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Existence of Minimizers for a Class of Anisotropic Free Discontinuity Problems (*).

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Summary. – *We prove the existence of minimizing pairs (K, u), K compact set of $\boldsymbol{R}^N$ and $u \in W^{1,p}(\Omega \backslash K)$, for the functional*

$$\mathcal{G}(K, u) = \int_{\Omega\backslash K} f(x, \nabla u) + \alpha \int_{\Omega\backslash K} |u-g|^q + \beta \mathcal{H}^{N-1}(K \cap \Omega)$$

when the integrand $f(x, z)$ is convex with respect to z, $|z|^p \leq f(x, z) \leq L|z|^p$, $p > 1$, and satisfies suitable assumptions of uniform continuity in x with respect to z.

1. – Introduction.

In recent years functionals involving volume and interfacial energies have been introduced as models in fracture mechanics, phase transition and image segmentation. In particular the following functional

$$\mathcal{F}(u, \Omega) = \int_{\Omega} f(x, \nabla u(x))\, dx + \alpha \int_{\Omega} |u-g|^q\, dx + \beta \mathcal{H}^{N-1}(S_u \cap \Omega)$$

includes those considered by [BZ], [DGCL], [Fo-Fr], [FF], [MS]. Here Ω is a bounded open set of $\boldsymbol{R}^N$, f is a continuous convex function of polynomial growth, $g \in L^\infty(\Omega)$, α, $\beta > 0$, $q \geq 1$ and u is a function of bounded variation. We recall that for a BV function u the jump set S_u coincides $\mathcal{H}^{N-1}$-a.e. with the complement of the Lebesgue points and the symbol ∇u stands for the approximate differential. In general the distributional derivative Du can be represented by $Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\, \nu \mathcal{H}^{N-1} \llcorner S_u + C(u)$, where $C(u)$ is the so called Cantor part of Du.

The results of De Giorgi and Ambrosio (see [DGA]) have showed that a natural class in which to minimize $\mathcal{F}$ is the class $SBV(\Omega)$ of those special functions $u \in BV(\Omega)$

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for which $C(u) = 0$. In particular the lower semicontinuity result of Ambrosio (see [A1], [A2], [A3]) implies the existence of minimizers for $\mathcal{F}$ in this class.

It is then natural to investigate if such minimizers are related to those of the «classical» counterpart of $\mathcal{F}$, i.e. the functional

$$\mathcal{G}(K, u) = \int_{\Omega \setminus K} f(x, \nabla u(x))\, dx + \alpha \int_{\Omega \setminus K} |u - g|^q\, dx + \beta \mathcal{H}^{N-1}(K \cap \Omega) \tag{1.1}$$

where now K is a closed subset in $\boldsymbol{R}^N$ and $u \in W^{1,p}(\Omega \setminus K)$. Notice that if $f(x, z) = |z|^2$ the above functional coincides with the model suggested by Mumford and Shah in the framework of image segmentation. For this model De Giorgi, Carriero and Leaci proved in [DGCL] that if u is a minimizer of $\mathcal{F}$ then the pair $(\overline{S}_u, u)$ minimizes $\mathcal{G}$ and $\mathcal{F}(u, \Omega) = \mathcal{G}(\overline{S}_u, u)$. This result has been obtained by means of a decay estimate of the energy in small balls which allows to prove that $\mathcal{H}^{N-1}((\overline{S}_u \setminus S_u) \cap \Omega) = 0$. This amounts to give a first information on the regularity of the jump set of the minimizers since in general, if $u \in SBV(\Omega)$, S_u can be any $(N-1)$-rectifiable set (see [A]). The proof given in [DGCL] makes a strong use of the scaling properties of the Mumford-Shah functionals and of the classical sup estimates for the gradient of harmonic functions.

This result has been later extended by [CL] to the case where $f(x, z) = |z|^p$, $p > 1$, and by [FF] to a class of convex integrands f not depending on x.

In this paper we allow f to depend also on x under suitable assumptions of uniform continuity in x with respect to z. Moreover f is supposed to be convex in z, but not necessarily differentiable. The main difficulty here is, as usual, in proving the decay estimate of energy, which is achieved by a typical blow-up argument. In our case the main point is to recover the convergence of the rescaled minimizers v_h to a $W^{1,p}$ minimizer v of the same functional without the area term. Differently from the case when f is independent of x, the minimizer v is not in general Lipschitz continuous, however a recent result of [CFP] provides Hölder continuity estimates on v which enables us to conclude with the decay estimate.

2. – Preliminary results.

Let $E \subset \boldsymbol{R}^N$, we denote by $\mathcal{H}^{N-1}(E)$ the $N-1$-dimensional Hausdorff measure of E. If $u\colon \Omega \to \boldsymbol{R}$ is a Borel function and if $x \in \Omega$ we say that $\tilde{u}(x) \in \boldsymbol{R} \cup \{\infty\}$ is the approximate limit of u at x if

$$g(\tilde{u}(x)) = \lim_{\varrho \to 0} |B_\varrho(x)|^{-1} \int_{B_\varrho(x)} g(u(y))\, dy$$

for every $g \in C(\boldsymbol{R} \cup \{\infty\})$.

We denote by S_u the set af all point $x \in \Omega$ in which the approximate limit does not exist. S_u is a Lebesgue negligible Borel set and the jump set of a BV function u, S_u, is $N-1$-rectifiable (see [DG] or [F]).

It is well known that if u belongs to the space $BV(\Omega)$ then Du, its distributional gradient, can be decomposed as $Du = \nabla u \mathcal{L}^N + D^s u$ where ∇u is the density of Du with respect to $\mathcal{L}^N$ and $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$. We also recall that a function u belongs to the space $SBV(\Omega)$ of the «special functions of bounded variations», introduced in [DGA], if u belongs to $BV(\Omega)$ and if $D^s u$ is such that $|D^s u|(\Omega \backslash S_u) = 0$.(For the study of the main properties of SBV functions we refer to [A1], [A2], [DGA]).

