Annali di Matematica pura ed applicata
(IV), Vol. CLXXVII (1999), pp. 143-172

Existence Results for Nonlinear Parabolic Equations via Strong Convergence of Truncations (*).

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Summary. – *We prove existence results for the initial-boundary value problem for parabolic equations of the type*

$$\begin{cases} u_t + A(u) + g(x, t, u)\,|\nabla u|^2 = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\boldsymbol{R}^N$ and $T > 0$, A is a pseudomonotone operator of Leray-Lions type defined in $L^2(0, T; H_0^1(\Omega))$, f belongs to $L^1(Q)$, u_0 is in $L^1(\Omega)$ and $g(x, t, s)$ is only assumed to be a Carathéodory function satisfying a sign condition. The result is achieved by proving the strong convergence in $L^2(0, T; H_0^1(\Omega))$ of trucations of solutions of approximating problems with L^1 converging data. To underline the importance of this tool, we show how it can be used for getting other existence theorems, dealing in particular with the following class of Cauchy-Dirichlet problems:

$$\begin{cases} u_t + A(u) = f + \operatorname{div}(\Phi(u)) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Phi \in C^0(\boldsymbol{R}, \boldsymbol{R}^N)$, and the data f and u_0 are still taken in $L^1(Q)$ and $L^1(\Omega)$ respectively.

1. – Introduction.

In this paper we deal with a class of nonlinear parabolic equations in a cylinder $Q = \Omega \times (0, T)$, where Ω is a bounded open subset of $\boldsymbol{R}^N$ and $T > 0$, whose simplest

(*) Entrata in Redazione il 2 aprile 1998.
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model is the following Cauchy-Dirichlet problem:

$$(1.1)\qquad \begin{cases} u_t - \Delta u + g(x, t, u)\,|\nabla u|^2 = f & \text{in } Q\,, \\ u = 0 & \text{on } \Sigma\,, \\ u(x, 0) = u_0(x) & \text{in } \Omega\,, \end{cases}$$

where Σ denotes the lateral surface of Q and $g\colon Q \times \boldsymbol{R} \to \boldsymbol{R}$ is a Carathéodory function satisfying the sign condition

$$(1.2)\qquad g(x, t, s)\, s \geq 0\,, \quad \forall s \in \boldsymbol{R}\,, \quad \text{a.e. } (x, t) \in Q\,.$$

Under this assumption on g, if $u_0 = 0$ the existence of a weak solution u of (1.1) belonging to $L^2(0, T; H_0^1(\Omega))$ was proved first in [LaMu] if f is in the dual space $L^2(0, T; H^{-1}(\Omega))$, then in [DO] if f belongs to $L^1(Q)$ and g satisfies the following additional hypothesis:

$$(1.3)\qquad \exists \delta,\ \sigma > 0\colon\quad g(x, t, s)\operatorname{sign}(s) \geq \delta > 0\,,\ \forall s \in \boldsymbol{R}\colon\ |s| \geq \sigma\,,\ \text{and a.e. } (x, t) \text{ in } Q\,.$$

In this latter paper, (1.3) plays a fundamental role since it allows to find *a priori* estimates in $L^2(0, T; H_0^1(\Omega))$ even with L^1 data (in the stationary case this was pointed out in [BG2]), hence a compactness result for approximating solutions provides the desired existence theorem. Note however that several examples of g satisfy (1.2) but not (1.3), mainly all functions $g(x, t, s)$ with a sign condition and such that $g(x, t, s)$ tends to zero as s tends to infinity, but also for istance $g(x, t, s) = s(\sin s)^2$ and similar oscillating functions are not included in (1.3).

Here we extend the results in [DO] in two different directions. First of all, assuming only (1.2), for every f in $L^1(Q)$ and u_0 in $L^1(\Omega)$ we prove the existence of a solution of (1.1), which belongs to $L^q(0, T; W_0^{1,q}(\Omega))$ for some $q < 2$ (indeed, it enjoys the same regularity of solutions of equations with measure data as it is stated in [BDGO]). Secondly, we show that if (1.3) holds true it can be found a solution in $L^2(0, T; H_0^1(\Omega))$ for every u_0 in $L^1(\Omega)$, and that this condition on the initial datum can not be weakened. In order words, we state that the problem

$$\begin{cases} u_t - \Delta u + g(x, t, u)\,|\nabla u|^2 = f & \text{in } Q\,, \\ u = 0 & \text{on } \Sigma\,, \\ u(x, 0) = \lambda & \text{in } \Omega\,, \end{cases}$$

with f in $L^1(Q)$, admits no solution u in $L^2(0, T; H_0^1(\Omega))$ such that $g(x, t, u)\,|\nabla u|^2$ belongs to $L^1(Q)$ if λ is singular with respect to Lebesgue measure.

The main point which allows to go further the previous works, in the sense that (1.3) is not essential to us, is the proof of a compactness result for the truncations of solutions of approximating problems with L^1-converging right hand sides, without an *a priori* bound in the space $L^2(0, T; H_0^1(\Omega))$. In order to underline the importance of this tool, we have chosen to plan the paper in the following way: in Section 2 we prove a first compactness theorem for simpler equations which do not contain the lower order term $g(x, t, u)\,|\nabla u|^2$; in this context it appears as a different proof of a result previously obtained in [Bl] (for elliptic equations see [Mu], [LM] and [LP]), and we show that the same method applies to more general operators in divergence form, in order to find a solution to Cauchy-Dirichlet problems whose model is the following one:

$$\text{(1.4)} \qquad \begin{cases} u_t - \operatorname{div}(A(x,t)(1+|u|)^m \nabla u + \Phi(u)) = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $0 < \alpha \leqslant A(x,t) \leqslant \beta$, $m \geqslant 0$ and Φ is only assumed to be continuous on $\boldsymbol{R}$, and as before f is in $L^1(Q)$ and u_0 in $L^1(\Omega)$. Since no growth assumptions are made on Φ and m, problem (1.4) will be studied in the framework of renormalized solutions, which were first introduced in [DL] in a different context, then used in this setting in [BDGM] for the stationary case and in [BlMu] for evolution equations (see also [Re]).

Section 3 will be devoted to the proof of our main result, concerning problem (1.1) or the more general model

$$\begin{cases} u_t - \operatorname{div}(A(x,t)(1+|u|)^m \nabla u) + g(x,t,u)|\nabla u|^2 = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

while in Section 4 we will prove the following trace result, showing some simple but interesting applications.

THEOREM 1.1. – *Let $p > 1$, p' its conjugate exponent $(1/p + 1/p' = 1)$, $a, b \in \boldsymbol{R}$, and define the space*

$$V_1^p((a,b)) \equiv \{u \colon \Omega \times [a,b] \to \boldsymbol{R} \colon u \in L^p(a,b; W_0^{1,p}(\Omega)),$$
$$u_t \in L^{p'}(a,b; W^{-1,p'}(\Omega)) + L^1(\Omega \times (a,b))\}.$$

Then we have, with continuous injection,

$$V_1^p \subset C([a,b]; L^1(\Omega)). \quad \blacksquare$$

2. – Strong convergence of truncations without lower order terms.

In this section we consider equations which do not contain lower order terms, in order to better show the method we use for proving the strong convergence of truncations. We will henceforth deal with the divergence form operator $A(u) \equiv \equiv -\operatorname{div}(a(x,t,u,\nabla u))$, where $a(x,t,s,\xi) \colon Q \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a Carathéodory function (i.e. it is measurable with respect to (x,t) and continuous with repect to s and ξ) such that:

$$\text{(2.1)} \qquad a(x,t,s,\xi)\cdot\xi \geqslant \alpha|\xi|^2, \qquad \alpha > 0,$$

$$\text{(2.2)} \qquad |a(x,t,s,\xi)| \leqslant b(|s|)[h(x,t) + |\xi|],$$

$$\text{(2.3)} \qquad [a(x,t,s,\xi) - a(x,t,s,\xi')]\cdot[\xi - \xi'] > 0,$$

for almost every (x, t) in Q, for every s in $\mathbf{R}$ and every ξ, ξ' in $\mathbf{R}^N (\xi \neq \xi')$, with $h(x, t)$ in $L^2(Q)$ and

$$b:[0, +\infty) \to (0, +\infty) \qquad \text{continuous}\,.$$

Note that our assumptions include the classical case of the laplacian, but also more general operators given, for istance, by $a(x, t, s, \xi) = (1 + |s|)^m \xi$ with $m \geq 0$. It should also be noticed that assumption (2.1) implies that $a(x, t, s, 0) = 0$ for every s in $\mathbf{R}$ and almost every (x, t) in Q.

Next we define, for $n \in \mathbf{N}$, $a_n(x, t, s, \xi) \equiv a(x, t, T_n(s), \xi)$, where, for every positive real number k, $T_k(s) = \min(k, \max(u, -k))$ denotes the truncation function. Thus, thanks to the continuity of b, we have that $A_n(u) \equiv -\operatorname{div}(a_n(x, t, u, \nabla u))$ is a bounded and coercive operator between the space $L^2(0, T; H_0^1(\Omega))$ and its dual $L^2(0, T; H^{-1}(\Omega))$. Hence the classical theory developed in [L] applies to give a weak solution u_n in $C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ to the initial-boundary value problem

$$\begin{cases} (u_n)_t + A_n(u_n) = f_n & \text{in } Q\,, \\ u_n = 0 & \text{on } \Sigma\,, \\ u_n(x, 0) = u_{0n}(x) & \text{in } \Omega\,, \end{cases} \tag{2.4}$$

where $\{f_n\}$ and $\{u_{0n}\}$ are sequences of smooth functions (for instance, $f_n \in C^\infty(Q)$ and $u_{0n} \in C^\infty(\Omega)$) which will converge respectively in $L^1(Q)$ and in $L^1(\Omega)$ to f and u_0. Here when we talk of weak solutions we mean solutions in distributional sense, as it is classically stated, with test functions that can be taken in $L^2(0, T; H_0^1(\Omega))$.

