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The Convergence of the Spectrum of a Weakly Connected Domain (*).

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Abstract. – *The paper deals with the convergence, as ε tends to zero, of the spectrum of the Neumann problem $-\Delta v_\varepsilon = \lambda(\varepsilon) v_\varepsilon$ in a «weakly connected» periodic domain Ω_ε of $\mathbb{R}^3$. The domain Ω_ε is composed of a finite number of disjoint connected domains linked by thin bridges (curved plates or tubes). Under a few assumptions on the characteristic sizes of these bridges, we give an explicit asymptotic formula for the eigenvalues which tend to zero and we prove that the rest of the spectrum converges to the spectrum of an elliptic coupled system.*

1. – Introduction.

The aim of this paper is to study the asymptotic behaviour with respect to a small parameter $\varepsilon > 0$ of the Neumann spectral problem

$$\text{(1.1)} \qquad \begin{cases} -\Delta v_\varepsilon = \lambda(\varepsilon)\, v_\varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

for some particular connected domain Ω_ε obtained by the perforation of a given bounded open set Ω of $\mathbb{R}^3$. Problem (1.1) can be associated to the Neumann problem

$$\text{(1.2)} \qquad \begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

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where f is a given function of $L^2(\Omega)$. One of our motivotions is to know if some information about the limiting behaviour of (1.2) can be deduced from the convergence of the spectrum of (1.1).

Problem (1.2) has been widely studied in the case where Ω_ε is a *strongly connected* domain in Hruslov's sense, *i.e.* there exists a bounded extension operator from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$. This extension property was introduced by Tartar [7] for the homogenization of perforated materials. Cioranescu and Saint Jean Paulin [4] first proved that domains with periodic isolated holes satisfy the extension property and that problem (1.2) then converges to the Neumann problem

$$\begin{cases} \operatorname{div}(A\nabla u) + \theta u = \theta f & \text{in } \Omega\,, \\ A\nabla u\cdot\nu = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\theta := \lim_{\varepsilon\to 0} |\Omega_\varepsilon|/|\Omega|$ and A is a constant positive definite matrix depending on the period of the structure. Acerbi, Chiadò Piat, Dal Maso, Percivale [1] on the one hand and Allaire, Murat [2] on the other hand then extended this result to the case of periodic domains with non isolated holes.

Following [4], Vanninathan [8] proved that when the extension property holds, the spectrum from (1.1) converges to the spectrum of problem

$$\begin{cases} -\operatorname{div}(A\nabla v) = \lambda\theta v & \text{in } \Omega\,, \\ A\nabla v\cdot\nu = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.4}$$

which is associated to problem (1.3).

On the other hand, Hruslov first noted (see also [3] for a new approach) that the limit problem of (1.2) can be more complicated than (1.3) if the domain Ω_ε is not strongly connected, *i.e.* it does not satisfy the extension property. For particular domains Ω_ε which are called *weakly connected* by Hruslov, problem (1.2) can converge in some suitable sense to a coupled system. This degeneracy compared to the expected limit (1.3) is due to the capacitary effect of very thin bridges which link several disjoint strongly connected regions of the domain Ω_ε.

We are interested here in the spectral convergence of (1.1) for particular weakly connected domains. More precisely, the domain Ω_ε has a periodic structure of period cell $\varepsilon Y_\varepsilon$. The rescaled period Y_ε is a connected open subset of the unit cube $Y := [0, 1[^3$, which is composed of $n \geq 2$ open subsets Y_k of Y with disjoint closure such that for each $1 \leq k \leq n$, the periodic set obtained by Y_k-repetition is a connected regular open set of $\mathbb{R}^3$, and for each $1 \leq k \leq n-1$, Y_k and Y_{k+1} are linked by a small bridge $Q_{k,\varepsilon}$. The set $Q_{k,\varepsilon}$ is a thin tube (along a curve) of length $l_k(\varepsilon)$ (which can tend to $+\infty$) and of cross section area $a_k(\varepsilon)$ which tends to 0. The domain Ω_ε is thus weakly connected thanks to the bridges $Q_{k,\varepsilon}$, $1 \leq k \leq n-1$, which link the consecutive disjoint connected parts Y_k, $1 \leq k \leq n$, of the period Y_ε (see Figures 1 and 3 in the Subsection 2.1).