In the sequel we consider a bounded open set of $\boldsymbol{R}^N$ Ω, a continuous function $f \colon \Omega \times \times \boldsymbol{R}^N \to [0, +\infty[$ satisfying the following assumptions

$$f(x, \cdot) \quad \text{is convex for all } x \in \Omega\,; \tag{2.1}$$

$$|z|^p \leq f(x, z) \leq L|z|^p \quad \text{for all } x \in \Omega\,, \quad \forall z \in \boldsymbol{R}^N\,, \tag{2.2}$$

with $L > 1$ and $p > 1$; there exists $\nu > 0$ such that for every $(x_0, z_0) \in \Omega \times \boldsymbol{R}^N$ and for every $\varphi \in C_0^1(\Omega)$ it results

$$\int_\Omega f(x_0, z_0 + \nabla\varphi(y))\, dy \geq \int_\Omega [f(x_0, z_0) + \nu(|z_0|^2 + |\nabla\varphi|^2)^{(p-2)/2} |\nabla\varphi|^2]\, dy\,; \tag{2.3}$$

there exists a continuous, bounded, increasing function $\omega \colon [0, +\infty[\to [0, +\infty[$, with $\omega(0) = 0$, such that

$$|f(x, z) - f(y, z)| \leq \omega(|x - y|)\, |z|^p\,. \tag{2.4}$$

We also assume that there exist $c_0 > 0$, $0 < m < p$ such that

$$|f(x, z) - f_p(x, z)| \leq c_0 |z|^{p-m} + 1 \quad \text{for a.e. } x \in \Omega\,, \quad \forall z \in \boldsymbol{R}^N\,, \tag{2.5}$$

where $f_p(x, z)$ is the p-recession function of f, i.e.

$$f_p(x, z) = \limsup_{t \to +\infty} \frac{f(x, tz)}{t^p} \quad \text{for all } x \in \Omega\,, \quad \forall z \in \boldsymbol{R}^N\,.$$

REMARK 2.1. – It is clear that $f_p(x, \cdot)$ is positively homogeneous of degree p and, if $f(x, \cdot)$ is convex, then $f_p(x, \cdot)$ is convex. Moreover if f verifies (2.2) then

$$|z|^p \leq f_p(x, z) \leq L|z|^p \quad \text{for a.e. } x \in \Omega\,; \quad z \in \boldsymbol{R}^N\,; \tag{2.6}$$

and if f verifies (2.4) then

$$|f_p(x, z) - f_p(y, z)| \leq \omega(|x - y|)\, |z|^p\,. \tag{2.7}$$

The proof of next lemma can be found, in a slightly different form, in Lemma 2.8 in [FF].

LEMMA 2.2. – *Under assumptions* (2.1), (2.2), (2.3) *and* (2.5) *it results*

$$\int_\Omega f_p(x_0, z_0 + \nabla\varphi(y))\, dy \geq \int_\Omega [f_p(x_0, z_0) + \nu(|z_0|^2 + |\nabla\varphi|^2)^{(p-2)/2} |\nabla\varphi|^2]\, dy$$

for every $(x_0, z_0) \in \Omega \times \boldsymbol{R}^N$ *and for every* $\varphi \in C_0^1(\Omega)$

Under assumptions (2.1) and (2.2) if u belongs to $SBV_{\text{loc}}(\Omega)$ and c is a positive constant we set, for every Borel set $E \subset \Omega$,

$$F(u, c, E) = \int_E f(x, \nabla u(x))\, dx + c\mathcal{H}^{N-1}(S_u \cap E), \tag{2.8}$$

if $c = 1$ we set $F(u, E) = F(u, 1, E)$.

DEFINITION 2.3. – *A function $u \in SBV_{loc}(\Omega)$ is a local minimizer in Ω of $F(u, c, E)$ if*

$$F(u, c, A) < +\infty, \qquad \forall A \subset\subset \Omega \tag{2.9}$$

and

$$F(u, c, A) \leq F(v, c, A)$$

for any $v \in SBV_{loc}(\Omega)$ such that $\operatorname{supp}(v-u) \subset\subset A \subset\subset \Omega$. *Similarly we say that $u \in W^{1,p}_{loc}(\Omega)$ is a local minimizer of the functional $\int_E f(x, \nabla u(x))\, dx$ if*

$$\int_A f(x, \nabla u(x))\, dx \leq \int_A f(x, \nabla v(x))\, dx$$

for any $v \in W^{1,p}_{loc}(\Omega)$ such that $\operatorname{supp}(v-u) \subset\subset A \subset\subset \Omega$.

Let us recall the definition of deviation from minimality (see [AP]).

DEFINITION 2.4. – *The deviation from minimality* $\operatorname{Dev}(u, c, \Omega)$ *of a function $u \in SBV_{loc}(\Omega)$ satisfying* (2.9) *is the smallest $\lambda \in [0, +\infty]$ such that*

$$\int_A f(x, \nabla u(x))\, dx + c\mathcal{H}^{N-1}(S_u \cap A) \leq \int_A f(x, \nabla v(x))\, dx + c\mathcal{H}^{N-1}(S_v \cap A) + \lambda$$

for any $v \in SBV_{loc}(\Omega)$ such that $\operatorname{supp}(v-u) \subset\subset A \subset\subset \Omega$.

The deviation from minimality estimates how far is u from being a minimizer. Obviously $\operatorname{Dev}(u, c, \Omega) = 0$ iff u is a local minimizer.