Our first theorem will concern the behaviour of the sequence $\{u_n)$ of solutions of (2.4) as n tends to infinity, and in order to deal with the time derivative of truncations, we introduce a time-regularization of a function v in $L^2(0, T; H_0^1(\Omega))$. Thus we define, for v in $L^2(0, T; H_0^1(\Omega))$, and $\nu > 0$,

$$v_\nu(x, t) = \int_{-\infty}^{t} \nu \tilde{v}(x, s)\, e^{\nu(s - t)}\, ds\,, \qquad \tilde{v}(x, s) = v(x, s)\, \chi_{(0, T)}(s)\,,$$

where χ_E denotes the characteristic function of a set E. This convolution function has been first used in [La], then in [DO] and [BDGO], and it enjoys the following properties: v_ν belongs to $C([0, T]; H_0^1(\Omega))$, $v_\nu(x, 0) = 0$ and v_ν converges strongly to v in $L^2(0, T; H_0^1(\Omega))$ as ν tends to infinity. Moreover, we have

$$(v_\nu)_t = \nu(v - v_\nu)$$

as a vector valued distribution, and finally if v belongs to $L^\infty(Q)$ then v_ν belongs to $L^\infty(Q)$ as well and

$$\|v_\nu\|_{L^\infty(Q)} \leq \|v\|_{L^\infty(Q)}\,, \qquad \forall \nu > 0\,. \tag{2.5}$$

Let us note that in the following of this work we will set

$$S_k(t) \equiv \int_0^t T_k(s)\, ds\,,$$

the primitive of the truncated function $T_k(t)$.

THEOREM 2.1. – *Let $a(x, t, s, \xi)$ satisfy assumptions* (2.1)-(2.3), *and let us also assume that*

$$f_n \to f \qquad \textit{weakly in } L^1(Q),$$
$$u_{0n} \to u_0 \quad \textit{strongly in } L^1(\Omega),$$

and let u_n be a solution of (2.4). *Then there exists a measurable function $u: Q \to \mathbf{R}$ such that $T_k(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$ and, up to a subsequence,*

$$T_k(u_n) \to T_k(u) \qquad \textit{strongly in } L^2(0, T; H_0^1(\Omega)) \textit{ for every fixed } k > 0\,.$$

PROOF. – First of all we choose, for τ in $(0, T)$, $T_k(u_n)\, \chi_{(0,\tau)}$ as test function in (2.4); integrating by parts, and denoting by $\langle \cdot, \cdot \rangle$ the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, we have, since $|S_k(t)| \le k|t|$,

$$\int_0^\tau \langle (u_n)_t, T_k(u_n) \rangle \ge \int_\Omega S_k(u_n(\tau))\, dx - k\|u_{0n}\|_{L^1(\Omega)}\,,$$

hence assumption (2.1) implies, for every τ in $(0, T)$,

$$\int_\Omega S_k(u_n(\tau))\, dx + \alpha \int_0^\tau \int_\Omega |\nabla T_k(u_n)|^2\, dx\, dt \le k(\|u_{0n}\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}) \le ck\,. \tag{2.6}$$

Inequality (2.6) gives the usual estimates for parabolic equations with L^1 data (see [BG], [BDGO], [ST]), that is to say u_n is bounded in $L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < (N+2)/(N+1)$ and in $L^\infty(0, T; L^1(\Omega))$, from which we can deduce that

$$\lim_{k \to +\infty} \text{meas}\,\{(x, t) \in Q: |u_n| > k\} = 0 \qquad \text{uniformly with respect to } n\,. \tag{2.7}$$

Moreover we have from (2.6) that $T_k(u_n)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$. Now, if we multiply the approximating equation by $\mathcal{T}_k'(u_n)$, where $\mathcal{T}_k(s)$ is a $C^2(\mathbf{R})$, nondecreasing function such that $\mathcal{T}_k(s) = s$ for $|s| \le k/2$ and $\mathcal{T}_k(s) = k$ for $|s| > k$, we get

$$(\mathcal{T}_k(u_n))_t - \text{div}\,(a_n(x, t, u_n, \nabla u_n)\, \mathcal{T}_k'(u_n)) + a_n(x, t, u_n, \nabla u_n)\, \nabla u_n\, \mathcal{T}_k''(u_n) = \mathcal{T}_k'(u_n)\, f\,,$$

in the sense of distributions. This implies, thanks to (2.6) and to the fact that $\mathcal{T}_k'$ has compact support, that $\mathcal{T}_k(u_n)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ while its time derivative $(\mathcal{T}_k(u_n))_t$ is bounded in $L^2(0, T; H^{-1}(\Omega)) + L^1(Q)$, hence a classical compactness result (see [Si]) allows us to conclude that $\mathcal{T}_k(u_n)$ is compact in $L^2(Q)$. Thus, for a subsequence, it also converges in measure and almost everywhere in Q. Since we have, for $\sigma > 0$,

$$\operatorname{meas}\{(x,t): |u_n - u_m| > \sigma\} \le \operatorname{meas}\left\{(x,t): |u_n| > \frac{k}{2}\right\} + $$
$$+\operatorname{meas}\left\{(x,t): |u_m| > \frac{k}{2}\right\} + \operatorname{meas}\{(x,t): |\mathcal{T}_k(u_n) - \mathcal{T}_k(u_m)| > \sigma\},$$

by (2.7) for every fixed $\varepsilon > 0$ we can choose $\bar{k}$ large enough to have

$$\operatorname{meas}\{(x,t): |u_n - u_m| > \sigma\} \le \operatorname{meas}\{(x,t): |\mathcal{T}_{\bar{k}}(u_n) - \mathcal{T}_{\bar{k}}(u_m)| > \sigma\} + \varepsilon, \quad \forall n, m \in N. \tag{2.8}$$

The fact that $\mathcal{T}_k(u_n)$ converges in measure for every $k > 0$ implies, using (2.8), that, up to subsequences, u_n also converges in measure and almost everywhere in Q. In particular, we have found out that there exists a measurable function u in $L^\infty(0, T; L^1(\Omega)) \cap \cap L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < (N+2)/(N+1)$ such that $T_k(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$, and for a subsequence, not relabeled,

$$T_k(u_n) \to T_k(u) \tag{2.9}$$

weakly in $L^2(0, T; H_0^1(\Omega))$, strongly in $L^2(Q)$ and a.e. in Q.

Let us take now a sequence $\{\psi_j\}$ of $C_c^\infty(\Omega)$ functions that strongly converges to u_0 in $L^1(\Omega)$, and set

$$n_{\nu,j}(u) \equiv T_k(u)_\nu + e^{-\nu t} T_k(\psi_j). \tag{2.10}$$

The definition of $\eta_{\nu,j}(u)$, which is a smooth approximation of $T_k(u)$, is needed to deal with a nonzero initial datum (see also [P]); note that this function has the following properties:

$$\begin{cases} (\eta_{\nu,j}(u))_t = \nu(T_k(u) - \eta_{\nu,j}(u)), \quad \eta_{\nu,j}(u)(0) = T_k(\psi_j), \quad |\eta_{\nu,j}(u)| \le k, \\ \eta_{\nu,j}(u) \to T_k(u) \quad \text{strongly in } L^2(0, T; H_0^1(\Omega)), \text{ as } \nu \text{ tends to infinity}. \end{cases} \tag{2.11}$$

Similarly to the elliptic case (see [LP]), we choose

$$w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\nu,j}(u)) \tag{2.12}$$

as test function in (2.4), with $h > k > 0$. We state separately, in next lemma, the behaviour of the term $\int_0^T \langle (u_n)_t, w_n \rangle$. Henceforward, we will denote by $\omega(n, \nu, j, h)$ all quantities (possibly different) such that

$$\lim_{h \to +\infty} \lim_{j \to +\infty} \lim_{\nu \to +\infty} \lim_{n \to +\infty} \omega(n, \nu, j, h) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then ν, j and finally h. Similarly we will write only $\omega(n)$, or $\omega(n, \nu)$, $\omega(n, \nu, h)$ to mean that the limits are made only on the specified parameters.

LEMMA 2.1. - *Under the previous assumptions we have*

$$\int_0^T \langle (u_n)_t, w_n \rangle \geq \omega(n, \nu, j, h).$$

PROOF. - First of all observe that, since $|\eta_{\nu,j}(u)| \leq k$, w_n can be written in the following way:

$$w_n = T_{h+k}(u_n - \eta_{\nu,j}(u)) - T_{h-k}(u_n - T_k(u_n)).$$

Hence, setting $G(t) = \int_0^t T_{h-k}(s - T_k(s))\,ds$, we have:

$$\int_0^T \langle (u_n)_t, w_n \rangle = \int_0^T \langle (\eta_{\nu,j}(u))_t, T_{h+k}(u_n - \eta_{\nu,j}(u)) \rangle + \tag{2.13}$$

$$+ \int_\Omega S_{h+k}(u_n - \eta_{\nu,j}(u))(T)\,dx - \int_\Omega G(u_n(T))\,dx + \int_\Omega G(u_{0n})\,dx - \int_\Omega S_{h+k}(u_{0n} - T_k(\psi_j))\,dx.$$

Define now the function $R(y) \equiv S_{h+k}(y - z) - G(y)$, with $|z| \leq k$; then

$$\begin{cases} R(y) = S_{h+k}(y + z) \geq 0 & \text{where } |y| \leq k, \\ R'(y) = T_{h+k}(y - z) - T_{h-k}(y - T_k(y)) \geq 0 & \text{where } y \geq k \geq z, \\ R'(y) \leq 0 & \text{where } y \leq -k \leq z. \end{cases}$$

