilon_{M(N)}} := d_N$ such that for any $(u, \lambda) \in \Sigma$

$$|u(0)| \leq d_N \Rightarrow \|u\|_1 \leq \varepsilon_{M(N)}.$$

Now, let us consider $(u, \lambda) \in \Sigma$ with $|u(0)| \leq d_N$: the equation in (3.11) can be written as

$$\begin{cases} u' = \phi^{-1}\left(\dfrac{y}{r^{\lambda(K-1)}}\right) \\ y' = -r^{\lambda(K-1)} f_\lambda(r, u). \end{cases} \tag{3.20}$$

We shall be concerned with the zeros $\{r_i\}_{i=1, \ldots, I}$ of u in the interval $[R/2, R]$. More precisely, we first estimate the distance between two successive zeros r_i and r_{i+1} of u in the case when

$$u'(r_i) > 0, \qquad u'(r_{i+1}) < 0 \quad \text{and} \quad u(r) > 0, \ \forall r \in (r_i, r_{i+1}).$$

From (3.20) we infer that $y'(r) < 0$ for every $r \in (r_i, r_{i+1})$; since $y(r_i) > 0$ and $y(r_{i+1}) < 0$, we deduce that there exists exactly one point $r^* \in (r_i, r_{i+1})$ such that $y(r^*) = 0$; again from (3.20), it follows that

$$u'(r) > 0 \ \forall r \in (r_i, r^*), \qquad u'(r) < 0 \ \forall r \in (r^*, r_{i+1}) \qquad \text{and} \qquad u'(r^*) = 0.$$

Let $B = (R/2)^{\lambda(K-1)}$; since $r \in [R/2, R]$, from (3.20) and (3.19) we deduce

$$\begin{cases} u' \leq \phi^{-1}\left(\dfrac{y}{B}\right) \\ y' \leq -BM(N)\, \phi(u). \end{cases} \tag{3.21}$$

Now, suppose $r \in (r_i, r^*)$; by multiplying the first inequality in (3.21) by $BM(N)\,\phi(u)$ and the second one by $\phi^{-1}(y/B)$ (which is positive in (r_i, r^*)) and adding up, we obtain:

$$BM(N)\, \phi(u(r))\, u'(r) + \phi^{-1}\left(\frac{y(r)}{B}\right) y'(r) < 0, \qquad \forall r \in (r_i, r^*);$$

this implies that the function $M(N)\,\Phi(u(r)) + \Phi_*(y(r)/B)$ is decreasing in (r_i, r^*). Hence

$$M(N)\,\Phi(u(r)) + \Phi_*\left(\frac{y(r)}{B}\right) > M(N)\,\Phi(u^*), \qquad \forall r \in (r_i, r^*), \quad u^* := u(r^*);$$

an easy computation gives

$$\phi(u'(r)) > c\Phi_*^{-1}(M(N)(\Phi(u^*) - \Phi(u(r)))), \qquad \forall r \in (r_i, r^*),$$

where $c = 1/2^{K-1}$. Solving with respect to $u'(r)$ and integrating on (r_i, r^*), we get

$$\int_{r_i}^{r^*} \frac{u'(r)}{\phi^{-1}\big(c\Phi_*^{-1}(M(N)(\Phi(u^*) - \Phi(u(r))))\big)}\,dr > r^* - r_i;$$

if we set $u(r) = u$, then we obtain

$$r^* - r_i < \int_0^{u^*} \frac{du}{\phi^{-1}\big(c\Phi_*^{-1}(M(N)(\Phi(u^*) - \Phi(u)))\big)}\,. \tag{3.22}$$

Now, an application of *(iii)*, *(ii)* in Proposition 2.2 yields the existence of $M_2(N) > M(N)$ such that

$$\int_0^{u^*} \frac{du}{\phi^{-1}\big(c\Phi_*^{-1}(M(N)(\Phi(u^*) - \Phi(u)))\big)} < \int_0^{u^*} \frac{du}{M_2(N)\,\phi^{-1}(\Phi_*^{-1}(\Phi(u^*) - \Phi(u)))}\,. \tag{3.23}$$

Hence, by (3.22), (3.23) and (3.18), we can conclude that

$$|r^* - r_i| < 1/N\,.$$

For the completion of the proof, it is now sufficient to observe that a computation analogue to the one developed above can be performed if we consider the interval (r^*, r_{i+1}) or an interval (r_i, r_{i+1}) where u is negative. ■

We are ready to prove Theorem 3.2.

PROOF OF THEOREM 3.2. – First of all, we recall from [21] that problem (3.11) can be put into the form (2.1) with respect to the Banach space $C^1_\#([0, R])$.