Let B be a ball in $\boldsymbol{R}^N$ with $N \geq 2$. If $u: B \to \boldsymbol{R}$ is measurable we can define (see [DGCL])

$$u_*(s, B) = \inf \{t \in \boldsymbol{R} : |\{u < t\} \cap B| \geq s\}, \qquad 0 \leq s \leq |B|$$

and the median of u in B

$$\operatorname{med}(u, B) = u_*\left(\frac{1}{2}|B|, B\right).$$

If u belongs to $SBV(B)$ and $(2\gamma_N \mathcal{H}^{N-1}(S_u \cap B))^{N/(N-1)} < \frac{1}{2}|B|$ we set, (see [DGCL]),

$$\tau'(u, B) = u_*((2\gamma_N \mathcal{H}^{N-1}(S_u \cap B))^{N/(N-1)}, B),$$

$$\tau''(u, B) = u_*(|B| - (2\gamma_N \mathcal{H}^{N-1}(S_u \cap B))^{N/(N-1)}, B),$$

where γ_N is the isoperimetric constant relative to the balls of $\boldsymbol{R}^N$. In the following if $u \in SBV(B)$ $\overline{u}$ stands for $(u \wedge \tau''(u, B)) \vee \tau'(u, B)$.

We also recall that if $u \in SBV(\Omega)$ then u belongs to $W^{1,p}(\Omega)$, with $p \geq 1$, iff $\mathcal{H}^{N-1}(S_u \cap \Omega) = 0$ and $\int_\Omega (|u|^p + |\nabla u|^p) < +\infty$.

The following proposition is a consequence of the Poincaré inequality, (see [DGCL]), and of the compactness theorem in SBV (see [A1] or [A2]). The proof is essentially contained in Theorem 3.5 and Remark 3.2 in [DGCL], (see also Theorem 2.6 in [CL]).

PROPOSITION 2.5. – *Let $B \subset \boldsymbol{R}^N$ be a ball, $f: B \times \boldsymbol{R}^N \to [0, +\infty[$ be a function verifying* (2.1) *and* (2.2) *and $\{u_h\}$ be a sequence in $SBV(B)$ such that*

$$\sup_{h \in N} \int_B f(x, \nabla u_h(x))\, dx < +\infty, \qquad \lim_h \mathcal{H}^{N-1}(S_{u_h} \cap B) = 0,$$

and let m_h be the medians of u_h in B. Then there exist a subsequence $\{u_{h_k}\}$ and a function $u \in W^{1,p}(B)$ such that

$$\overline{u}_{h_k} - m_{h_k} \to u \quad in\ L^p(B)$$

and

$$\int_B f(x, \nabla u(x))\, dx \leq \liminf_k \int_B f(x, \nabla \overline{u}_{h_k}(x))\, dx.$$

PROOF. – For simplicity we assume that $1 < p < N$, the case $p \geq N$ can be dealt in a similar way taking into account Remark 3.3 in [DGCL]. From the Poincaré inequality and from the assumptions we get

$$\|\overline{u}_h - m_h\|_{p^*} \leq c(N, p)\|\nabla u_h\|_p \leq c\left(\int_B f(x, \nabla u_h(x))\, dx\right)^{1/p}.$$

Moreover

$$\|\nabla(\overline{u}_h - m_h)\|_p \leq \|\nabla u_h\|_p$$

and

$$\mathcal{H}^{N-1}(S_{\overline{u}_h} \cap B) \leq \mathcal{H}^{N-1}(S_{u_h} \cap B).$$

Then there exists a subsequence $\overline{u}_{h_k}$ such that $\overline{u}_{h_k} - m_{h_k}$ converges strongly in $L^p(B)$ to a function u belonging to $GSBV(\Omega)$, i.e. a function such that $u^M \in SBV(\Omega)$ for every M, where u^M is the truncated function at level M. For every M let $(\overline{u}_{h_k} - m_{h_k})^M$ be the truncated function of $\overline{u}_{h_k} - m_{h_k}$ at level M. The compactness theorem implies that $(\overline{u}_{h_k} - m_{h_k})^M \to u^M$ strongly in $L^p(B)$ and that $\nabla(\overline{u}_{h_k} - m_{h_k})^M \to \nabla u^M$ weakly in $L^p(B)$. Finally we get

$$\int_B f(x, \nabla u^M)\, dx \leq \liminf_k \int_B f(x, \nabla((\overline{u}_{h_k} - m_{h_k})^M)\, dx \leq \liminf_k \int_B f(x, \nabla \overline{u}_{h_k})$$

and

$$\mathcal{H}^{N-1}(S_{u^M} \cap B) \leq \liminf_k \mathcal{H}^{N-1}(S_{(\overline{u}_{h_k} - m_{h_k})^M} \cap B) \leq \liminf_k \mathcal{H}^{N-1}(S_{u_{h_k}} \cap B) = 0\,.$$

Then we deduce that u^M belongs to $W^{1,p}(B)$ and that ∇u^M is equibounded in $L^p(B)$. If $M \to +\infty$ we deduce that u belongs to $W^{1,p}(B)$ and the thesis follows. ∎

The next theorem describes the limit behaviour of a sequence $\{u_h\}$ in SBV when the deviations from minimality and the area terms $\mathcal{H}^{N-1}(S_{u_h})$ go to zero.

THEOREM 2.6. – *Let $B_r \subset \mathbf{R}^N$ be a ball centered at the origin with radius r, $f: B \times \mathbf{R}^N \to [0, +\infty[$ be a function verifying* (2.1) *and* (2.2), *$\{u_h\} \subset SBV(B_r)$, m_h be the medians of u_h and $\{c_h\} \subset (0, +\infty)$. Assume that*

(i) $\sup_{h \in N} F(u_h, c_h, B_r) < +\infty$,

(ii) $\lim_h \mathcal{H}^{N-1}(S_{u_h} \cap B_r) = 0$,

(iii) $\lim_h \mathrm{Dev}(u_h, c_h, B_r) = 0$.