Hence for every z: $|z| \leq k$, we have $R(y) \geq 0$ for every y in $\mathbf{R}$, and since $|\eta_{\nu,j}(u)| \leq k$ we get, from (2.13),

$$\int_0^T \langle (u_n)_t, w_n \rangle \geq \int_0^T \langle (\eta_{\nu,j}(u))_t, T_{h+k}(u_n - \eta_{\nu,j}(u)) \rangle + \tag{2.14}$$

$$+ \int_\Omega G(u_{0n})\,dx - \int_\Omega S_{h+k}(u_{0n} - T_k(\psi_j))\,dx.$$

Using (2.11) we have:

$$\int_0^T \langle (\eta_{\nu,j}(u))_t, T_{h+k}(u_n - \eta_{\nu,j}(u)) \rangle = \nu \int_Q (T_k(u) - \eta_{\nu,j}(u))\, T_{h+k}(u_n - \eta_{\nu,j}(u))\,dx\,dt,$$

so that as n tends to infinity we find:

$$\int_0^T \langle (\eta_{\nu,j}(u))_t,\, T_{h+k}(u_n - \eta_{\nu,j}(u))\rangle =$$

$$= \omega(n) + \nu \int_Q (T_k(u) - \eta_{\nu,j}(u))\, T_{h+k}(u - \eta_{\nu,j}(u))\, dx\, dt =$$

$$= \omega(n) + \nu \int_{\{|u| \le k\}} (u - \eta_{\nu,j}(u))\, T_{h+k}(u - \eta_{\nu,j}(u))\, dx\, dt +$$

$$+ \int_{\{u > k\}} (k - \eta_{\nu,j}(u))\, T_{h+k}(u - \eta_{\nu,j}(u))\, dx\, dt +$$

$$+ \int_{\{u < -k\}} (-k - \eta_{\nu,j}(u))\, T_{h+k}(u - \eta_{\nu,j}(u))\, dx\, dt\,.$$

Since $|\eta_{\nu,j}(u)| \le k$, last three terms are positive, hence we deduce from (2.14), letting n and j go to infinity,

$$\int_0^T \langle (u_n)_t,\, w_n\rangle \ge \omega(n) + \int_\Omega G(u_0) - \int_\Omega S_{h+k}(u_0 - T_k(u_0))\, dx + \omega(n, j)\,.$$

Since we have $|G(u_0) - S_{h+k}(u_0 - T_k(u_0))| \le 2k|u_0|\chi_{\{|u_0| > h\}}$, it follows that

$$\lim_{h \to +\infty} \int_\Omega G(u_0) - \int_\Omega S_{h+k}(u_0 - T_k(u_0))\, dx = 0\,,$$

and so

$$\int_0^T \langle (u_n)_t,\, w_n\rangle \ge \omega(n, \nu, j, h)\,. \qquad \blacksquare$$

Henceforward, the proof will follow the lines of the elliptic case treated in [LP]; first of all, Lemma 2.1 allows us to write:

$$\int_Q a_n(x, u_n, \nabla u_n)\, \nabla w_n\, dx\, dt \le \int_Q f_n w_n + \omega(n, \nu, j, h)\,. \tag{2.15}$$

Now, note that $\nabla w_n = 0$ if $|u_n| > h + 4k$; then, if we set $M = h + 4k$, splitting the integral on the left hand side of (2.15) on the sets $\{|u_n| > k\}$ and $\{|u_n| \le k\}$, using the fact that $a(x, t, s, \xi)\cdot\xi \ge 0$ and $a(x, t, s, 0) = 0$, we have, for n large (for simplicity we will omit hereafter the dependence on x and t in the function $a(x, t, s, \xi)$):

$$
(2.16) \qquad \int_Q a_n(u_n, \nabla u_n)\nabla w_n\,dx\,dt = \int_Q a(T_M(u_n), \nabla T_M(u_n))\nabla w_n\,dx\,dt \geq
$$

$$
\geq \int_Q a(T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - \eta_{\nu,j}(u))\,dx\,dt -
$$

$$
- \int_{\{|u_n|>k\}} |a(T_M(u_n), \nabla T_M(u_n))|\,|\nabla \eta_{\nu,j}(u)|\,dx\,dt\,.
$$

Last term can be dealt with in the following way:

$$
\int_{\{|u_n|>k\}} |a(T_M(u_n), \nabla T_M(u_n))|\,|\nabla \eta_{\nu,j}(u)|\,dx\,dt \leq
$$

$$
\leq \int_{\{|u_n|>k\}} |a(T_M(u_n), \nabla T_M(u_n))|\,|\nabla T_k(u)|\,dx\,dt +
$$

$$
+ \int_Q |a(T_M(u_n), \nabla T_M(u_n))|\,|\nabla \eta_{\nu,j}(u) - \nabla T_k(u)|\,dx\,dt\,.
$$

Recalling that $M = h + 4k$, we note that, for fixed h, in virtue of (2.9) and the growth assumption (2.2), $a(T_M(u_n), \nabla T_M(u_n))$ is bounded in $L^2(Q)^N$ with respect to n, while $|\nabla T_k(u)|\,\chi_{\{|u_n|>k\}}$ strongly converges to zero in $L^2(Q)$. Moreover we can use (2.11) to obtain:

$$
\int_{\{|u_n|>k\}} |a(T_M(u_n), \nabla T_M(u_n))|\,|\nabla \eta_{\nu,j}(u)| \leq \omega(n, \nu)\,.
$$

Last inequality, together wit (2.15) and (2.16), allows us to deduce:

$$
\int_Q a(T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - \eta_{\nu,j}(u))\,dx\,dt \leq \int_Q f_n w_n\,dx\,dt + \omega(n, \nu, j, h)\,,
$$

which yields, thanks to (2.9) and (2.11),

$$
\int_Q a(T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))\,dx\,dt \leq \int_Q f_n w_n\,dx\,dt + \omega(n, \nu, j, h)\,,
$$

where $\omega(n, \nu, j, h)$ includes, at every new step, all the terms which will go to zero once the parameters will tend to infinity in the prescribed order. As the right hand side is concerned, we have, simply using the Lebesgue theorem,

$$\int_Q f_n w_n \, dx\, dt = \int_Q f T_{2k}(u - T_h(u) + T_k(u) - \eta_{\nu,j}(u))\, dx\, dt + \omega(n) =$$

$$= \int_Q f T_{2k}(u - T_h(u))\, dx\, dt + \omega(n, \nu) = \omega(n, \nu, h).$$

Therefore we can conclude:

$$\int_Q [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx\, dt \leq$$

$$\leq \omega(n, \nu, j, h) - \int_Q a(T_k(u_n), \nabla T_k(u))\nabla (T_k(u_n) - T_k(u))\, dx\, dt.$$

Using (2.9) for the last term in the right hand side and letting first n tend to infinity, then respectively ν, j and h, we can finally write:

$$\lim_{n\to+\infty} \int_Q [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx\, dt = 0, \tag{2.17}$$

which is enough to obtain, using assumption (2.3) (see Lemma 5 of [BMP]), that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^2(0, T; H_0^1(\Omega)),$$

for every fixed $k > 0$. ■

REMARK 2.1. – In the proof of Theorem 2.1 it can not be assumed that u_{0n} converges to u_0 only weakly in $L^1(\Omega)$, in fact the strong convergence of u_{0n} has been essentially used in the proof of Lemma 2.1. Let us also remark that the use of the approximation ψ_j of u_0 is also necessary, we can not take the same u_{0n} instead of ψ_j since we need to pass first to the limit in n in all the integrals, and of course ∇u_{0n} does not converge in $L^2(Q)^N$.

REMARK 2.2. – The previous proof works exactly in the same way under a milder coercivity assumption on $a(x, t, s, \xi)$. To be more precise, assume that instead of (2.1) the following condition holds:

$$a(x, t, s, \xi)\cdot\xi \geq \alpha_k |\xi|^2, \quad \forall s: |s| \leq k, \quad \alpha_k > 0. \tag{2.18}$$

Clearly, (2.18) includes the classical case where $\alpha_k \geq \alpha > 0$ for every k, but it goes further since it admits the possibility that the sequence α_k converges to zero as k tends to infinity, a case which corresponds to a degenerating coerciveness assumption, and it is satisfied for instance if $a(x, t, s, \xi) = \xi/(1 + |s|)^\lambda$ with $\lambda > 0$.

With this hypothesis (2.6) becomes

$$\int_\Omega S_k(u_n(\tau))\,dx + \alpha_k \int_0^\tau \int_\Omega |\nabla T_k(u_n)|^2\,dx\,dt \le k(\|u_{0n}\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}) \le ck\,, \qquad \forall \tau \in (0, T),$$

which still implies that $T_k(u_n)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ for every $k>0$ and u_n is bounded in $L^\infty(0, T; L^1(\Omega))$, hence

$$\lim_{k\to+\infty} \text{meas}\,\{(x, t): |u_n| > k\} = 0 \quad \text{uniformly with respect to } n\,.$$

As in the proof of Theorem 2.1, this is enough to deduce that there exists a function u in $L^\infty(0, T; L^1(\Omega))$ such that, for a subsequence, u_n converges to u almost everywhere in Q and $T_k(u_n)$ converges to $T_k(u)$ weakly in $L^2(0, T; H_0^1(\Omega))$, strongly in $L^2(Q)$ and almost everywhere in Q. Hereafter, the rest of the proof of Theorem 2.1 applies straightforwardly in this setting too; it only remains to point out that Lemma 5 in [BMP], which gives the conclusion by (2.17), is applied to the coercive function $a(x, t, T_k(s), \xi)$, for every $k>0$. ∎

Let us give an application of Theorem 2.1, in order to obtain for the following initial boundary value problem similar results to those proved in [LP] and [B] for the stationary case:

$$(2.19)\qquad \begin{cases} u_t - \text{div}\,(a(x, t, u, \nabla u) + \Phi(u)) = f & \text{in } Q\,,\\ u = 0 & \text{on } \Sigma\,,\\ u(x, 0) = u_0(x) & \text{in } \Omega\,,\end{cases}$$

under assumptions (2.1)-(2.3), and with $\Phi \in C^0(\boldsymbol{R}, \boldsymbol{R}^N)$. The absence of growth conditions on Φ may imply that the term $\Phi(u)$ does not belong to $L^1(Q)^N$, so problem (2.19) can not be formulated in the sense of distributions. For the same reason, in the elliptic case (see [BDGM]) it has been adapted to this setting the definition of renormalized solution introduced in [DL] in a different context. Then this notion has been extended to parabolic equations in [BlMu]; it formally consists in multiplying (2.19) pointwise by $S'(u)$, where S is a smooth function on $\boldsymbol{R}$ such that S' has compact support, so that all the integrals in the weak formulation are in fact taken on the set $\{|u| \le L\}$, where L is such that $\text{supp}(S') \subset [-L, L]$.