The result is based on the solutions $\lambda \in \mathbb{R}_+$ and $\vec{v} = (v_1, \ldots, v_n) \in H^1(\Omega)^n$ of the following eigenvalue problem

$$
(1.5)\quad \begin{cases} \operatorname{div}(A_1 \nabla v_1) + \delta_1 (v_1 - v_2) = \lambda |Y_1| v_1 & \text{in } \Omega, \\ -\operatorname{div}(A_j \nabla v_j) + \delta_{j-1}(v_j - v_{j-1}) + \delta_j (v_j - v_{j+1}) = \lambda |Y_j| v_j & \text{in } \Omega, 2 \leqslant j \leqslant n-1, \\ -\operatorname{div}(A_n \nabla v_n) + \delta_{n-1}(v_n - v_{n-1}) = \lambda |Y_n| v_n & \text{in } \Omega, \\ A_j \nabla v_j \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\delta_j \geqslant 0$ for $1 \leqslant j \leqslant n-1$, and A_j for $1 \leqslant j \leqslant n$, is the matrix which appears in problem (1.3) for the strongly connected domain of period εY_j. Since the matrices A_j are positive definite, problem (1.5) is an elliptic coupled system of Neumann's type and its spectrum is thus a sequence of non-negative numbers: $0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$ Note that if $(m-1)$ numbers δ_j are equal to zero, the multiplicity of the eigenvalue zero is equal to m since the solutions of (1.5) satisfy

$$
\int_\Omega \sum_{j=1}^{n} A_j \nabla v_j \cdot \nabla v_j + \int_\Omega \sum_{j=1}^{n-1} \delta_j (v_j - v_{j+1})^2 = \lambda \int_\Omega \sum_{j=1}^{n} v_j^2 .
$$

The main result of the paper is then the following:

THEOREM 1.1. – *Denote by* $0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \ldots$ *the spectrum of problem* (1.1). *Assume that for a given* $1 \leqslant m \leqslant n$ *one has*

$$
\frac{\alpha_1(\varepsilon)}{\varepsilon^2 l_1(\varepsilon)} \ll \ldots \ll \frac{\alpha_{m-1}(\varepsilon)}{\varepsilon^2 l_{m-1}(\varepsilon)} \ll 1 \quad \textit{and} \quad \frac{\alpha_k(\varepsilon)}{\varepsilon^2 l_k(\varepsilon)} \xrightarrow[\varepsilon \to 0]{} \delta_k > 0 \ \textit{for}\ m \leqslant k \leqslant n-1 .
$$

Then, for any $1 \leqslant k \leqslant m-1$, *the eigenvalue* $\lambda_k(\varepsilon)$ *satisfy the estimate*

$$
\lambda_k(\varepsilon) \underset{\varepsilon \to 0}{\sim} \left(\frac{1}{|Y_k|} + \frac{1}{\sum_{j>k} |Y_j|} \right) \frac{\alpha_k(\varepsilon)}{\varepsilon^2 l_k(\varepsilon)} .
$$

Moreover, for any $m \geqslant k$, *the eigenvalue* $\lambda_k(\varepsilon)$ *converges, up to a subsequence, to the eigenvalue* λ_k *of the spectrum of problem* (1.5) *in which* $\delta_1 = \ldots = \delta_{m-1} := 0$ *and* $\delta_j > 0$ *for* $j \geqslant m$.

In particular, if $m = 1$, or equivalently if

$$
\frac{\alpha_k(\varepsilon)}{\varepsilon^2 l_k(\varepsilon)} \xrightarrow[\varepsilon \to 0]{} \delta_k > 0 \quad \text{for any } 1 \leqslant k \leqslant n-1 ,
$$

no eigenvalue of the spectrum of problem (1.1) tends to 0. However, the limit spectral problem (1.5) is a coupled system which is not of the classical form (1.4). This shows that the non-degeneracy of the spectrum from (1.1) does not imply the classical limit (1.3) of (1.2).