We also assume that

$$u_h(x) - m_h \to u(x) \in W^{1,p}(B_r) \qquad a.e.\ in\ B_r\,.$$

Then u is a local minimizer of the functional $\int_{B_r} f(x, \nabla v(x))\, dx$ in $W^{1,p}(B_r)$ and

$$\lim_h F(u_h, c_h, B_\varrho) = \int_{B_\varrho} f(x, \nabla u(x))\, dx\,, \qquad \forall \varrho \in (0, r)\,.$$

PROOF. – The proof is essentially contained in Theorem 4.8 in [DGCL] and Theorem 3.11 in [FF], where instead of (ii) it is assumed that $c_h \to +\infty$. However what is really needed for the proof is that

$$\liminf_h c_h (\mathcal{H}^{N-1}(S_{u_h} \cap B_r))^{N/(N-1)} = 0\,. \tag{2.10}$$

This is clear if $\liminf_h c_h < +\infty$, while, if $\liminf_h c_h = +\infty$, (2.10) follows from assumption (i) and from the fact that $\mathcal{H}^{N-1}(S_{u_h} \cap B_r)$ is infinitesimal. ∎

3. – The Decay Lemma.

In this section we are going to prove a lemma which estimates the decay of the functional F in small balls. We assume that the integrand $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ satisfies assumption (2.1), (2.2), (2.3) and (2.4). Moreover we assume that

$$f(x, tz) = t^p f(x, z) \quad \text{for a.e. } x \in \Omega\,; \quad \forall z \in \mathbf{R}^N; \quad \forall t > 1\,. \tag{3.1}$$

We first recall a regularity result proved in [CFP].

THEOREM 3.1. – *Let f satisfy* (2.1), (2.2), (2.3) *and* (2.4). *For every $\alpha \in (0, 1)$ there exists a constant c_α, depending only on $N, p, L, \nu, \omega, \alpha$, such that if u is a local minimizer of the functional $\int_\Omega f(x, \nabla w(x))\,dx$ in $W^{1,p}_{\text{loc}}(\Omega)$, for every $B_R(x_0) \subset\subset \Omega$ and for every $0 < \varrho < R$*

$$\int_{B_\varrho(x_0)} |\nabla u|^p\,dx \le c_\alpha \left(\frac{\varrho}{R}\right)^{N-p+\alpha p} \int_{B_R(x_0)} |\nabla u|^p\,dx\,.$$

REMARK 3.2 (Scaling). – If $u \in SBV(\Omega)$, $B_\varrho(x_0) \subset \Omega$, it can be easily checked that, the rescaled function

$$u_\varrho(y) = \varrho^{(1-p)/p} u(x_0 + \varrho y)$$

belongs to $SBV(\Omega_\varrho)$ where $\Omega_\varrho = \varrho^{-1}(\Omega - x_0)$, and that

$$\mathcal{H}^{N-1}(S_{u_\varrho} \cap B_\sigma) = \varrho^{1-N}\mathcal{H}^{N-1}(S_u \cap B_{\sigma\varrho}(x_0)) \quad \text{for } 0 < \sigma \le 1\,.$$

Moreover if f verifies assumption (3.1), then

$$\int_{B_\sigma} f(x_0 + \varrho y, \nabla u_\varrho(y))\,dy = \varrho^{1-N} \int_{B_{\sigma\varrho}(x_0)} f(x, \nabla u(x))\,dx$$

and

$$\operatorname{Dev}(u_\varrho, c, B_\sigma) = \varrho^{1-N}\operatorname{Dev}(u, c, B_{\sigma\varrho}(x_0))\,.$$

These scaling properties allow us, with a typical blow-up argument, to prove the following Decay Lemma:

LEMMA 3.3 (Decay). – *Let $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ be a function verifying assumptions* (2.1), (2.2), (2.3), (2.4) *and* (3.1). *Let α be a given number in $(0, 1)$ and $\Omega' \subset\subset \Omega$. For every $c > 0$ and $0 < \tau < 1$ there exist $\theta = \theta(\alpha, c, \tau, \Omega')$ and $\varepsilon = \varepsilon(\alpha, c, \tau, \Omega')$ such that if $x \in \Omega'$, $0 < R < 1/2\operatorname{dist}(\Omega', \partial\Omega)$, $u \in SBV_{\text{loc}}(\Omega)$ and*

$$F(u, c, B_R(x)) \le \varepsilon R^{N-1}\,, \qquad \operatorname{Dev}(u, c, B_R(x)) \le \theta F(u, c, B_R(x))\,,$$

then

$$F(u, c, B_{\tau R}(x)) \leq (Lc_\alpha + 1)\, \tau^{N-p+\alpha p} F(u, c, B_R(x)),$$

where c_α *is the constant appearing in Theorem* 3.1.

PROOF. – Let us fix $c > 0$, $0 < \tau < 1$ and $0 < \alpha < 1$ and argue by contradiction.

If the decay property is not true there will exist two sequences $\{\varepsilon_h\}$, $\{\theta_h\}$ such that $\lim_h \varepsilon_h = \lim_h \theta_h = 0$, functions $u_h \in SBV_{\mathrm{loc}}(\Omega)$ and balls $B_{r_h}(x_h) \subset \Omega$ with $x_h \in \Omega'$ and $0 < r_h < 1/2 \operatorname{dist}(\Omega', \partial\Omega)$, such that

$$F(u_h, c, B_{r_h}(x_h)) = \varepsilon_h r_h^{N-1}, \qquad \operatorname{Dev}(u_h, c, B_{r_h}(x_h)) = \theta_h F(u_h, c, B_{r_h}(x_h))$$

and

$$F(u_h, c, B_{\tau r_h}(x_h)) > (Lc_\alpha + 1)\, \tau^{N-p+\alpha p} F(u_h, c, B_{r_h}(x_h)).$$

Up to a subsequence we may assume that $\lim_h x_h = x_0 \in \overline{\Omega}'$ and $\lim_h r_h = r_0 \leq 1/2 \operatorname{dist}(\Omega', \partial\Omega)$.