DEFINITION 2.1. – *A measurable function u in $L^1(0, T; W_0^{1,1}(\Omega))$ will be said a renormalized solution of (2.19) if $T_k(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ for every $k>0$,*

$$(2.20)\qquad \lim_{h\to+\infty} \int_{\{(x,t):\, h\le|u|\le h+1\}} a(x, t, u, \nabla u)\,\nabla u\,dx\,dt = 0\,,$$

u satisfies in the sense of distributions

$$(2.21)\qquad (S(u))_t - \text{div}\,(a(x, t, u, \nabla u)\,S'(u) + \Phi(u)\,S'(u)) + a(x, t, u, \nabla u)\,\nabla u\,S''(u) = \\ = S'(u)\,f - S''(u)\,\Phi(u)\,\nabla u\,,$$

for every $S \in C^\infty(\mathbf{R})$ such that S' has compact support, and u satisfies the initial condition in the sense that $S(u)$ belongs to $C^0([0, T]; L^1(\Omega))$.

Let us note that all the terms in (2.21) have distributional meaning since S' has compact support and we have asked that $T_k(u)$ is in $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$; as far as (2.20) is concerned, it is usually required in order to have uniqueness. Thus we are able to obtain the following existence result, which estends those obtained if $\Phi = 0$ and with standard growth conditions in [BlMu], [Pr] and [BDGO] (here also with a measure as right hand side).

THEOREM 2.2. – *Assume that $a(x, t, s, \xi)$ satisfies* (2.1)-(2.3), *and that Φ belongs to $C^0(\mathbf{R}, \mathbf{R}^N)$, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$. Then there exists a renormalized solution u of* (2.19).

PROOF. – We set $\Phi_n(s) \equiv \Phi(T_n(s))$, and we consider the approximating problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a_n(x, t, u_n, \nabla u_n) + \Phi_n(u_n)) = f_n & \text{in } Q\,, \\ u_n = 0 & \text{on } \Sigma\,, \\ u_n(x, 0) = u_{0n}(x) & \text{in } \Omega\,, \end{cases} \tag{2.22}$$

where f_n and u_{0n} are sequences of smooth functions converging strongly to f in $L^1(Q)$ and to u_0 in $L^1(\Omega)$ respectively. The existence of a solution u_n in $L^2(0, T; H_0^1(\Omega)) \cap \cap C([0, T]; L^2(\Omega))$ of (2.22) (a distribution solution with test functions in $L^2(0, T; H_0^1(\Omega)))$ can be proved easily by means of Schauder's fixed point theorem; then we take $T_k(u_n)$ as test function in (2.22) and we observe that for n sufficiently large we have:

$$\int_Q \Phi_n(u_n)\, \nabla T_k(u_n)\, dx\, dt = \int_Q \Phi(T_k(u_n))\, \nabla T_k(u_n)\, dx\, dt\,,$$

so that, by the divergence theorem in Sobolev spaces, denoting $\widetilde{\Phi}(y) = \int_0^y \Phi(z)\, dz$, it follows:

$$\int_Q \Phi_n(u_n)\, \nabla T_k(u_n)\, dx\, dt = \int_0^T dt \int_\Omega \operatorname{div}(\widetilde{\Phi}(T_k(u_n)))\, dx = \int_0^T dt \int_{\partial\Omega} \widetilde{\Phi}(T_k(u_n)) \cdot \bar{\nu}\, d\sigma = 0$$

where $\bar{\nu}$ denotes the unit outward normal to $\partial\Omega$. Therefore we easily see that again we find estimate (2.6), which implies, like in the proof of Theorem 2.1, that there exists a measurable function u in $L^\infty(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < (N+2)/(N+1)$ such that $T_k(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$ and up to subsequences

$$u_n \to u \qquad \text{a.e. in } Q\,,$$

$$T_k(u_n) \to T_k(u) \qquad \text{weakly in } L^2(0, T; H_0^1(\Omega)) \text{ for every } k > 0\,.$$

In order to prove the strong convergence of truncations, we can repeat the proof of Theorem 2.1, since the added term $-\operatorname{div}(\Phi_n(u_n))$ can be dealt with as follows: when we take w_n, defined in (2.12), as test function in (2.22), since $\nabla w_n \equiv 0$ on the set where $|u_n| > h + 4k$, setting $M \equiv h + 4k$ we can write

$$\int_Q \Phi_n(u_n)\, \nabla w_n \, dx\, dt = \int_Q \Phi(T_M(u_n))\, \nabla w_n \, dx\, dt\,.$$

Since $\Phi(T_M(u_n))$ strongly converges to $\Phi(T_M(u))$ in $L^2(Q)$ while ∇w_n weakly converges to $\nabla T_{2k}(u - T_h(u) + T_k(u) - \eta_{\nu,j}(u))$ in $L^2(Q)$ as n tends to infinity, we obtain:

$$\int_Q \Phi_n(u_n)\, \nabla w_n \, dx\, dt = \int_Q \Phi(T_M(u))\, \nabla T_{2k}(u - T_h(u) + T_k(u) - \eta_{\nu,j}(u))\, dx\, dt + \omega(n)\,.$$

Then, letting ν go to infinity we get:

$$\int_Q \Phi_n(u_n)\, \nabla w_n \, dx\, dt = \int_Q \Phi(T_M(u))\, \nabla T_{2k}(u - T_h(u))\, dx\, dt + \omega(n, \nu)\,. \tag{2.23}$$

Applying again the divergence theorem to the function $\Psi(y) = \int_0^y \Phi(z)\, \chi_{\{h<|z|<h+2k\}}\, dz$ we conclude from (2.23)

$$\int_Q \Phi_n(u_n)\, \nabla w_n \, dx\, dt = \omega(n, \nu) + \int_0^T dt \int_\Omega \operatorname{div}(\Psi(T_M(u)))\, dx = \omega(n, \nu)\,.$$

This is the only change required in order to apply the proof of Theorem 2.1 to the solutions of (2.22), then we can conclude that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^2(0, T; H_0^1(\Omega)) \text{ for every } k > 0\,. \tag{2.24}$$

Moreover, choosing $T_1(u_n - T_h(u_n))$ as test function in (2.22) and reasoning as above we get:

$$\int_{\{h \leq |u_n| \leq h+1\}} a(x, t, u_n, \nabla u_n)\, \nabla u_n \, dx\, dt \leq \int_Q f_n\, T_1(u_n - T_h(u_n))\, dx\, dt +$$

$$+ \int_{\{|u_{0n}|>h)} |u_{0n}|\, dx \leq \int_{\{|u_n|>h)} |f_n|\, dx\, dt + \int_{\{|u_{0n}|>h)} |u_{0n}|\, dx\,,$$

which implies, passing to the limit first in n then in h (for the term in the left hand side we use Fatou lemma), that u satisfies (2.20).

In order to show that u is a renormalized solution of (2.19), we multiply (2.22) by $S'(u_n)$, with $S \in C^\infty(\boldsymbol{R})$ and S' having compact support, say $\operatorname{supp}(S') \subset [-L, L]$. Since we have $S'(u_n)(u_n)_t = (S(u_n))_t$, we obtain the following equality in the sense of distributions:

$$(S(u_n))_t - \operatorname{div}(a_n(x, t, u_n, \nabla u_n) S'(u_n) + \Phi_n(u_n) S'(u_n)) + \\ + a_n(x, t, u_n, \nabla u_n) \nabla u_n S''(u_n) = S'(u_n) f - S''(u_n) \Phi_n(u_n) \nabla u_n . \tag{2.25}$$

Now observe that

$$a_n(x, t, u_n, \nabla u_n) S'(u_n) = a(x, t, T_L(u_n), \nabla T_L(u_n)) S'(T_L(u_n)) ,$$

so that thanks to (2.24), and the growth assumption (2.2), we have

$$a_n(x, t, u_n, \nabla u_n) S'(u_n) \to a(x, t, T_L(u), \nabla T_L(u)) S'(T_L(u)) = a(x, t, u, \nabla u) S'(u) ,$$

and the convergence is strong in $L^2(Q)^N$. With an identical reasoning we have that

$$a_n(x, t, u_n, \nabla u_n) \nabla u_n S''(u_n) \to a(x, t, u, \nabla u) \nabla u S''(u) \qquad \text{strongly in } L^1(Q),$$

and the other terms are dealt with in the same way, using always (2.24) and the fact that S' has compact support. Thus, passing to the limit in (2.25) as n tends to infinity, we find that u is a renormalized solution of (2.19) since it satisfies (2.21). As far as the initial condition is concerned, it is enough to observe that $S(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ and from the equation, $(S(u))_t$ belongs to $L^1(Q) + L^2(0, T; H^{-1}(\Omega))$, hence by Theorem 1.1 we have that $S(u)$ belongs to $C([0, T]; L^1(\Omega))$. ∎

3. – Equations with lower order terms having natural growth conditions.

In this section we consider the following Cauchy-Dirichlet problem:

$$\begin{cases} u_t - \operatorname{div}(a(x, t, u, \nabla u)) + g(x, t, u, \nabla u) = f & \text{in } Q , \\ u = 0 & \text{on } \Sigma , \\ u(x, 0) = u_0(x) & \text{in } \Omega , \end{cases} \tag{3.1}$$

where $a(x, t, s, \xi)$ satisfies assumptions (2.1)-(2-3), and the function $g(x, t, u, \nabla u)$ plays the role of an Hamiltonian with quadratic growth with respect to the gradient, that is $g(x, t, s, \xi)$ is a Carathéodory function (i.e. $g(x, t, s, \xi)$ is measurable with respect to (x, t) and continuous in s and ξ) for which there exist positive constants γ_k such that:

$$|g(x, t, s, \xi)| \le \gamma_k (1 + |\xi|^2) , \quad \forall s \in \boldsymbol{R}: |s| \le k , \quad \forall \xi \in \boldsymbol{R}^N , \\ \text{and a.e. } (x, t) \in Q , \quad \forall k > 0 , \tag{3.2}$$

$$g(x, t, s, \xi) s \ge 0 , \quad \forall s \in \boldsymbol{R} , \quad \forall \xi \in \boldsymbol{R}^N \text{ and a.e. } (x, t) \text{ in } Q . \tag{3.3}$$

Note that no growth assumptions from above are made either on $a(x, t, s, \xi)$ or on $g(x, t, s, \xi)$ as functions of s; as far as the data are concerned, we will take

$$f \in L^1(Q), \quad f \geq 0, \quad u_0 \in L^1(\Omega), \quad u_0 \geq 0.$$

We remark that the assumption of positivity on the data is not necessary to find a solution of (3.1), on the other hand the extension to nonpositive f and u_0 does not add substantial difficulty to the problem apart from rather tedious technicalities.