2. – Statement of the result.

2.1. *The geometry of the problem.* – Let $\varepsilon = 1/N$, $N \in \mathbb{N}^*$.
Let Ω be an open subset of $\mathbb{R}^3$ which satisfies

(H1) Ω is a rectangle parallelotop of integer coordinates or a finite union of such parallelotops.

Let $n \geq 2$ be an integer and let $Y := [0, 1[^3$ be the unit cube of $\mathbb{R}^3$.

We consider n open subsets Y_k of Y, $1 \leq k \leq n$, and the open subsets E_k of $\mathbb{R}^3$ obtained by Y-repetition of Y_k, *i.e.*

$$E_k := \bigcup_{\kappa \in \mathbb{Z}^3} (Y + \kappa), \tag{2.1}$$

which satisfy

$$\text{(H2)} \quad \begin{cases} Y_k \text{ and } E_k \text{ are connected sets with Lipschitz continuous boundary,} \\ \overline{Y_j} \cap \overline{Y_k} = \emptyset, \quad \text{for } j \neq k. \end{cases}$$

We also consider $(n-1)$ subsets $Q_{k,\varepsilon}$ of $\overset{\circ}{Y}$, $1 \leq k \leq n-1$, which are designed to provide thin bridges connecting Y_k and Y_{k+1} and which satisfy

$$\text{(H3)} \quad \begin{cases} Q_{k,\varepsilon} \subset \overset{\circ}{Y}, \qquad |Q_{k,\varepsilon}| \to 0, \\ Q_{k,\varepsilon} \cup Y_k \cup Y_{k+1} \text{ is a connected set,} \\ \overline{Q_{j,\varepsilon}} \cap \overline{Q_{k,\varepsilon}} = \emptyset, \quad \text{for } j \neq k \text{ and } \overline{Y_j} \cap \overline{Q_{k,\varepsilon}} = \emptyset, \quad \text{for } j \notin \{k, k+1\}. \end{cases}$$

The geometry of the bridges will be specified more precisely below.

We define the period of the weakly connected domain by

$$Y_\varepsilon := \left(\bigcup_{k=1}^{n} Y_k \right) \cup \left(\bigcup_{k=1}^{n-1} Q_{k,\varepsilon} \right).$$

We consider the n open subsets $\Omega_{k,\varepsilon}$ of Ω, $1 \leq k \leq n$, obtained by εY-repetition of the sets εY_k in Ω, *i.e.* $\Omega_{k,\varepsilon} := \Omega \cap \varepsilon E_k$. In the same way, we consider the $(n-1)$ subsets $\omega_{k,\varepsilon}$ of Ω, $1 \leq k \leq n-1$, obtained by εY-repetition of the sets $\varepsilon Q_{k,\varepsilon}$ in Ω. The weakly connected domain is then defined by

$$\Omega_\varepsilon := \bigcup_{k=1}^{n} \Omega_{k,\varepsilon} \cup \omega_\varepsilon \quad \text{where} \quad \omega_\varepsilon := \bigcup_{k=1}^{n-1} \omega_{k,\varepsilon}. \tag{2.3}$$

Note that Ω_ε is also obtained by εY-repetition of the period $\varepsilon Y_\varepsilon$, *i.e.*

$$\Omega_\varepsilon = \Omega \cap \varepsilon E_\varepsilon \quad \text{where} \quad E_\varepsilon := \bigcup_{\kappa \in \mathbb{Z}^3} (\varepsilon Y_\varepsilon + \varepsilon\kappa).$$

The domain Ω_ε is an open subset of Ω which is connected by the set of small measure ω_ε since $|\omega_\varepsilon| \to 0$ by assumption (H3).