For every $h \in N$ and $y \in B_1$, we set

$$v_h(y) = r_h^{(1-p)/p} \varepsilon_h^{-1/p} u_h(x_h + r_h y).$$

Moreover we set

$$F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) = \int_{B_1} f(x_0 + r_0 y, \nabla v_h(y))\, dy + \frac{c}{\varepsilon_h} \mathcal{H}^{N-1}(S_{v_h} \cap B_1),$$

$$F_h\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) = \int_{B_1} f(x_h + r_h y, \nabla_y v_h(y))\, dy + \frac{c}{\varepsilon_h} \mathcal{H}^{N-1}(S_{v_h} \cap B_1),$$

while the symbols $\operatorname{Dev}_{0,h}$ and Dev_h will denote the deviation from the minimality relative to the functionals $F_{0,h}$ and F_h respectively. From Remark 3.2 we easily obtain that

$$F_h\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) = 1, \qquad \operatorname{Dev}_h\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) = \theta_h$$

and that

$$F_h\left(v_h, \frac{c}{\varepsilon_h}, B_\tau\right) > (Lc_\alpha + 1)\, \tau^{N-p+\alpha p}. \tag{3.2}$$

Notice that if $w \in SBV(B_1)$, $\operatorname{supp}(w - v_h) \subset B_1$ and

$$F_{0,h}\left(w, \frac{c}{\varepsilon_h}, B_1\right) \leq F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_1\right), \tag{3.3}$$

by (2.4) we have

$$(3.4)\qquad F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) \leq F_h\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) + \omega_h \int_{B_1} |\nabla v_h|^p \leq$$

$$\leq F_h\left(w, \frac{c}{\varepsilon_h}, B_1\right) + \theta_h + \omega_h \int_{B_1} |\nabla v_h|^p \leq$$

$$\leq F_{0,h}\left(w, \frac{c}{\varepsilon_h}, B_1\right) + \theta_h + \omega_h \left(\int_{B_1} |\nabla v_h|^p + \int_{B_1} |\nabla w|^p \right),$$

where $\omega_h = \omega(|x_h - x_0| + |r_h - r_0|)$. Assumption (2.2) implies that

$$(3.5)\qquad \int_{B_1} |\nabla v_h|^p \leq F_h\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) = 1$$

then (2.2), (3.3), the first inequality in (3.4) and (3.5) imply that

$$\int_{B_1} |\nabla w|^p \leq 1 + \omega_h .$$

Then we have

$$F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) \leq F_{0,h}\left(w, \frac{c}{\varepsilon_h}, B_1\right) + \theta_h + \omega_h(2 + \omega_h).$$

If (3.3) does not hold the above inequality is trivial. In conclusion we have proved that

$$\mathrm{Dev}_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_1\right) \leq \omega_h(2 + \omega_h) + \theta_h .$$

Notice also that since $c/\varepsilon_h \to +\infty$ as $h \to +\infty$ we have that $\lim_h \mathcal{H}^{N-1}(S_{v_h} \cap B_1) = 0$. Thus, by Proposition 2.5 and Theorem 2.6, we deduce that there exists $v \in W^{1,p}(B_1)$ local minimizer of $\int_{B_1} f(x_0 + r_0 y, \nabla w(y))\, dy$ such that

$$\int_{B_\varrho} f(x_0 + r_0 y, \nabla v(y))\, dy = \lim_h F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_\varrho\right) \leq 1 , \qquad \forall 0 < \varrho < 1 .$$

Finally Theorem 3.1 implies that there exists $c_\alpha > 0$ such that

$$(3.6)\qquad \int_{B_\varrho} |\nabla v|^p \leq c_\alpha \varrho^{N-p+\alpha p} \int_{B_1} |\nabla v|^p \leq c_\alpha \varrho^{N-p+\alpha p} .$$

Assumption (2.2) and inequality (3.6) imply

$$\lim_h F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_\tau\right) = \int_{B_\tau} f(x_0 + r_0 y, \nabla v(y))\, dy \le L c_\alpha \tau^{N-p+\alpha p},$$

hence, since

$$F_h\left(v_h, \frac{c}{\varepsilon_h}, B_\tau\right) \le F_{0,h}\left(v_h, \frac{c}{\varepsilon_h}, B_\tau\right) + \omega_h,$$

we get a contradiction with (3.2), thus proving the desired estimate. ■

4. – The density lower bound.

In order to prove the existence of minimizers of the functional $\mathcal{G}$ let us introduce the following relaxed functional defined on SBV:

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, \nabla u(x))\, dx + \alpha \int_\Omega |u - g|^q\, dx + \mathcal{H}^{N-1}(S_u \cap \Omega)$$

where $g \in L^\infty(\Omega)$ and $q \ge 1$.

If we denote by $u^M = -M \vee (u \wedge M)$ the truncate of u at the level $M = \|g\|_{L^\infty}$, it results that u^M belongs to the space $SBV_{\text{loc}}(\Omega)$, $S_{u^M} \subset S_u$ and $\nabla u^M = \nabla u \chi_{\{|u| < M|\}}$ (see [ADM]); thus

$$\mathcal{F}(u^M, \Omega) \le \mathcal{F}(u, \Omega).$$

It is then clear that, in order to minimize $\mathcal{F}$, we may restrict to those functions u such that $\|u\|_{L^\infty} \le \|g\|_{L^\infty}$. The existence of minimizers for $\mathcal{F}$ follows from compactness and lower semicontinuity results contained in [A1].