First of all, we prove a compactness result on the truncatures of the approximating solutions, which extends the one obtained in [DO], since we will not assume to have an *a priori* bound in the space $L^2(0, T; H_0^1(\Omega))$.

THEOREM 3.1. – *Assume that* (2.1)-(2.3) *and* (3.2)-(3.3) *are satisfied and that* $\{f_n\}$ *and* $\{u_{0n}\}$ *are two sequences of smooth functions such that*

$$f_n \geq 0, \quad f_n \to f \quad \text{weakly in } L^1(Q),$$
$$u_{0n} \geq 0, \quad u_{0n} \to u_0 \quad \text{strongly in } L^1(\Omega).$$

Then if $\{u_n\} \subset L^2(0, T; H_0^1(\Omega))$ *is a sequence of solutions of*

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x, t, u_n, \nabla u_n)) + g(x, t, u_n, \nabla u_n) = f_n & \text{in } Q, \\ u_n = 0 & \text{on } \Sigma, \\ u_n(x, 0) = u_{0n}(x) & \text{in } \Omega, \end{cases} \tag{3.4}$$

there exists a measurable function u *such that* $T_k(u)$ *belongs to* $L^2(0, T; H_0^1(\Omega))$ *for every* $k > 0$ *and, up to subsequences,*

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^2(0, T; H_0^1(\Omega)) \text{ for every } k > 0,$$

$$g(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u) \quad \text{strongly in } L^1(Q).$$

PROOF. – We divide the proof in four steps.

Step 1:

Here we find the usual *a priori* estimates; first of all observe that thanks to the sign condition assumed on g it is easily proved (it suffices to take u_n^-, the negative part of u_n, as test function in (3.4)) that since f_n and u_{0n} are positive then u_n is positive as well.

Moreover if we take $T_k(u_n)$ as test function in (3.4) the term with $g(x, t, u_n, \nabla u_n)$ can be dropped out by (3.3), and we obtain the estimate (hereafter we will denote by c_i positive constants not depending on n):

$$\int_\Omega S_k(u_n)(\tau)\, dx + \int_0^\tau \int_\Omega |\nabla T_k(u_n)|^2 \leq kc_0, \quad \forall \tau \in (0, T), \quad \forall k > 0. \tag{3.5}$$

Like in the proof of Theorem 2.1, we deduce that there exists a measurable function u

in $L^\infty(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < (N+2)/(N+1)$ such that $T_k(u)$ is in $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$ and, up to a subsequence,

$$T_k(u_n) \to T_k(u) \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \text{ strongly in } L^2(Q) \text{ and a.e. in } Q. \tag{3.6}$$

Consider now a function $\varrho_h^\varepsilon \in C^1(\boldsymbol{R})$ such that

$$\begin{aligned} &\varrho_h^\varepsilon(s) \equiv 0 && \text{if } |s| \leqslant h, \\ &\varrho_h^\varepsilon(s) = \operatorname{sign}(s) && \text{if } |s| \geqslant h + \varepsilon, \\ &(\varrho_h^\varepsilon)'(s) \geqslant 0, && \forall s \in \boldsymbol{R}. \end{aligned}$$

Using $\varrho_h^\varepsilon(u_n)$ as test function in (3.4) we have, setting $R_h^\varepsilon(s) \equiv \int_0^s \varrho_h^\varepsilon(t)\,dt$,

$$\int_\Omega R_h^\varepsilon(u_n(T))\,dx + \int_Q a(x, t, u_n, \nabla u_n)\,\nabla u_n(\varrho_h^\varepsilon)'(u_n)\,dx\,dt +$$

$$+ \int_Q g(x, t, u_n, \nabla u_n)\,\varrho_h^\varepsilon(u_n)\,dx\,dt \leqslant \int_Q f_n \varrho_h^\varepsilon(u_n)\,dx\,dt + \int_\Omega R_h^\varepsilon(u_{0n})\,dx.$$

Using (2.1), the fact that ϱ_h^ε is non decreasing and that $0 \leqslant R_h^\varepsilon(t) \leqslant |t| \chi_{\{|t|>h\}}$ we get:

$$\int_Q g(x, t, u_n, \nabla u_n)\,\varrho_h^\varepsilon(u_n)\,dx\,dt \leqslant \int_{\{|u_n|>h\}} |f_n|\,dx\,dt + \int_{\{|u_{0n}|>h\}} |u_{0n}|\,dx.$$

In virtue of the sign condition on g we can apply Fatou lemma to obtain, as ε tends to zero,

$$\int_{\{|u_n|>h\}} |g(x, t, u_n, \nabla u_n)|\,dx\,dt \leqslant \int_{\{|u_n|>h\}} |f_n|\,dx\,dt + \int_{\{|u_{0n}|>h\}} |u_{0n}|\,dx. \tag{3.7}$$

Step 2:

Let again $\eta_{\nu,j}(u) = T_k(u)_\nu + e^{-\nu t} T_k(\psi_j)$ be the regularization of $T_k(u)$ which has been defined in (2.10) (hence ψ_j is smooth and converges strongly to u_0 in $L^1(\Omega)$), and consider the auxiliary function $\varphi_\lambda(s) = s e^{\lambda s^2}$, already used in [BMP] (and in several papers afterwards) to deal with an Hamiltonian term growing quadratically with respect to the gradient. Now we take $\varphi_\lambda(u_n - \eta_{\nu,j}(u))^-$ as test function in (3.4), with λ to be chosen later; note that this is an admissible test function since $\eta_{\nu,j}$ is bounded by k. From the equation we get (henceforward we will omit to write explicitly in all the integrals the dependence on x and t):

$$\int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - \eta_{\nu_j}(u))^- \rangle + \int_Q a(u_n, \nabla u_n)\,\nabla(u_n - \eta_{\nu,j}(u))^-\,\varphi_\lambda'(u_n - \eta_{\nu,j}(u))^-\,dx\,dt +$$

$$+ \int_Q g(u_n, \nabla u_n)\,\varphi_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt = \int_Q f_n \varphi_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt,$$

from which it follows:

$$(3.8)\quad \int\limits_{\{u_n \le \eta_{\nu,j}(u)\}} \{(a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\}\cdot$$

$$\cdot\, \varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt \le -\int_Q f_n \varphi_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt +$$

$$+\int_Q g(u_n, \nabla u_n)\,\varphi_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt + \int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - \eta_{\nu_j}(u))^-\rangle +$$

$$+\int_Q a(u_n, \nabla \eta_{\nu,j}(u))\nabla(u_n - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt\,.$$

Observe that since $\eta_{\nu,j}(u)$ is bounded by k, we have that $\varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \equiv 0$ on the set where $u_n > k$, hence we have

$$\int_Q a(u_n, \nabla \eta_{\nu,j}(u))\,\nabla(u_n - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt =$$

$$= \int_Q a(T_k(u_n), \nabla \eta_{\nu,j}(u))\,\nabla(T_k(u_n) - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt\,.$$

Using now (3.6) and (2.2), we have that $a(T_k(u_n), \nabla \eta_{\nu,j}(u))\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-$ strongly converges in $L^2(Q)^N$, as n tends to infinity, in virtue of Lebesgue theorem, hence we can pass to the limit to get:

$$\lim_{n\to+\infty}\int_Q a(u_n, \nabla \eta_{\nu,j}(u))\,\nabla(u_n - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt =$$

$$= \int_Q a(T_k(u), \nabla \eta_{\nu,j}(u))\,\nabla(T_k(u) - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u - \eta_{\nu,j}(u))^-\,dx\,dt\,.$$

Recalling that $\eta_{\nu,j}(u)$ converges to $T_k(u)$ as ν tends to infinity strongly in $L^2(0, T; H_0^1(\Omega))$ and almost everywhere in Q, we obtain, adopting the same notation as in Section 2,

$$\int_Q a(u_n, \nabla \eta_{\nu,j}(u))\,\nabla(u_n - \eta_{\nu,j}(u))^-\,\varphi'_\lambda(u_n - \eta_{\nu,j}(u))^-\,dx\,dt = \omega(n, \nu)\,.$$

By the same arguments it can be proved that the first term in the right hand side of (3.8) goes to zero as first n and then ν go to infinity, so that (3.8) implies

$$\int_{\{u_n \leq \eta_{\nu,j}(u)\}} \{(a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\} \cdot \tag{3.9}$$

$$\cdot \varphi'_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt \leq \int_Q g(u_n, \nabla u_n)\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt +$$

$$+ \int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \rangle + \omega(n, \nu) \leq \omega(n, \nu) +$$

$$+ \gamma_k \int_Q |\nabla u_n|^2 \varphi_\lambda(u_n - \eta_{\nu_j}(u))^- \, dx\, dt + \int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \rangle .$$

We also have, by (2.1), and since $\varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \equiv 0$ if $u_n > k$,

$$\alpha \int_Q |\nabla u_n|^2 \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt \leq$$

$$\leq \int_Q a(u_n, \nabla u_n)\, \nabla(u_n - \eta_{\nu,j}(u))\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt +$$

$$+ \int_Q a(u_n, \nabla u_n)\, \nabla \eta_{\nu,j}(u)\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt \leq$$

$$\leq \int_{\{u_n \leq \eta_{\nu,j}(u)\}} \{(a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\}\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt +$$

$$+ \int_Q a(T_k(u_n), \nabla(T_k(u_n))\, \nabla \eta_{\nu,j}(u)\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt +$$

$$+ \int_Q a(u_n, \nabla \eta_{\nu,j}(u))\, \nabla(u_n - \eta_{\nu,j}(u))\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt .$$