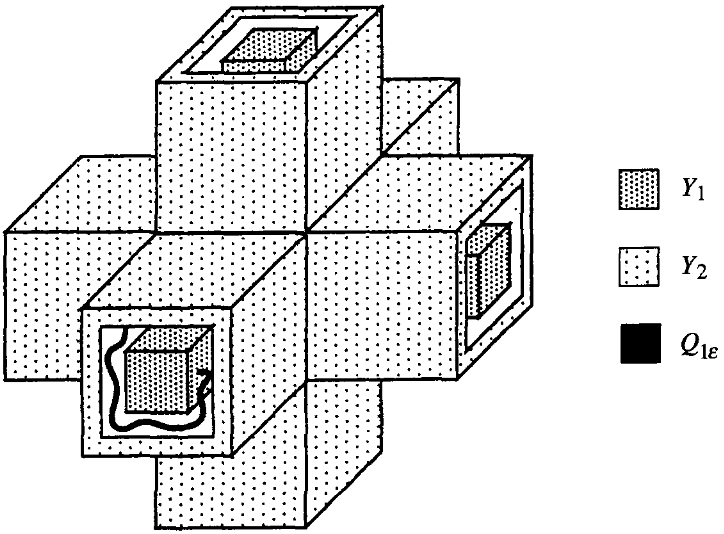

Fig. 1. – A period Y_ε for $n = 2$.

ILLUSTRATIONS. – Figure 1 shows a period Y_ε for $n = 2$, Figure 2 a cross section of this period and Figure 3 the weakly connected domain Ω_ε obtained by εY repetition of 8 periods $\varepsilon Y_\varepsilon$. Figure 4 shows a cross section of a period Y_ε for $n = 3$.

Let us now describe each bridge $Q_{k,\varepsilon}$ for $1 \leq k \leq n - 1$. Let $F_{k,\varepsilon}: [0, 1] \rightarrow]0, 1[^3$ be a simple curve (without multiple points) of class C^d, which is plane if $d = 2$ and skew if

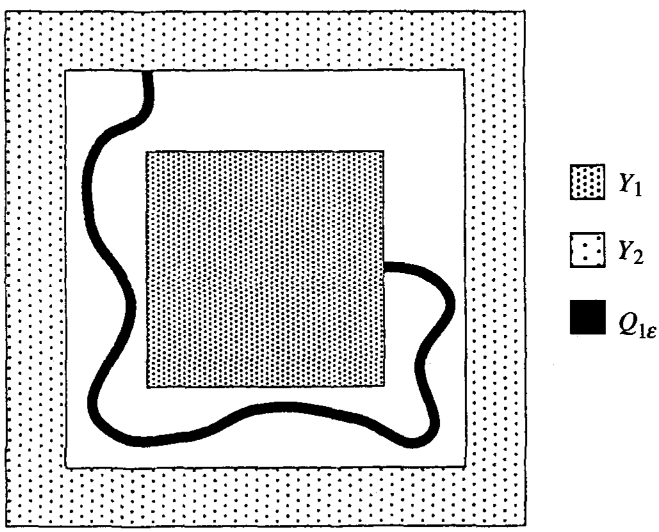

Fig. 2. – A cross section of a period Y_ε for $n = 2$.