THEOREM 4.1. – *If* $\alpha > 0$, $g \in L^\infty(\Omega)$ *and* f *is a continuous function verifying* (2.1), (2.2), *there exists* $u \in SBV_{\text{loc}}(\Omega)$ *minimizer of* $\mathcal{F}$. *Moreover* $\|u\|_{L^\infty} \le \|g\|_{L^\infty}$.

REMARK 4.2. – If u is a minimizer of $\mathcal{F}$, $B_\varrho(x) \subset \Omega$ and $v \in SBV(B_\varrho(x))$ is such that $\operatorname{supp}(v - u) \subset\subset B_\varrho(x)$ then

$$\mathcal{F}(u, B_\varrho(x)) \le \mathcal{F}(v^M, B_\varrho(x))$$

and so

$$\int_{B_\varrho(x)} f(x, \nabla u(x))\, dx + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \le$$

$$\le \int_{B_\varrho(x)} f(x, \nabla v(x))\, dx + \mathcal{H}^{N-1}(S_v \cap B_\varrho(x)) + 2^q \alpha \omega_n \varrho^N \|g\|_{L^\infty},$$

then

$$\mathrm{Dev}(u, B_\varrho(x)) \leq 2^q a\omega_N \varrho^N \|g\|_{L^\infty}$$

for all balls $B_\varrho(x) \subset \Omega$.

Let us now recall the definition of quasi minimizer of the functional $F(v, \Omega)$ (see [AP] or [AFP]).

DEFINITION 4.3 (quasi minimizers). – *A function $u \in SBV_{\mathrm{loc}}(\Omega)$ is a quasi minimizer of the functional $F(v, \Omega)$ if there exists $\omega \geq 0$ such that*

$$\mathrm{Dev}(u, B_\varrho(x)) \leq \omega\varrho^s$$

for some $s \in (N-1, N]$.

We will denote by $M_{s,\omega}(\Omega)$ the class of all quasi minimizer verifying the above inequality.

LEMMA 4.4 (Energy upper bound). – *Let $f\colon \Omega \times \boldsymbol{R}^N \to [0, +\infty[$ be a function verifying assumptions* (2.1) *and* (2.2). *If $u \in M_{s,\omega}(\Omega)$ then, for every ball $B_\varrho(x) \subset \Omega$ it results*

$$\int_{B_\varrho(x)} f(y, \nabla u(y))\,dy + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \leq N\omega_N \varrho^{N-1} + \omega\varrho^s .$$

PROOF. – Let us fix $\varrho' < \varrho$ and let us consider the function $w(y) = u(y)\,\chi_{B_\varrho(x)\setminus B_{\varrho'}(x)}$.

The quasi minimality of u implies that

$$F(u, B_\varrho(x)) \leq F(w, B_\varrho(x)) + \omega\varrho^s ,$$

hence

$$\int_{B_{\varrho'}(x)} f(y, \nabla u(y))dy + \mathcal{H}^{N-1}(S_u \cap B_{\varrho'}(x)) \leq N\omega_N \varrho^{N-1} + \omega\varrho^s ;$$

if $\varrho' \uparrow \varrho$ we get the thesis. ∎

In the following if $u \in SBV_{\mathrm{loc}}(\Omega)$ we set

$$F_p(u, E) = \int_E f_p(x, \nabla u(x))\,dx + \mathcal{H}^{N-1}(S_u \cap E)$$

for every Borel set $E \subset \Omega$, f_p being the p recession function of f defined in Section 2, and we set $\mathrm{Dev}_p(u, E)$ for the deviation from minimality of u with respect to the functional F_p.

Let us now prove two lemmas.

LEMMA 4.5. – *Let* $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ *be a function verifying assumptions* (2.1), (2.2), *and* (2.5). *Let* $B_\varrho(x) \subset \Omega$, *then for every* $\beta > 0$ *there exists a constant* $\bar{c} = \bar{c}(\beta, N)$ *such that for all* $u \in SBV(B_\varrho(x))$ *it results*

$$F_p(u, B_\varrho(x)) \leq (1+\beta) F(u, B_\varrho(x)) + \bar{c}\varrho^N$$

and

$$F(u, B_\varrho(x)) \leq (1+\beta) F_p(u, B_\varrho(x)) + \bar{c}\varrho^N .$$

PROOF. – Let $u \in SBV(B_\varrho(x))$ and let $\gamma > 0$. We have

$$F_p(u, B_\varrho(x)) = F(u, B_\varrho(x)) + \int_{B_\varrho(x)} [f_p(y, \nabla u(y)) - f(y, \nabla u(y))]\, dy ,$$

assumption (2.5) implies

$$F_p(u, B_\varrho(x)) \leq F(u, B_\varrho(x)) + c_0 \int_{B_\varrho(x)} |\nabla u|^{p-m} + c_0 \omega_N \varrho^N .$$

Hölder and Young inequalities imply that there exists $\tilde{c} = \tilde{c}(\gamma)$ such that

$$\int_{B_\varrho(x)} |\nabla u|^{p-m} \leq \gamma \int_{B_\varrho(x)} |\nabla u|^p + N\omega_N \tilde{c}(\gamma)\, \varrho^N .$$

Then

$$F_p(u, B_\varrho(x)) \leq F(u, B_\varrho(x)) + c_0 \gamma \int_{B_\varrho(x)} |\nabla u|^p + c_0 \tilde{c}(\gamma)\, N\omega_N \varrho^N + c_0 \omega_N \varrho^N$$

by (2.2) we get

$$F_p(u, B_\varrho(x)) \leq (1 + c_0\gamma) F(u, B_\varrho(x)) + \bar{c}(\gamma, N)\, \varrho^N$$

setting $\beta = c_0 \gamma$ the first estimate is proved. The second one follows similarly. ∎