Now, let $n_0 = \max(n^*, 2k_0)$ (for the definition of k_0 see Theorem 5.4 in the Appendix). Next, let us consider $n > n_0$ and the number d_n arising from Proposition 3.7. In order to prove the existence of the solutions with exactly n zeros by an application of Theorem 2.1 let us introduce the sets

$$B = \{(u, \lambda) \in \operatorname{dom} \mathcal{N}: u(0) < \bar{d}\} \tag{3.24}$$

($\bar{d}$ as in Proposition 3.6) and

$$A_n = \{(u, \lambda) \in \operatorname{dom} \mathcal{N}: u(0) < d_n\}. \tag{3.25}$$

Moreover, the functional

$$\boldsymbol{k}: \Sigma \cap (\bar{B} \setminus A_n) \to \mathbb{N}$$

will be defined by

$$\boldsymbol{k}(u, \lambda) = \boldsymbol{n}(u).$$

Let us now prove that conditions (2.2) and (2.3) are satisfied. Indeed, it is sufficient to observe that

$$\partial(\bar{B} \setminus A_n) = \{(u, \lambda): u(0) = \bar{d}\} \cup \{(u, \lambda): u(0) = d_n\};$$

if $(u, \lambda) \in \Sigma$ and $u(0) = d_n$ then, by Proposition 3.7 we get $\boldsymbol{n}(u) > n$; on the other hand, if $(u, \lambda) \in \Sigma$ and $u(0) = \bar{d}$ then, by Proposition 3.6, we have $\boldsymbol{n}(u) < n^*$. Hence, being $n^* < n$, condition (2.2) is satisfied.

As far as the boundedness of $\boldsymbol{k}^{-1}(n)$ is concerned, if $(u, \lambda) \in \boldsymbol{k}^{-1}(n) \subset \Sigma \cap (\bar{B} \setminus A_n)$, then $u(0) < \bar{d} \leqslant d_0$: Lemma 2.3 implies that $\|u\|_1 \leqslant \varepsilon_0$ and so also (2.3) is fulfilled.

Finally, we have to choose an open set on which to compute the degree; to this aim, we refer to the discussion contained in the Appendix. Here, we only give some details; more precisely, for every $\alpha \in (0, \varepsilon_0)$, let us define

$$\Omega^\alpha = \{u \in C^1_{\sharp}([0, R]): \mathcal{L}(u'(r)) + G(u(r)) < G(\alpha), \ \forall r \in [0, R]\} \tag{3.26}$$

and $p = [n/2]$.

In the Appendix it will be proved that there exist $\alpha_p > 0$ and $\varepsilon > 0$ such that, when we set

$$\Omega_0 = \Omega^{\alpha_p + \varepsilon} \setminus \overline{\Omega^{\alpha_p - \varepsilon}}, \tag{3.27}$$

it follows that

$$(\boldsymbol{k}^{-1}(n))_0 \subset \Omega_0. \tag{3.28}$$

Finally, we define

$$U_0^n = \Omega_0 \cap (\bar{B} \setminus A_n) \tag{3.29}$$

and state the following:

CLAIM 1. – The degree $\deg(I - \mathcal{N}_0, U_0^n)$ is well defined and

$$\deg(I - \mathcal{N}_0, U_0^n) \neq 0. \tag{3.30}$$

The proof of the above claim (being essentially based on arguments already developed in [7, 8, 12]) can be found in the Appendix.

Hence, an application of Theorem 2.1 provides the existence of a solution u_n of problem (3.1) with

$$n(u_n) = n \quad \text{and} \quad u_n(0) > 0 .$$

We stress the fact that for this solution u_n we have $\|u_n\|_1 \leq \varepsilon_0$.

A similar argument, considering the sets

$$B = \{(u, \lambda) \in \text{dom } N \colon u(0) > -\bar{d}\}$$

($\bar{d}$ as in Proposition 3.6) and

$$A_n = \{(u, \lambda) \in \text{dom } N \colon u(0) > -d_n\}$$

shows that there exists at least one solution v_n of (3.1) such that

$$n(v_n) = n \quad \text{and} \quad v_n(0) < 0 . \qquad \blacksquare$$

4. – Nonlinearities which are superlinear at infinity.

In this section, we deal with nonlinearities f which satisfy (H_f) together with a rapid growth at infinity. More precisely, let us consider the following boundary value problem:

$$\begin{cases} \nabla\cdot(b(|\nabla u|)\,\nabla u) + f(|x|, u) = 0 & \text{in } \mathcal{B} \\ u = 0 & \text{on } \partial\mathcal{B} \end{cases} \tag{4.1}$$

where $\mathcal{B}$ is the open ball of center 0 and radius $R > 0$ in $\mathbb{R}^K$ $(K > 1)$; let $b\colon [0, +\infty) \to \to [0, +\infty)$ and set $\psi(s) = sb(|s|)$, for $s \in \mathbb{R}$, and $F(r, s) = \int_0^s f(r, u)\, du$.