Recalling the convergence of $\eta_{\nu,j}(u)$ to $T_k(u)$ in $L^2(0, T; H_0^1(\Omega))$ and using (3.6) and the fact that $\varphi_\lambda(u - T_k(u))^- \equiv 0$ we find, as n and ν tend to infinity,

$$\alpha \int_Q |\nabla u_n|^2 \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt \leq \omega(n, \nu) +$$

$$+ \int_{\{u_n \leq \eta_{\nu,j}(u)\}} \{(a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\}\, \varphi_\lambda(u_n - \eta_{\nu,j}(u))^- \, dx\, dt ,$$

and so (3.9) becomes

$$\int_{\{u_n \leq \eta_{\nu,j}(u)\}} \{(a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\}\left(\varphi'_\lambda - \frac{\gamma_k}{\alpha}\varphi_\lambda\right) dx\, dt \leq$$

$$\leq \omega(n, \nu) + \int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle .$$

Choosing λ large enough so that $\varphi'_\lambda(s) - \frac{\gamma_k}{\alpha}\varphi_\lambda(s) \geq \frac{\alpha}{2}$ for every s in $\boldsymbol{R}$, we get:

$$\frac{\alpha}{2} \int_{\{u_n \leq \eta_{\nu,j}(u)\}} (a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u)))\nabla(u_n - \eta_{\nu,j}(u))\, dx\, dt \leq$$

$$\leq \omega(n, \nu) + \int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle .$$

This step will be concluded after the proof of the following lemma.

LEMMA 3.1. – *We have*:

$$\int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle \leq \omega(n, \nu, j).$$

PROOF. – Let $\Phi_\lambda^-(s) = \int_0^s \varphi_\lambda(t^-)\, dt$; then we have

$$\int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle = \int_\Omega \Phi_\lambda^-(u_n - \eta_{\nu,j}(u))(T)\, dx -$$

$$- \int_\Omega \Phi_\lambda^-(u_{0n} - T_k(\psi_j))\, dx + \int_0^T \langle (\eta_{\nu,j}(u))_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle,$$

which yields, since $\Phi_\lambda^-(s) \leq 0$ and by definition of the ν-regularization,

$$\int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle \leq - \int_\Omega \Phi_\lambda^-(u_{0n} - T_k(\psi_j))\, dx +$$

$$+ \nu \int_Q (T_k(u) - \eta_{\nu,j}(u))\varphi_\lambda (u_n - \eta_{\nu,j}(u))^-\, dx\, dt .$$

Now we can pass to the limit as n tends to infinity by means of the Lebesgue theorem; also using that $\varphi_\lambda(t^-)\, t \leq 0$, we get:

$$\int_0^T \langle (u_n)_t, \varphi_\lambda (u_n - \eta_{\nu,j}(u))^- \rangle \leq - \int_\Omega \Phi_\lambda^-(u_0 - T_k(\psi_j))\, dx + \omega(n).$$

Thus the conclusion of the lemma follows by letting j go to infinity and recalling that $\Phi_\lambda^-(s) \equiv 0$ if s is positive. ■

Using this result we have thus proved that

$$\int_{\{u_n \leq \eta_{\nu,j}(u)\}} (a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u))) \nabla (u_n - \eta_{\nu,j}(u))\, dx\, dt \leq \omega(n, \nu, j). \tag{3.10}$$

Step 3.

In this step we will closely follow the technique already used in the proof of Theorem 2.1; indeed we take

$$w_n \equiv T_{2k}(u_n - T_h(u_n) + (T_k(u_n) - \eta_{\nu,j}(u))^+)$$

as test function in (3.4), with $h > k$. Since w_n is positive, we easily obtain from the equation

$$\int_0^T \langle (u_n)_t, w_n \rangle + \int_Q a(u_n, \nabla u_n) \nabla w_n\, dx\, dt \leq \int_Q f_n w_n\, dx\, dt. \tag{3.11}$$

In order to deal with the first term of (3.11) we observe that the function w_n can be written as $w_n = T_{h+k}(u_n - \eta_{\nu,j}(u))^+ - T_{h-k}(u_n - T_k(u_n))$. Defining the function of real variable

$$S_{h+k}^+(t) = \int_0^T T_{h+k}(s^+)\, ds \quad \text{and} \quad G(t) = \int_0^T T_{h-k}(s - T_k(s))\, ds,$$

we have

$$\int_0^T \langle (u_n)_t, w_n \rangle = \int_\Omega S_{h+k}^+(u_n - \eta_{\nu,j}(u))(T)\, dx - \int_\Omega G(u_n(T))\, dx +$$

$$+ \int_\Omega G(u_{0n})\, dx - \int_\Omega S_{h+k}^+(u_{0n} - T_k(\psi_j)\, dx + \int_0^T \langle (\eta_{\nu,j}(u))_t, T_{h+k}(u_n - \eta_{\nu,j}(u))^+ \rangle.$$

Since $S_{h+k}^+(y - z) - G(y) \geq 0$ for every $y \in \boldsymbol{R}$ if $|z| \leq k$, we get:

$$\int_0^T \langle (u_n)_t, w_n \rangle \geq \nu \int_Q (T_k(u) - \eta_{\nu,j}(u))\, T_{h+k}(u_n - \eta_{\nu,j}(u))^+\, dx\, dt +$$

$$+ \int_\Omega G(u_{0n})\, dx - \int_\Omega S_{h+k}^+(u_{0n} - T_k(\psi_j))\, dx \geq \nu \int_Q (T_k(u) - \eta_{\nu,j}(u))^+\, T_{h+k}(u - \eta_{\nu,j}(u))^+\, dx\, dt +$$

$$+ \int_\Omega G(u_0)\, dx - \int_\Omega S_{h+k}^+(u_0 - T_k(\psi_j))\, dx + \omega(n).$$

Proceeding as in the proof of Lemma 2.1 we then find, as j and h tend to infinity,

$$\int_0^T \langle (u_n)_t, w_n \rangle \geq \omega(n, \nu, j, h),$$

which together with (3.11) implies

$$\int_Q a(u_n, \nabla u_n) \nabla w_n \, dx\, dt \leq \int_Q f_n w_n \, dx\, dt + \omega(n, \nu, j, h). \tag{3.12}$$

With the same arguments used in Theorem 2.1 it can be proved that

$$\int_Q a(u_n, \nabla u_n) \nabla w_n \, dx\, dt \geq \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla (T_k(u_n) - \eta_{\nu,j}(u))^+ \, dx\, dt + \omega(n, \nu),$$

hence by (3.12) we have

$$\int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla \eta_{\nu,j}(u))) \nabla (T_k(u_n) - \eta_{\nu,j}(u))^+ \, dx\, dt \leq$$

$$\leq \int_Q f_n w_n - \int_Q a(T_k(u_n), \nabla \eta_{\nu,j}(u)) \nabla (T_k(u_n) - \eta_{\nu,j}(u))^+ \, dx\, dt + \omega(n, \nu, j, h).$$

Using (2.11) and (3.6), together with assumption (2.2), last term go to zero as first n and then ν and h tend to infinity, so we deduce:

$$\int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla \eta_{\nu,j}(u))) \nabla (T_k(u_n) - \eta_{\nu,j}(u))^+ \, dx\, dt \leq \omega(n, \nu, j, h). \tag{3.13}$$

Step 4.

Since $\eta_{\nu,j}(u)$ is an approximation of $T_k(u)$ in the strong topology of $L^2(0, T; H_0^1(\Omega))$, and $T_k(u_n)$ strongly converges to $T_k(u)$ in $L^2(Q)$, we can write

$$\int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))) \nabla (T_k(u_n) - T_k(u)) \, dx\, dt \leq$$

$$\leq \int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla \eta_{\nu,j}(u))) \nabla (T_k(u_n) - \eta_{\nu,j}(u)) \, dx\, dt + \omega(n, \nu),$$

from which it follows (recall that $(u_n - \eta_{\nu,j}(u))^- = (T_k(u_n) - \eta_{\nu,j}(u))^-$):

$$\int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))) \nabla (T_k(u_n) - T_k(u)) \, dx\, dt \leq$$

$$\leq \int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla \eta_{\nu,j}(u))) \nabla (T_k(u_n) - \eta_{\nu,j}(u)) \, dx\, dt +$$

$$+ \int_{\{u_n \leq \eta_{\nu,j}(u)\}} (a(u_n, \nabla u_n) - a(u_n, \nabla \eta_{\nu,j}(u))) \nabla (u_n - \eta_{\nu,j}(u)) \, dx\, dt + \omega(n, \nu).$$

Thus thanks to (3.10) and (3.13) we have proved that

$$\lim_{n\to+\infty}\int_Q (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))\nabla(T_k(u_n) - T_k(u))\,dx\,dt = 0\,,$$

so by Lemma 5 in [BMP] (it is here that we use assumption (2.3)) we obtain:

(3.14) $T_k(u_n)\to T_k(u)$ strongly in $L^2(0, T; H_0^1(\Omega))$ for every $k>0$.