LEMMA 4.6. – *Let* $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ *be a function verifying assumptions* (2.1), (2.2), *and* (2.5). *Let* $B_\varrho(x) \subset \Omega$. *If* $u \in M_{s,\omega}(\Omega)$, *then for every* $\beta > 0$ *there exists a constant* $\tilde{c} = \tilde{c}(\beta, N)$ *such that*

$$\mathrm{Dev}_p(u, B_\varrho(x)) \leq \beta(2+\beta) F_p(u, B_\varrho(x)) + \tilde{c}\varrho^N + (1+\beta)\,\omega\varrho^s .$$

PROOF. – Let $v \in SBV(\Omega)$ such that $\mathrm{supp}(v-u) \subset B_\varrho(x)$. If $F_p(v, B_\varrho(x)) \leq F_p(u, B_\varrho(x))$ given $\beta > 0$, Lemma 4.5 and the quasi minimality of u imply that there exists $\bar{c} = \bar{c}(\beta, N)$ such that

$$\begin{aligned} F_p(u, B_\varrho(x)) &\leq (1+\beta) F(u, B_\varrho(x)) + \bar{c}\varrho^N \leq \\ &\leq (1+\beta) F(v, B_\varrho(x)) + (1+\beta)\,\omega\varrho^s + \bar{c}\varrho^N \leq \\ &\leq (1+\beta)^2 F_p(v, B_\varrho(x)) + \bar{c}(1+\beta)\,\varrho^N + (1+\beta)\,\omega\varrho^s + \bar{c}\varrho^N . \end{aligned}$$

If $F_p(v, B_\varrho(x)) \geq F_p(u, B_\varrho(x))$ the above inequality is trivial.

Therefore setting $\tilde{c} = \bar{c}(2+\beta)$, we have

$$\mathrm{Dev}_p(u, B_\varrho(x)) \le \beta(2+\beta)\, F_p(u, B_\varrho(x)) + \tilde{c}\varrho^N + (1+\beta)\,\omega\varrho^s .$$

LEMMA 4.7 (Density lower bound). – *Let $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ be a function verifying asumptions* (2.1), ..., (2.5). *Let $\Omega' \subset\subset \Omega$; then there exist θ_0 and ϱ_0 depending on N, p, L, ν, Ω' such that if $u \in M_{s,\omega}(\Omega)$ then*

$$F_p(u, B_\varrho(x)) > \theta_0 \varrho^{N-1}$$

for all balls $B_\varrho(x) \subset\subset \Omega' \subset\subset \Omega$, with center $x \in \overline{S}_u \cap \Omega'$ and radius $\varrho < \varrho_0$.

PROOF. – Let us fix $\alpha \in (0, 1)$ and $\delta > 0$ such that $1 - \delta - p(1-\alpha) > 0$ and such that $s - N + 1 > \delta > 0$.

Let $\tau \in (0, 1)$ such that $(Lc_\alpha + 1)\tau^{1-\delta-p(1-\alpha)} < 1$, where c_α is the constant given in Theorem 3.1. We set $\varepsilon_0 = \varepsilon(\alpha, \tau, \Omega')$ and $\theta = \theta(\alpha, \tau, \Omega')$ where $\varepsilon(\alpha, \tau, \Omega')$ and $\theta(\alpha, \tau, \Omega')$ are as in Decay Lemma. Finally we fix $\beta = \beta(\alpha, \tau, \Omega') > 0$ such that $(\beta(2+\beta))/\theta < 1/2$ and $\varrho_0 > 0$ such that

$$\varrho_0 < \min\left\{ \frac{\theta\varepsilon_0}{4\tilde{c}}\, \tau^{N-1+\delta} ; \left(\frac{\theta\varepsilon_0}{4(1+\beta)\,\omega}\, \tau^{N-1+\delta} \right)^{1/(s-N+1)} \right\}$$

where $\tilde{c}$ is the constant given in Lemma 4.6. We can assume that the point x in S_u is the origin.

We claim that if $\varrho < \varrho_0$ and $B_\varrho \subset\subset \Omega' \subset\subset \Omega$ then inequality

$$F_p(u, B_\varrho) \le \varepsilon_0 \varrho^{N-1} \tag{4.1}$$

implies

$$F_p(u, B_{\tau^h\varrho}) \le \varepsilon_0 \tau^{h\delta} (\tau^h \varrho)^{N-1}, \qquad \forall h \in \mathbf{N} . \tag{4.2}$$

Assume now (4.2) true for a given $h > 0$ and assume

$$\mathrm{Dev}_p(u, B_{\tau^h\varrho}) \le \theta F_p(u, B_{\tau^h\varrho}), \tag{4.3}$$

then the Decay Lemma implies

$$\begin{aligned} F_p(u, B_{\tau^{h+1}\varrho}) &\le (Lc_\alpha + 1)\, \tau^{N-p+\alpha p} F_p(u, B_{\tau^h\varrho}) \le \\ &\le \varepsilon_0 (Lc_\alpha + 1)\, \tau^{N-p+\alpha p} \tau^{h\delta} (\tau^h \varrho)^{N-1} = \\ &= \varepsilon_0 (Lc_\alpha + 1)\, \tau^{\alpha p - p - \delta + 1} \tau^{(h+1)\delta} (\tau^{h+1}\varrho)^{N-1} \le \varepsilon_0 \tau^{(h+1)\delta} (\tau^{h+1}\varrho)^{N-1} . \end{aligned}$$