We assume the following hypotheses on ψ, f and F:

(K_ψ) $\psi\colon \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism such that

$$\liminf_{s\to 0} \frac{\psi(\sigma s)}{\psi(s)} > 1 , \qquad \limsup_{s\to 0} \frac{\psi(\sigma s)}{\psi(s)} < +\infty , \qquad \forall \sigma > 1 \tag{4.2}$$

and

$$\liminf_{s\to +\infty} \frac{\psi(\sigma s)}{\psi(s)} > 1 , \qquad \limsup_{s\to +\infty} \frac{\psi(\sigma s)}{\psi(s)} < +\infty , \qquad \forall \sigma > 1 . \tag{4.3}$$

We set $\Psi(x) = \int_0^x \psi(s)\, ds$ for every $x \in \mathbb{R}$ and, like in [21], we introduce the constant

$\Gamma \in (0, 1)$ defined by

$$\Gamma = \limsup_{s \to +\infty} \frac{\Psi(s)}{s\psi(s)}.$$

(K_f) $f: [0, R] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{u \to 0} \frac{f(r, u)}{\psi(u)} = +\infty, \quad \text{uniformly in } r \in [0, R] \tag{4.4}$$

and

$$\lim_{|u| \to +\infty} \frac{f(r, u)}{\psi(u)} = +\infty, \quad \text{uniformly in } r \in [0, R]. \tag{4.5}$$

(K_F) $F(r, s)$ is differentiable with respect to $r \in [0, R]$ and there exists a continuous function $\alpha: [0, R] \to \mathbb{R}^+$ such that

$$\left| \frac{\partial F}{\partial r}(r, s) \right| \leq \alpha(r) F(r, s), \qquad \forall r \in [0, R], \ \forall s \in \mathbb{R}. \tag{4.6}$$

Now, for every $x \in \mathbb{R}$, let us define $F^0(x) = \min\{F(r, x): r \in [0, R]\}$ and, according to (3.6), let us consider the function $\hat{f}: \mathbb{R} \to \mathbb{R}$ defined by:

$$\hat{f}(s) = \begin{cases} \sup\{f(r, u): r \in [0, R], \ u \in [0, s]\} & \text{if } s > 0 \\ \inf\{f(r, u): r \in [0, R], \ u \in [s, 0]\} & \text{if } s < 0. \end{cases} \tag{4.7}$$

Finally, like in [21], for every $\theta \in (0, 1)$, let us set

$$\delta_\theta = \liminf_{|s| \to +\infty} \frac{F^0(\theta s)}{s \hat{f}(s)}.$$

We will prove:

THEOREM 4.1. – *Assume* (K_ψ), (K_f) *and* (K_F). *Suppose also that there exists* $Z > 0$ *such that*

$$\frac{\partial F}{\partial r}(r, s) \geq 0, \qquad \forall r \in [0, R], \ \forall |s| > Z \tag{4.8}$$

and that there exists $\theta \in (0, 1)$ *such that* $\delta_\theta > 0$ *and*

$$K\delta_\theta > K\Gamma - 1. \tag{4.9}$$

Then, there exists $\bar{n}$ *such that for every* $n > \bar{n}$ *problem* (4.1) *has at least four radial solutions* u_n, v_n, w_n *and* z_n *with* $u_n(0) > 0$, $v_n(0) < 0$, $w_n(0) > 0$ *and* $z_n(0) < 0$, *all having exactly* n

zeros in $[0, R)$. *Moreover, we have*

$$\lim_{n\to+\infty} |u_n(r)| + |u_n'(r)| = 0 = \lim_{n\to+\infty} |v_n(r)| + |v_n'(r)|, \text{ uniformly in } r\in[0,R] \tag{4.10}$$

and

$$\lim_{n\to+\infty} |w_n(r)| + |w_n'(r)| = +\infty = \lim_{n\to+\infty} |z_n(r)| + |z_n'(r)|, \text{ uniformly in } r\in[0,R]. \tag{4.11}$$

REMARK 4.2. – *Assumption* (4.9) *is a condition of subcritical growth at infinity on the lines of* [21]. *In particular, when* $f(r, s) \sim a(r)\,|s|^{\nu-2}s$ *near infinity (with* $a > 0$ *and* $a' \geq 0$*) and* $\psi(x) = |x|^{p-2}x$, $p > 1$, *then* (4.9) *reduces to* (cf. [15])

$$\frac{a(R)}{a(0)}\nu < \frac{pK}{K-p} = p^*.$$

For example, if we set $\Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, *then Theorem* 4.1 *guarantees that the Dirichlet problem for the equation* $\Delta_p u + a_1|u|^{\mu-2}u + a_2|u|^{\nu-2}u = 0$ $(a_1, a_2 > 0)$ *has, for each n sufficiently large, at least four radially symmetric solutions with n zeros in* $[0, R)$, *provided that* $0 < \mu < p < \nu < p^*$.