Now observe that we have, for every $\sigma>0$,

$$\text{meas}\{(x,t)\colon |\nabla u_n - \nabla u| > \sigma\} \leq \text{meas}\{(x,t)\colon |u_n|>k\} + {}$$
$$+\text{meas}\{(x,t)\colon |u|>k\} + \text{meas}\{(x,t)\colon |\nabla(T_k(u_n)) - \nabla(T_k(u))|>\sigma\}\,,$$

then as a consequence of (3.14) we also have, that ∇u_n converges to ∇u in measure, and therefore, always reasoning for subsequences,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q\,,$$

which implies

$$g(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u) \quad \text{a.e. in } Q\,.$$

In virtue of Vitali theorem, in order to prove the strong L^1 compactness of $g(x, t, u_n, \nabla u_n)$, it is enough to show that it is an equi-integrable sequence. Indeed we have, for a subset E of Q,

$$\int_E |g(u_n, \nabla u_n)|\,dx\,dt = \int_{E\cap\{|u_n|\leq k\}} |g(u_n, \nabla u_n)|\,dx\,dt + \int_{E\cap\{|u_n|>k\}} |g(u_n, \nabla u_n)|\,dx\,dt \leq$$
$$\leq \int_E \gamma_k(1+|\nabla T_k(u_n)|^2)\,dx\,dt + \int_{\{|u_n|>k\}} |f_n|\,dx\,dt + \int_{\{|u_{0n}|>k\}} |u_{0n}|\,dx\,,$$

where we have used estimate (3.7). Recalling that $|\nabla T_k(u_n)|^2, f_n$ and u_{0n} are all strongly convergent sequences, it is possible to fix a k sufficiently large in the previous inequality to get that, as meas(E) tends to zero, all the three terms of the right hand side go to zero uniformly with respect to n, that is $g(u_n, \nabla u_n)$ is equi-integrable. Thus we have obtained that

$$g(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u) \quad \text{strongly in } L^1(Q). \quad \blacksquare$$

Reasoning as in Theorem 2.2, the convergence of $g(x, t, u_n, \nabla u_n)$ in $L^1(Q)$ and the strong convergence of truncations in $L^2(0, T; H_0^1(\Omega))$ allow to deduce the following existence result.

THEOREM 3.2. – *Let assumptions* (2.1)-(2.3) *and* (3.2)-(3.3) *be satisfied and let f and u_0 be positive functions belonging to $L^1(Q)$ and to $L^1(\Omega)$ respectively. Then there exists a positive renormalized solution u of* (3.1) *in the sense that u is in $L^\infty(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$ for every $q<(N+2)/(N+1)$, $T_k(u)$ belongs to*

$L^2(0, T; H_0^1(\Omega))$ for every $k > 0$,

$$\lim_{h \to +\infty} \int_{\{(x,t):\, h \le u \le h+1\}} a(x, t, u, \nabla u) \nabla u \, dx \, dt = 0 , \tag{3.15}$$

u satisfies in the sense of distributions

$$(S(u))_t - \operatorname{div}(a(x, t, u, \nabla u) S'(u)) + \\ + a(x, t, u, \nabla u) \nabla u S''(u) + g(x, t, u, \nabla u) S'(u) = S'(u) f \tag{3.16}$$

for every $S \in C^\infty(\mathbf{R})$ such that S' has compact support, and u satisfies the initial condition in the sense that $S(u)$ belongs to $C^0(0, T; L^1(\Omega))$. ∎

REMARK 3.1. – With minor modifications it is possible to find a solution if the right hand side is of the form $f + \chi$ with f in $L^1(Q)$ and χ in $L^2(0, T; H^{-1}(\Omega))$.

REMARK 3.2. – The Dirichlet problem (3.1) (as well as problem (2.19) actually), under assumptions (3.2)-(3.3), can also be formulated in the framework of entropy solutions. We recall that this notion has been first introduced in [BBGGPV] for elliptic equations, then extended to evolution problems in [Pr] (see also [AMST], [P]); let us precise this definition ($T_k(s)$ and its primitive $S_k(s)$ are defined as before).

DEFINITION 3.1. – *A measurable function u in $L^1(0, T; W_0^{1,1}(\Omega))$ is an entropy solution of (3.1) if u belongs to $L^\infty(0, T; L^1(\Omega))$, $T_k(u)$ belongs to $L^2(0, T; H_0^1(\Omega))$ for every $k > 0$, $S_k(u(\cdot, t))$ belongs to $L^1(\Omega)$ for every $t \in [0, T]$ and every $k > 0$, $g(x, t, u, \nabla u)$ is in $L^1(Q)$ and u satisfies:*

$$\int_\Omega S_k(u - \varphi)(\tau) \, dx + \int_0^\tau \langle \varphi_t T_k(u - \varphi) \rangle \, dt + \int_0^\tau \int_\Omega a(x, t, u, \nabla u) \nabla T_k(u - \varphi) \, dx \, dt + \\ + \int_0^\tau \int_\Omega g(x, t, u, \nabla u) T_k(u - \varphi) \, dx \, dt \le \int_0^\tau \int_\Omega f T_k(u - \varphi) \, dx \, dt + \int_\Omega S_k(u_0 - \varphi(0)) \, dx , \tag{3.17}$$

for every $\tau \in [0, T]$, $k > 0$, and for all φ in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ such that φ_t belongs to $L^1(Q) + L^2(0, T; H^{-1}(\Omega))$.

It is worth noting, like for the renormalized solutions, that all the terms in (3.17) (the duality $\langle \cdot, \cdot \rangle$ in the second integral is between $L^1(Q) + L^2(0, T; H^{-1}(\Omega))$ and $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$) make sense since $T_k(u - \varphi)$ belongs to $L^2(0, T; H_0^1(\Omega))$ (indeed $\nabla T_k(u - \varphi) \equiv 0$ if $|u| > k + \|\varphi\|_{L^\infty(Q)}$). Moreover the trace result in Theorem 1.1 implies that φ is in $C([0, T]; L^1(\Omega))$, and since $|S_k(u - \varphi)(t)| \le |S_k(u(t))| + k|\varphi(t)| \in L^1(\Omega)$, the first and last terms are well defined.

The existence of one entropy solution can be deduced from our previous result on renormalized solutions. Indeed, the solution obtained by approximation as before is in $L^\infty(0, T; L^1(\Omega))$ and $S_k(u(\cdot, t))$ belongs to $L^1(\Omega)$ for every $k > 0$. Then we choose in (3.16) $S(s) = H_n(s)$ with $H_n(s) = \int_0^s h_n(t)\,dt$ and

$$h_n(s) = \begin{cases} 1\,, & 0 \leqslant s \leqslant n\,, \\ n+1-s\,, & n \leqslant s \leqslant n+1\,, \\ 0\,, & n+1 \leqslant s\,, \\ h_n(-s)\,, & s \leqslant 0\,, \end{cases}$$

and we take $T_k(H_n(u)-\varphi)$ as test function in (3.16), with φ having the properties asked above, and $k>0$. Integrating by parts we obtain:

$$\begin{aligned}(3.18)\qquad \int_\Omega S_k(H_n(u)-\varphi)(\tau)\,dx &+ \int_0^\tau \langle \varphi_t, T_k(H_n(u)-\varphi)\rangle\,dt + \\ &+ \int_0^\tau\int_\Omega h_n(u)\,a(x,t,u,\nabla u)\,\nabla T_k(H_n(u)-\varphi)\,dx\,dt + \\ &+ \int_0^\tau\int_\Omega h_n'(u)\,a(x,t,u,\nabla u)\,\nabla u T_k(H_n(u)-\varphi)\,dx\,dt + \\ &+ \int_0^\tau\int_\Omega g(x,t,u,\nabla u)\,T_k(H_n(u)-\varphi)\,dx\,dt \leqslant \\ &+ \int_0^\tau\int_\Omega f T_k(H_n(u)-\varphi)\,dx\,dt + \int_\Omega S_k(H_n(u_0)-\varphi(0))\,dx\,.\end{aligned}$$

Note that $h_n(u)$ converges to 1, while $H_n(u)$ converges to u, almost everywhere in Q, and moreover $h_n'(u) = -\operatorname{sign}(u)\,\chi_{\{n\leqslant|u|\leqslant n+1\}}$. We get from (3.18):

$$\begin{aligned}(3.19)\qquad \int_\Omega S_k(H_n(u)-\varphi)(\tau)\,dx &+ \int_0^\tau \langle \varphi_t, T_k(H_n(u)-\varphi)\rangle\,dt + \\ &+ \int_0^\tau\int_\Omega g(x,t,u,\nabla u)\,T_k(H_n(u)-\varphi)\,dx\,dt + \\ &+ \int_0^\tau\int_\Omega h_n(u)\,a(x,t,u,\nabla u)\,\nabla T_k(H_n(u)-\varphi)\,dx\,dt \leqslant \\ &\leqslant \int_0^\tau\int_\Omega f T_k(H_n(u)-\varphi)\,dx\,dt + \int_\Omega S_k(H_n(u_0)-\varphi(0))\,dx + \\ &\qquad + k \int_{\{n\leqslant|u|\leqslant n+1\}} a(x,t,u,\nabla u)\,\nabla u\,dx\,dt\,.\end{aligned}$$

Since $|H_n(u)| \leqslant |u|$ we have, by the Lipschitz continuity of $S_k(s)$, that

$$0 \leqslant S_k(H_n(u) - \varphi)(t) \leqslant S_k(u(t)) + k|\varphi(t)|,$$

hence by Lebesgue theorem

$$\lim_{n\to+\infty} \int_\Omega S_k(H_n(u) - \varphi)(\tau)\, dx = \int_\Omega S_k(u - \varphi)(\tau)\, dx, \tag{3.20}$$

and similarly

$$\lim_{n\to+\infty} \int_\Omega S_k(H_n(u_0) - \varphi(0))\, dx = \int_\Omega S_k(u_0 - \varphi(0))\, dx. \tag{3.21}$$

Moreover, we have that $\nabla T_k(H_n(u) - \varphi) = 0$ if $|H_n(u)| > k + \|\varphi\|_{L^\infty(Q)}$, but $|H_n(u)| > n$ if $|u| > n$ and $H_n(u) = u$ if $|u| \leqslant n$, hence for $n > k + \|\varphi\|_{L^\infty(Q)}$ we can write, setting $M = k + \|\varphi\|_{L^\infty(Q)}$:

$$\int_0^\tau \int_\Omega h_n(u)\, a(x, t, u, \nabla u)\, \nabla T_k(H_n(u) - \varphi)\, dx\, dt =$$

$$= \int_0^\tau \int_\Omega h_n(u)\, a(x, t, T_M(u), \nabla T_M(u))\, \nabla T_k(T_M(u) - \varphi)\, dx\, dt,$$

which yields

$$\lim_{n\to+\infty} \int_0^\tau \int_\Omega h_n(u)\, a(x, t, u, \nabla u)\, \nabla T_k(H_n(u) - \varphi)\, dx\, dt = \tag{3.22}$$

$$= \int_0^\tau \int_\Omega a(x, t, T_M(u), \nabla T_M(u))\, \nabla T_k(T_M(u) - \varphi)\, dx\, dt =$$

$$= \int_0^\tau \int_\Omega a(x, t, u, \nabla u)\, \nabla T_k(u - \varphi)\, dx\, dt.$$