If (4.3) is not true by Lemma 4.6 we have

$$F_p(u, B_{\tau^h\varrho}) \le \frac{\mathrm{Dev}_p(u, B_{\tau^h\varrho})}{\theta} \le \frac{\beta(2+\beta)}{\theta} F_p(u, B_{\tau^h\varrho}) + \frac{\tilde{c}}{\theta} (\tau^h \varrho)^N + \frac{(1+\beta)\,\omega}{\theta} (\tau^h\varrho)^s$$

so that

$$F_p(u, B_{\tau^h \varrho}) \leq \frac{2\tilde{c}}{\theta} (\tau^h \varrho)^N + \frac{2(1+\beta)\,\omega}{\theta} (\tau^h \varrho)^s$$

and thus

$$F_p(u, B_{\tau^{h+1}\varrho}) \leq \varepsilon_0 (\tau^{h+1}\varrho)^{N-1} \tau^{(h+1)\delta} \times$$

$$\times \left[\frac{2\tilde{c}}{\theta\varepsilon_0} \varrho \tau^{h(1-\delta)} \tau^{-(N-1+\delta)} + \frac{2(1+\beta)\,\omega}{\theta\varepsilon_0} \varrho^{s-N+1} \tau^{h(s-N+1-\delta)} \tau^{-(N-1+\delta)} \right] \leq$$

$$\leq \varepsilon_0 (\tau^{(h+1)}\varrho)^{N-1} \tau^{(h+1)\delta} \left[\frac{2\tilde{c}}{\theta\varepsilon_0} \varrho \tau^{-(N-1+\delta)} + \frac{2(1+\beta)\,\omega}{\theta\varepsilon_0} \varrho^{s-N+1} \tau^{-(N-1+\delta)} \right] \leq$$

$$\leq \varepsilon_0 (\tau^{h+1}\varrho)^{N-1} \tau^{(h+1)\delta} .$$

This proves (4.2). If $F_p(u, B_\varrho) \leq \varepsilon \varrho^{N-1}$ for some $\varrho < \varrho_0$ then

$$\lim_{\varrho \to 0} \varrho^{1-N} F_p(u, B_\varrho) = 0$$

and Lemma 2.6 and Theorem 3.6 in [DGCL] imply that $0 \notin S_u$ so we fall in a contradiction. This proves the thesis for every $x \in S_u \cap \Omega'$, and by a density argument we get the thesis for every $x \in \overline{S}_u \cap \Omega'$. ■

PROPOSITION 4.8. – *Let* $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ *be a function verifying* (2.1), (2.2), (2.3), (2.4), (2.5) *and let* $u \in M_{s,\omega}(\Omega)$. *Then*

$$\mathcal{H}^{N-1}(\Omega \cap \overline{S}_u \setminus S_u) = 0 .$$

PROOF. – Let $\Omega' \subset\subset \Omega$ and let $x \in \overline{S}_u \cap \Omega'$. Lemma 4.7 and assumptions (2.2) imply that

$$\liminf_{\varrho \downarrow 0} \left[\int_{B_\varrho(x)} |\nabla u|^p + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \right] \varrho^{1-N} > 0 .$$

Setting

$$\Omega'_0 = \left\{ x \in \Omega' : \liminf_{\varrho \downarrow 0} \left[\int_{B_\varrho(x)} |\nabla u|^p + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \right] \varrho^{1-N} = 0 \right\}$$

we have

$$\Omega' \cap \overline{S}_u \subset \Omega' \setminus \Omega'_0$$

and then

$$(\Omega' \cap \overline{S}_u)\backslash S_u \subset (\Omega' \backslash \Omega'_0)\backslash S_u \subset (\Omega' \backslash S_u)\backslash \Omega'_0 .$$

Being $\mathcal{H}^{N-1}((\Omega'\backslash S_u)\backslash \Omega'_0) = 0$ (see Lemma 2.6 in [DGCL]), we get

$$\mathcal{H}^{N-1}((\overline{S}_u \cap \Omega')\backslash S_u) = 0 .$$

Letting $\Omega' \uparrow \Omega$ we get the thesis. ■

THEOREM 4.9. – *Let* $f\colon \Omega \times \mathbf{R}^N \to [0, +\infty[$ *be a function verifying* (2.1), (2.2), (2.3), (2.4), (2.5); $\alpha > 0$; $g \in L^\infty(\Omega)$. *If* $u \in SBV_{\rm loc}(\Omega)$ *is a minimizer of* $\mathcal{F}(u, \Omega)$, *then the pair* $(\overline{S}_u, u)$ *is a minimizer of* $\mathcal{G}$, *i.e.*

$$\mathcal{G}(\overline{S}_u, u) \leq \mathcal{G}(K, v)$$

for any closed set $K \subset \mathbf{R}^N$ *and any* $v \in W^{1,p}_{\rm loc}(\Omega\backslash K)$.

PROOF. – If u is a minimizer of $\mathcal{F}$ in Ω then

$$\mathcal{F}(u, \Omega) \leq \mathcal{F}(0, \Omega) = \|g\|_q^q < +\infty$$

so $|\nabla u| \in L^p(\Omega)$ and thus u belongs to $W^{1,p}_{\rm loc}(\Omega\backslash \overline{S}_u)$. Let now v be a function belonging to $W^{1,p}_{\rm loc}(\Omega\backslash K)$ such that $\mathcal{G}(v, K) < +\infty$. We can assume that v is bounded, v belongs to $SBV_{\rm loc}(\Omega)$ and $\overline{S}_v \subset K$, (see Lemma 3.2 in [FF] or Lemma 2.3 in [DGCL]). From the minimality of u and Proposition 4.8 we get

$$\mathcal{G}(\overline{S}_u, u) = \mathcal{F}(u, \Omega) \leq \mathcal{F}(v, \Omega) \leq \mathcal{G}(K, v),$$

so the theorem follows. ■

REMARK 4.10. – Notice that by Theorem 4.9 and Lemma 4.4 it follows that if $1 < p < N$ and if u is a minimizer of $\mathcal{F}(u, \Omega)$ then, for every $x_0 \in \Omega\backslash \overline{S}_u$, $\int_{B_\varrho(x_0)} |\nabla u|^p$ decays like ϱ^{N-1} as ϱ goes to zero. Therefore we may deduce, by the classical Morrey estimates, that $u \in C^{0,(p-1)/p}_{\rm loc}(\Omega\backslash \overline{S}_u)$.

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