Assumption (4.8) *can be omitted if a stronger growth restriction on f at infinity is required.*

REMARK 4.3. – *Note that, by* (K_ψ), (K_f) *and* (K_F), *we can repeat all the arguments in Section* 3 *(in particular, the proof of Theorem* 3.2).

Now, let us consider a continuous function $g^*: \mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} g^*(x)\,x > 0\,, & \forall x \neq 0 \\ \lim_{x\to 0} \dfrac{g^*(x)}{\psi(x)} = +\infty \end{cases} \tag{4.12}$$

and $g^*(x) = f(R, x)$ for $|x|$ large. Let us define the homotopy

$$f_\lambda^*(r, x) = \lambda f(r, x) + (1-\lambda)\,g^*(x)\,, \qquad \forall (r, x, \lambda) \in [0, R]\times\mathbb{R}\times[0, 1].$$

We consider the boundary value problem

$$\begin{cases} (r^{\lambda(K-1)}\psi(u'))' + r^{\lambda(K-1)} f_\lambda^*(r, u) = 0 \\ u'(0) = 0 = u(R)\,. \end{cases} \tag{4.13}$$

By the choice of g^*, the nonlinearity f_λ^* satisfies the same assumptions of f.

Now, as in Section 2, for a solution u of (4.13) with $u(0) = d$, we introduce the function $E_\lambda(r, d) = \mathcal{L}(u'(r)) + F_\lambda^*(r, u(r))$; moreover, we use the same notation of Section 3. In particular, for every $d \neq 0$, Σ_d is still defined by (3.17), according to the new boundary value problem considered.

In what follows, we present some estimates on solutions of (4.13) which are based on arguments developed in [21]; we point out that in that paper the authors deal with the case $f(|x|, u) = \bar{g}(u) - q(|x|, u)$ (with $\bar{g}$ growing faster than ϕ at infinity and q bounded). However, all the computations can be adapted to our more general situation.

LEMMA 4.4 ([21, Corollary 5.1]). – *Assume* (K_ψ), (K_F), (4.5), (4.8) *and* (4.9). *Then, given any* $M_0 > 0$, *there exists* $\widetilde{M}(M_0) \geq M_0$ *such that for any solution* u *of* (4.13) *with* $u(0) \geq \widetilde{M}$ *we have*

$$|u(r)|^2 + |\psi(u'(r))|^2 \geq M_0, \qquad \forall r \in [0, R].$$

Now, we state a lower estimate on the number of zeros of solutions to (4.13) which follows from the asymptotic behaviour of f_λ^* at infinity; this estimate constitutes the counterpart of Proposition 3.7 where the analogous condition in zero is concerned.

PROPOSITION 4.5 ([21, Lemma 5.3]). – *Assume* (K_ψ), (K_F), (4.5), (4.8) and (4.9). *Then for every* $N > 0$ *there exists* $M_N^* > 0$, $M_N^* > \bar{d}$, *such that for any solution* $(u, \lambda) \in \Sigma_d$ *(for some* d*) we have*

$$|u(0)| \geq M_N^* \Rightarrow \boldsymbol{n}(u) > N.$$

PROOF OF THEOREM 4.1. – First of all, we observe that all the assumptions of Theorem 3.2 are satisfied. Hence, there is n_0 such that for $n > n_0$ problem (4.1) admits the two solutions (of small norm) u_n and v_n, with $u_n(0) > 0$, $v_n(0) < 0$, both having n zeros in $[0, R)$ and satisfying (4.10).

The existence of the pair of solutions w_n and z_n follows from [21], according to our previous remarks where we have noticed that the estimates of [21] extend to the class of nonlinearities we are treating. However, in order to enter in the setting of Theorem 2.1, we consider $n > 2\tilde{n}$ ($\tilde{n}$ as in Theorem 5.5 in the Appendix) and the corresponding M_n^* given in Proposition 4.5; then, take $B^1 = \operatorname{dom} \mathcal{N}$ and

$$A_n^1 = \{(u, \lambda) \in \operatorname{dom} \mathcal{N}\colon u(0) > M_n^* > \bar{d}\}.$$

Arguing as in the proof of Theorem 3.2, it follows, by Proposition 4.5, that condition (2.2) is satisfied. The boundedness of $\boldsymbol{k}^{-1}(n)$ $(n > 2\tilde{n})$ (as well as (4.11)) follows from the elastic property stated in Lemma 4.4 with the same arguments already developed in the proof of Theorem 4.1 in [21] for nonlinearities which «grow faster than ϕ at $\pm\infty$».

We omit the details concerning the choice of the open set on which to prove the degree condition (2.4), since they coincide with their analogue in the proof of Theorem 3.2.