Similarly we see that $T_k(H_n(u) - \varphi)$ converges to $T_k(u - \varphi)$ strongly in $L^2(0, T; H_0^1(\Omega))$ and weakly-$*$ in $L^\infty(Q)$, so that we deduce

$$\lim_{n\to+\infty} \int_0^\tau \langle \varphi_t, T_k(H_n(u) - \varphi)\rangle\, dt = \int_0^\tau \langle \varphi_t, T_k(u - \varphi)\rangle > dt. \tag{3.23}$$

Putting together (3.20)-(3.23), using (3.15) and the Lebesgue theorem, we pass to the limit in (3.19) as n tends to infinity and find that u is an entropy solution of (3.1). ∎

REMARK 3.2. – Both the result of this section and those of Section 2 can be proved in exactly the same way for similar divergence form operators which have a growth of order p with respect to the gradient, on the model of the p-laplacian. For the sake of simplicity, we have chosen here to set our problems in the space $L^2(0, T; H_0^1(\Omega))$ instead of $L^p(0, T; W_0^{1,p}(\Omega))$.

4. – A trace result.

Here we are going to give the proof of Theorem 1.1; let us point out that this trace result, as far as we know, is new and represents the more natural extension to the L^1 framework of the classical theorem which states that if u belongs to $L^2(a, b; H_0^1(\Omega))$ and u_t belongs to $L^2(a, b; H^{-1}(\Omega))$ then u is in $C([a, b]; L^2(\Omega))$.

PROOF OF THEOREM 1.1. – Let u be a function of the space

$$V_1^p((a, b)) \equiv \{u\colon \Omega \times [a, b] \to \mathbf{R}\colon u \in L^p(a, b; W_0^{1,p}(\Omega)),$$

$$u_t \in L^{p'}(a, b; W^{-1,p'}(\Omega)) + L^1(\Omega \times (a, b))\}.$$

By classical arguments (for instance the proofs of Lemma 1 and Lemma 2 in Chapter XVII, n. 2 of [DaL] can be rewritten for our case in the same way) we have the following things: first, there exists a continuous prolongation operator P from $V_1^p((a, b))$ to $V_1^p(\mathbf{R})$, so that we can find a function Pu in $L^p(\mathbf{R}; W_0^{1,p}(\Omega))$ with $(Pu)_t$ in $L^{p'}(\mathbf{R}; W^{-1,p'}(\Omega)) + L'(\Omega \times \mathbf{R})$ such that $Pu = u$ in $[a, b]$; in addition, it is possible to find a sequence $\{\psi_n\} \subset C_c^\infty(\mathbf{R}; W_0^{1,p}(\Omega))$ such that ψ_n strongly converges to Pu in $V_1^p(\mathbf{R})$, that is

$$\begin{cases} \psi_n \to Pu & \text{strongly in } L^p(\mathbf{R}; W_0^{1,p}(\Omega)), \\ (\psi_n)_t \to (Pu)_t & \text{strongly in } L^{p'}(\mathbf{R}; W^{-1,p'}(\Omega)) + L^1(\Omega \times \mathbf{R}). \end{cases} \tag{4.1}$$

Let now $S_1(s) = \int_0^s T_1(t)\, dt$; we have

$$\int_\Omega S_1(\psi_n - \psi_m)(t)\, dx = \int_{-\infty}^t \frac{d}{d\sigma} \int_\Omega S_1(\psi_n - \psi_m)(\sigma)\, dx\, d\sigma =$$

$$= \int_{-\infty}^t \int_\Omega T_1(\psi_n - \psi_m)(\sigma)((\psi_n)_t - (\psi_m)_t)(\sigma)\, dx\, d\sigma.$$

Since by (4.1) ψ_n is a Cauchy sequence in $V_1^p(\boldsymbol{R})$, we can write:

$$\int_\Omega S_1(\psi_n - \psi_m)(s)\,dx \leq \omega(n, m), \qquad \forall s \in R\,, \tag{4.2}$$

where $\omega(n, m)$ denotes a term which goes to zero as n and m go to infinity. On the other hand we have:

$$\int_\Omega S_1(\psi_n - \psi_m)(s)\,dx = \int_{\{|\psi_n - \psi_m| < 1\}} \frac{|(\psi_n - \psi_m)(s)|^2}{2}\,dx +$$

$$+ \int_{\{|\psi_n - \psi_m| \geq 1\}} \left(|(\psi_n - \psi_m)(s)| - \frac{1}{2} \right) dx \geq$$

$$\geq \int_{\{|\psi_n - \psi_m| < 1\}} \frac{|(\psi_n - \psi_m)(s)|^2}{2}\,dx + \int_{\{|\psi_n - \psi_m| \geq 1\}} \frac{|(\psi_n - \psi_m)(s)|}{2}\,dx\,,$$

which yelds, by (4.2),

$$\int_\Omega |\psi_n - \psi_m|(s)\,dx = \int_{\{|\psi_n - \psi_m| < 1\}} |\psi_n - \psi_m|(s)\,dx + \int_{\{|\psi_n - \psi_m| \geq 1\}} |\psi_n - \psi_m|(s)\,dx \leq$$

$$\leq \left(\int_{\{|\psi_n - \psi_m| < 1\}} |\psi_n - \psi_m|^2(s)\,dx \right)^{1/2} |\Omega|^{1/2} + 2\omega(n, m) \leq$$

$$\leq (2\,|\Omega|\,\omega(n, m))^{1/2} + 2\omega(n, m)\,.$$

Therefore ψ_n is a Cauchy sequence in $C_c^0(\boldsymbol{R}; L^1(\Omega))$ (the space of continuous functions from $\boldsymbol{R}$ in $L^1(\Omega)$ having compact support) equipped with the topology of uniform convergence, and since the limit of ψ_n in $V_1^p(\boldsymbol{R})$ is Pu we have that

$$\psi_n \to Pu \quad \text{in } C_c^0(\boldsymbol{R}; L^1(\Omega))\,,$$

which implies, since $Pu = u$ in $[a, b]$, that u belongs to $C([a, b]; L^1(\Omega))$. ■

An immediate application of this trace result is the following one.

THEOREM 4.1. – *Assume that $A(x, t, s)$ is a Carathéodory function such that, for positive constants α, β:*

$$0 < \alpha \leq A(x, t, s) \leq \beta\,, \qquad \forall s \in \boldsymbol{R}\,, \quad a.e.\ (x, t) \in Q \tag{4.3}$$

and let (3.2)-(3.3) *be satisfied together with the following assumption:*

$$\text{(4.4)} \qquad \exists L\,, \quad \delta > 0 \colon g(x,t,s,\xi)\,\mathrm{sign}(s) \geq \delta |\xi|^2\,,$$

$$\forall s\colon |s| \geq L > 0\,, \qquad \forall \xi \in \mathbf{R}^N\,, \textit{ a.e. } (x,t) \textit{ in } Q\,.$$

If f is in $L^1(Q)$, then there exists a solution u of

$$\text{(4.5)} \qquad \begin{cases} u_t - \mathrm{div}(A(x,t,u)\nabla u) + g(x,t,u,\nabla u) = f & \textit{in } Q\,,\\ u = 0 & \textit{on } \Sigma\,,\\ u(x,0) = u_0 & \textit{in } \Omega\,,\end{cases}$$

such that u is in $L^2(0,T;H_0^1(\Omega))$, $g(x,t,u,\nabla u)$ is in $L^1(Q)$ and the equation is satisfied in distributional sense, if and only if u_0 belongs to $L^1(\Omega)$.

PROOF. – If there exists a distributional solution u in $L^2(0,T;H_0^1(\Omega))$ with $g(x,t,u,\nabla u)$ in $L^1(\Omega)$, then by Theorem 1.1 u belongs to $C([0,T];L^1(\Omega))$, hence u_0 must be in $L^1(\Omega)$. On the other hand, the existence of at least one solution of this kind has been obtained by approximation in Section 3. It is enough to observe that the sequence $\{u_n\}$ of solutions of

$$\text{(4.6)} \qquad \begin{cases} (u_n)_t - \mathrm{div}(A(x,t,u_n)\nabla u_n) + g(x,t,u_n,\nabla u_n) = f_n & \text{in } Q\,,\\ u_n = 0 & \text{on } \Sigma\,,\\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega\,,\end{cases}$$

is bounded in $L^2(0,T;H_0^1(\Omega))$, so that the limit function, which is a distributional solution of (4.5), belongs to $L^2(0,T;H_0^1(\Omega))$ as well. Indeed, taking $T_L(u_n)$ as test function in (4.6) we get:

$$\int_\Omega S_1(u_n(T))\,dx + \alpha\int_Q |\nabla T_L(u_n)|^2\,dx\,dt + \int_Q g(x,t,u_n,\nabla u_n)\,T_L(u_n)\,dx\,dt \leq$$

$$\leq L\|f_n\|_{L^1(Q)} + \int_Q S_L(u_{0n})\,dx\,,$$

where $S_L(s)$ denotes, as before, the primitive of $T_L(s)$. Since $\{f_n\}$ and $\{u_{0n}\}$ are bounded in $L^1(Q)$ and in $L^1(\Omega)$ respectively, using also (4.4) we obtain:

$$\alpha \int_{\{(x,t)\colon |u_n|\leq L\}} |\nabla u_n|^2\,dx\,dt + \delta L \int_{\{(x,t)\colon |u_n|\geq L\}} |\nabla u_n|^2\,dx\,dt \leq Lc\,,$$

which gives the desired estimate. ■

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