Then, take $\bar{n} = \max(n_0, 2\tilde{n})$. The solutions u_n and w_n, for $n > \bar{n}$, are distinct, since (see the proof of Theorem 3.2 and the definition of A_n^1) $u_n(0) < \bar{d}$ and $w_n(0) > \bar{d}$. Analogously, we have $v_n(0) > -\bar{d}$ and $z_n(0) < -\bar{d}$: thus, also v_n and z_n are different. This shows that problem (4.1) has at least four radial solutions with n zeros in $[0, R)$. ∎

5. – Appendix.

In this appendix we study, with the notation of Section 2 and by means of a time-map technique, the autonomous problem

$$
\left\{ \begin{array}{l} (\phi(u'))' + g(u) = 0 \\ u'(0) = 0 = u(R)\,. \end{array} \right. \tag{5.1}
$$

In the sequel, we follow an approach like in [7, 8, 12].

Let us assume

(H_ϕ) $\phi:[-\varepsilon_1, \varepsilon_1] \to [-\phi(\varepsilon_1), \phi(\varepsilon_1)]$ is an odd increasing homeomorphism such that

$$
\liminf_{s \to 0} \frac{\phi(\sigma s)}{\phi(s)} > 1\,, \qquad \forall \sigma > 1 \tag{5.2}
$$

and

$$
\limsup_{s \to 0} \frac{\phi(\sigma s)}{\phi(s)} < +\infty\,, \qquad \forall \sigma > 1\,. \tag{5.3}
$$

(H_g) $g:[-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ is a continuous function such that

$$
\lim_{s \to 0} \frac{g(s)}{\phi(s)} = +\infty\,.
$$

As it was done in Section 3, from (H_g) we deduce the existence of $\varepsilon_0 > 0$ such that $g(s)\,s > 0$, for all $0 < |s| \le \varepsilon_0$. Then, we recall that the solutions of the equation

$$
(\phi(u'))' + g(u) = 0 \tag{5.4}
$$

satisfy the energy relation

$$
H(u(r), u'(r)) = \mathcal{L}(u'(r)) + G(u(r)) = const\,, \qquad \forall r \in \mathbb{R}\,,
$$

where, as before, $G(x) = \int_0^x g(s)\,ds$ and $\mathcal{L}(x) = x\phi(x) - \int_0^x \phi(t)\,dt$. Then (recalling (2.9)), in the phase-plane $(x, y) = (u, \phi(u'))$ every (nontrivial) orbit of (5.4) is periodic and corresponds to the closed curve defined by the equation

$$
H(x, y) = \Phi_*(y) + G(x) = G(\alpha)
$$

for some $\alpha > 0$. In particular, for $\alpha \in (0, \varepsilon_0)$ the (unique) solution of the Cauchy problem

$$
\left\{ \begin{array}{l} (\phi(u'))' + g(u) = 0 \\ u(0) = \alpha\,, \qquad u'(0) = 0\,, \end{array} \right. \tag{5.5}
$$

which will be denoted by $u(\cdot;\alpha,0)$, is defined on $\mathbb{R}$ and satisfies the relation

$$H(u(r),u'(r)) = (\Phi_* \circ \phi)(u'(r)) + G(u(r)) = G(\alpha), \qquad \forall r \in [0,R]. \tag{5.6}$$

Moreover, we can define the functions $T_i\colon (0,\varepsilon_0) \to (0,+\infty)$ $(i=1,2)$ by

$$T_1(\alpha) = \int_0^\alpha \frac{ds}{(\phi^{-1}\circ \Phi_*^{-1})(G(\alpha)-G(s))} \tag{5.7}$$

and

$$T_2(\alpha) = \int_{-\alpha_1}^0 \frac{ds}{(\phi^{-1}\circ \Phi_*^{-1})(G(\alpha)-G(s))}, \tag{5.8}$$

where $G(-\alpha_1) = G(\alpha)$. It is straightforward to check, integrating (5.6), that they represent the time needed for a quarter of rotation along the orbit of energy $G(\alpha)$ in the upper (lower) half plane: from the point $(0,(\phi^{-1}\circ\Phi_*^{-1})(G(\alpha)))$ to the point $(\alpha,0)$ (from $(\alpha,0)$ to $(0,-(\phi^{-1}\circ\Phi_*^{-1})(G(\alpha)))$) and from the point $(-\alpha_1,0)$ to the point $(0,(\phi^{-1}\circ\Phi_*^{-1})\cdot$ $\cdot(G(-\alpha_1)))$ (from $(0,-(\phi^{-1}\circ\Phi_*^{-1})(G(-\alpha_1)))$ to $(-\alpha_1,0)$), respectively.

Now, we introduce some notation; let us define the set $\mathcal{F} = \{(x,y)\in Q \mid a_n\colon 2nx+(2n+1)y=R$ or $b_n\colon (2n+1)x+2ny=R$ or $c_n\colon 2(n+1)x+(2n+1)y=R$ or $d_n\colon (2n+1)x+2(n+1)y=R$ for some $n\in\mathbb{N}\}$, where $Q=\{(x,y)\in\mathbb{R}^2\colon x>0,\ y>0\}$. Moreover, let us consider the following open disjoint subsets of Q (see Figure 1):

$$A_{1n} = \{(x,y)\in Q\colon 2nx+(2n+1)y>R,\ (2n+1)x+2ny>R,\ 2nx+(2n+1)y<R,\ (2n+1)x+2ny<R\},$$

$$A_{2n} = \{(x,y)\in Q\colon 2nx+(2n+1)y<R,\ (2n+1)x+2ny>R\},$$

$$A_{3n} = \{(x,y)\in Q\colon 2nx+(2n+1)y<R,\ (2n+1)x+2ny<R,\ 2(n+1)x+(2n+1)y>R,\ (2n+1)x+2(n+1)y>R\},$$

$$A_{4n} = \{(x,y)\in Q\colon 2(n+1)x+(2n+1)y>R,\ (2n+1)x+2(n+1)y<R\},$$

$$A_{5n} = \{(x,y)\in Q\colon 2nx+(2n+1)y>R,\ (2n+1)x+2ny<R\},$$

$$A_{6n} = \{(x,y)\in Q\colon 2(n+1)x+(2n+1)y<R,\ (2n+1)x+2(n+1)y>R\}.$$

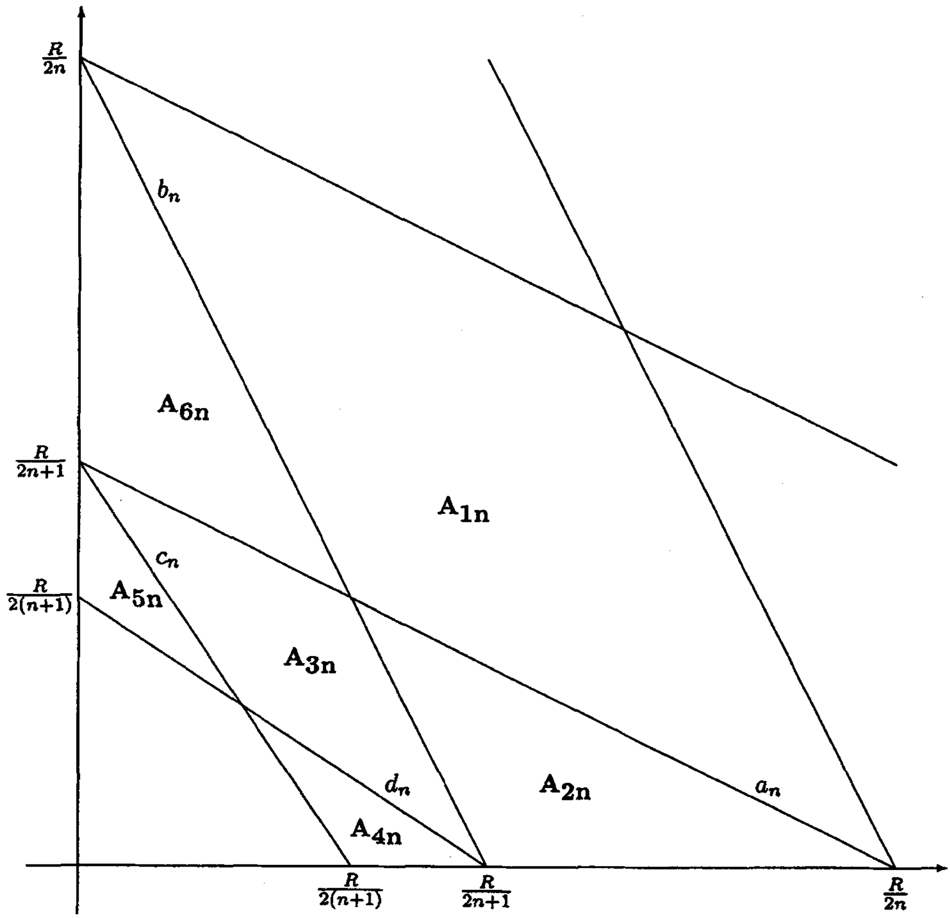

Figure 1. – Some of the regions A_{in}.

Arguing as in [8] and [12], we have the following facts, which are crucial for the proofs of Theorem 5.4 below and of our main results (Theorem 3.2 and Theorem 4.1), respectively:

PROPOSITION 5.1. – *Problem* (5.1) *has a solution of energy* $G(\alpha)$ *if and only if there exists an integer* $n \in \mathbb{N}$, $n \geq 0$, *such that*

$$2nT_1(\alpha) + (2n+1)\,T_2(\alpha) = R$$

or

$$(2n+1)\,T_1(\alpha) + 2nT_2(\alpha) = R$$

or

$$2(n+1)\,T_1(\alpha) + (2n+1)\,T_2(\alpha) = R$$

or

$$(2n+1)\,T_1(\alpha) + 2(n+1)\,T_2(\alpha) = R\,.$$

Moreover, for any solution u of (5.1) *with energy $G(\alpha)$ we have:*

(1) *u has exactly $2n$ zeros in $[0, R)$ and $u(0) < 0 \Leftrightarrow (T_1(\alpha), T_2(\alpha)) \in a_n$.*

(2) *u has exactly $2n$ zeros in $[0, R)$ and $u(0) > 0 \Leftrightarrow (T_1(\alpha), T_2(\alpha)) \in b_n$;*

(3) u has exactly $2n+1$ zeros in $[0, R)$ and $u(0) < 0 \Leftrightarrow (T_1(\alpha), T_2(\alpha)) \in c_n$;

(4) *u has exactly $2n+1$ zeros in $[0, R)$ and $u(0) > 0 \Leftrightarrow (T_1(\alpha), T_2(\alpha)) \in d_n$.*

By the above Proposition, in order to obtain existence and nodal properties of solutions to (5.1), we are led to study the intersections between the support of the curve $T\colon (0, \varepsilon_0) \to \mathbb{R}^2$ (defined by $T(\alpha) = (T_1(\alpha), T_2(\alpha))$) and the set $\mathcal{T}$ previously introduced. Now, we can prove the following:

PROPOSITION 5.2. – *For $i = 1, 2$ we have*

$$\lim_{\alpha \to 0} T_i(\alpha) = 0\,. \tag{5.9}$$

PROOF. – Consider the case $i = 1$ (the case $i = 2$ is similar); for simplicity, we denote by T_1^g the time-map relative to the equation $(\phi(u'))' + g(u) = 0$. Due to (H_g) and the analogue of (3.19), for every $\Lambda > 0$ one has (for α small enough) $T_1^g(\alpha) \leq T_1^{\Lambda\phi}(\alpha)$. Repeating the arguments in the proof of Proposition 3.7 (in particular, the application of (*ii*) in Proposition 2.2), we deduce that

$$\lim_{\alpha \to 0} T_1^{\Lambda\phi}(\alpha) := \chi(\Lambda)\,,$$

with $\lim_{\Lambda \to +\infty} \chi(\Lambda) = 0$. The above facts are sufficient to obtain (5.9). ∎

REMARK 5.3 *In the paper* [32] *it was proved, for the linear operator $u \mapsto -u''$, that the time-map is infinitesimal for $\alpha \to +\infty$ or for $\alpha \to 0$ when g is superlinear at infinity or sublinear in zero, respectively. For nonlinearities g such that $\lim_{|x| \to +\infty} g(x)/\phi(x) = +\infty$, the former fact has been generalized in* [20]; *for the case when $\lim_{x \to 0} g(x)/\phi(x) = +\infty$, the latter is extended in Proposition 5.2 above.*

Now, we prove a multiplicity result for the autonomous problem (5.1):

THEOREM 5.4. – *There exists $k_0 \in \mathbb{N}$ such that for every $k \geq 2k_0$ problem* (5.1) *has at least two solutions u_k and v_k with $u_k(0) > 0$ and $v_k(0) < 0$, both having exactly k zeros in $[0, R)$.*

PROOF. – The asymptotic behaviour of T_i ($i = 1, 2$), described in Proposition 5.2, guarantees that the support of the curve T «emanates» from the point $P_0 = (0, 0)$. In other words, recalling the definition of the straight lines a_k, b_k, c_k and d_k, there exists k_0 s.t. the support of T intersects all these lines, for $k \geq k_0$; since each intersection gives rise to solutions of (5.1), the first part of the statement is proved. The nodal properties of these solutions easily follow from the second part of Proposition 5.1. ■

Now, assume that $g\colon \mathbb{R} \to \mathbb{R}$ satisfies $g(x)\,x > 0$ for every $x \in \mathbb{R}$, $x \neq 0$, and

$$\lim_{|x| \to +\infty} \frac{g(x)}{\phi(x)} = +\infty = \lim_{x \to 0} \frac{g(x)}{\phi(x)}. \tag{5.10}$$

Then, we can prove:

THEOREM 5.5. – *Suppose that g satisfies* (5.10). *Then, there exists* $\tilde{n} \in \mathbb{N}$ *such that for every integer* $n \geq 2\tilde{n}$ *problem* (5.1) *has at least four solutions* u_n, v_n, w_n *and* z_n *with* $u_n(0) > 0$, $v_n(0) < 0$, $w_n(0) > 0$, $z_n(0) < 0$, *having exactly n zeros in* $[0, R)$.

PROOF. – According to Remark 5.3 and [21], one has $\lim_{\alpha \to +\infty} T_i(\alpha) = 0$, $i = 1, 2$; then, we can repeat the proof of Theorem 5.4 noting that we have $P_0 = P_\infty = (0, 0)$. In other words, whenever the support of the curve $\alpha \mapsto T(\alpha)$ intersects one of the lines of the set $\mathcal{F}$ for some $\alpha_1 > 0$, then it will necessarily intersect this same line for some $\alpha_2 \neq \alpha_1$. ■

We end this Appendix by going back to the fact (used in Sections 3 and 4) that some «local» degree relative to the autonomous problem (5.1) is different from zero; more precisely, in the proof of Theorem 3.2 we stated the following:

CLAIM 1. – The degree $\deg(I - \mathcal{N}_0, U_0^n)$ is well defined and

$$\deg(I - \mathcal{N}_0, U_0^n) \neq 0. \tag{5.11}$$

In the proof of Claim 1 we will use the next result.

PROPOSITION 5.6. – *Let* Ω^α *be defined as in* (3.26); *if* $\alpha > 0$ *is such that* $T(\alpha) \notin \mathcal{F}$ *then, for some* n,

$$\deg(I - \mathcal{N}_0, \Omega^\alpha) = \deg_B(\tau, (-\alpha, \alpha), 0) = \begin{cases} +1 & \text{if } T(\alpha) \in A_{1n} \\ 0 & \text{if } T(\alpha) \in A_{2n} \cup A_{4n} \cup A_{5n} \cup A_{6n} \\ -1 & \text{if } T(\alpha) \in A_{3n}, \end{cases}$$

where $\tau\colon \mathbb{R} \to \mathbb{R}$ *is the shooting map defined by* $\tau(\xi) = u(R; \xi, 0)$.

PROOF OF CLAIM 1. – We give the proof for n even, $n = 2p$; we also recall that $n > n^*$ and $n > 2k_0$.

First, observe that Claim 1 was stated in Section 3 when we were considering functions u_n s.t. $u_n(0)>0$. With the notation introduced in the proof of Theorem 3.2, we observe that, by the choice of the sets $\bar{B}$ and A_n (recall (3.24)-(3.25)) and because $T(\alpha_p+\varepsilon)\notin\mathcal{F}$, $T(\alpha_p-\varepsilon)\notin\mathcal{F}$, $\deg(I-\mathcal{N}_0, U_0^n)$ is well defined.

By Proposition 5.1 there exists at least a positive real number α_p such that

$$T(\alpha_p)\in b_p.$$

By the continuity of T_i $(i=1,2)$, there exists $\varepsilon>0$ such that (see Figure 1)

$$\begin{cases} T(\alpha_p+\varepsilon)\in A_{1p} \\ T(\alpha_p-\varepsilon)\in A_{5p} \end{cases} \tag{5.12}$$

or

$$\begin{cases} T(\alpha_p+\varepsilon)\in A_{2p} \\ T(\alpha_p-\varepsilon)\in A_{3p} \end{cases} \tag{5.13}$$

or

$$\begin{cases} T(\alpha_p+\varepsilon)\in A_{1p} \\ T(\alpha_p-\varepsilon)\in A_{3p}. \end{cases} \tag{5.14}$$

Moreover, following a continuity argument developed in the proof of Th. 5.1 in [20], the real number ε can be chosen such that, for Ω_0 defined in (3.27), we have

$$(k^{-1}(n))_0\subset\Omega_0. \tag{5.15}$$

Then, if U_0^n is defined by (3.29), by (5.15), we deduce that

$$(k^{-1}(n))_0\cap\Omega_0=(k^{-1}(n))_0\cap U_0^n.$$

When (5.12) or (5.13) occurs, from Proposition 5.6 we infer

$$\deg(I-\mathcal{N}_0, U_0^n)=1\,.$$

When (5.14) occurs, again by Proposition 5.6 and by the additivity property of the degree we have

$$2=\deg(I-\mathcal{N}_0,\Omega_0)=\deg_B(\tau,(-\alpha_p-\varepsilon,\ -\alpha_p+\varepsilon)\cup(\alpha_p-\varepsilon,\ \alpha_p+\varepsilon),0)=$$

$$=\deg_B(\tau,(-\alpha_p-\varepsilon,\ -\alpha_p+\varepsilon),0)+\deg_B(\tau,(\alpha_p-\varepsilon,\ \alpha_p+\varepsilon),0)\,.$$

Hence

$$\deg_B(\tau,(-\alpha_p-\varepsilon,\ -\alpha_p+\varepsilon),0)=\deg_B(\tau,(\alpha_p-\varepsilon,\ \alpha_p+\varepsilon),0)=1\,.$$

Thus, we conclude that, in any case,

$$\deg(I - \mathcal{N}_0, U_0^n) = \deg_B(\tau, (\alpha_p - \varepsilon, \alpha_p + \varepsilon), 0) = 1 . \quad \blacksquare$$

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