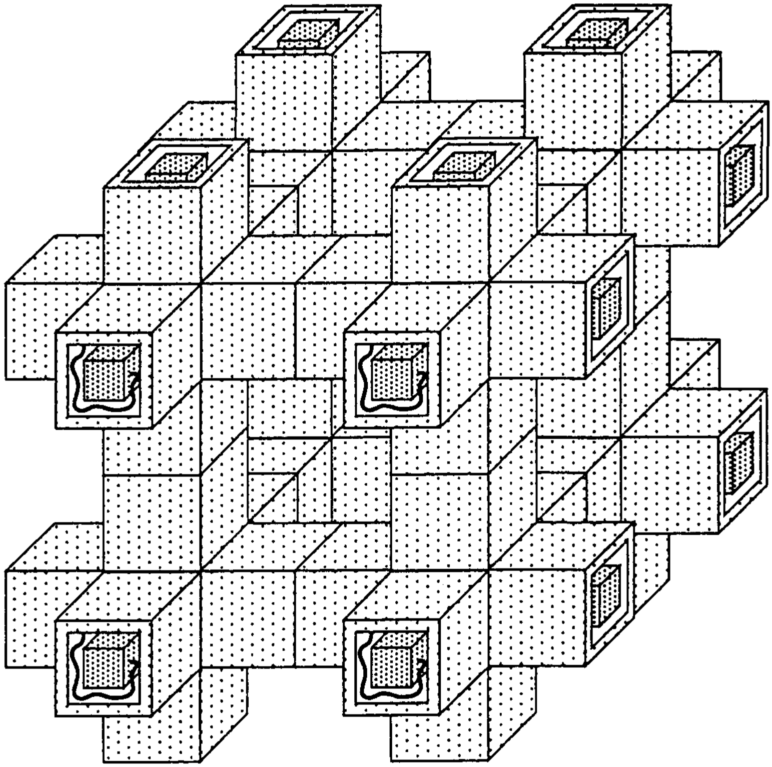

Fig. 3. – A weakly connected domain Ω_ε composed of 8 cells.

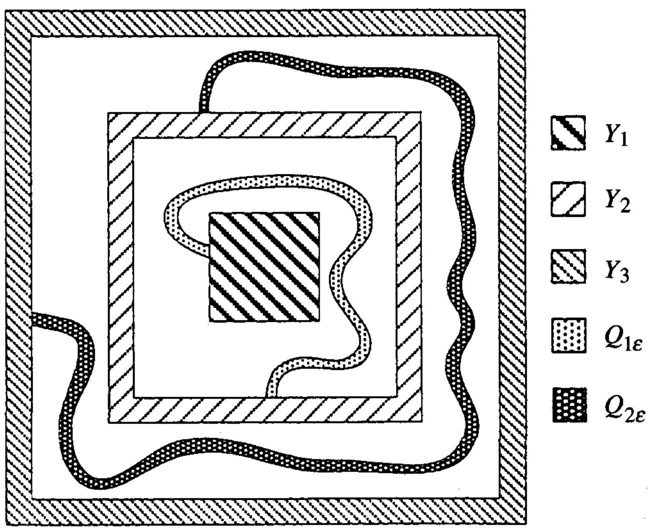

Fig. 4. – A cross section of a period Y_ε for $n = 3$.

$d=3$. Assume that the curve is strongly regular in the sense where for any $t\in[0,1]$, the d vectors $(F'_{k,\varepsilon}(t), F''_{k,\varepsilon}(t), F^{(d)}_{k,\varepsilon}(t))$ span the space $\mathbb{R}^d$. We can then define the Frenet curvilinear basis $(T_{k,\varepsilon}(t), N_{k,\varepsilon}(t), B_{k,\varepsilon}(t))$ associated to $F_{k,\varepsilon}$. Note that $B_{k,\varepsilon}(t)$ is a constant vector for a plane curve, *i.e.* $d=2$. Let $\Sigma_{k,\varepsilon}$ be a regular connected bounded open set of $\mathbb{R}^2$ such that $\alpha_k(\varepsilon) := |\Sigma_{k,\varepsilon}| \to 0$. The bridge $Q_{k,\varepsilon}$ is defined by

$$(2.4)\quad Q_{k,\varepsilon} := \{y=\Phi_{k,\varepsilon}(t,\nu,\beta) := F_{k,\varepsilon}(t)+\nu N_{k,\varepsilon}(t)+\beta B_{k,\varepsilon}(t);\ t\in[0,1],(\nu,\beta)\in\Sigma_{k,\varepsilon}\},$$

i. e. $Q_{k,\varepsilon}$ is a tube of cross section $\Sigma_{k,\varepsilon}$ along the curve $F_{k,\varepsilon}$. The set $Q_{k,\varepsilon}$ satisfies the assumption

$$(\text{H4})\quad \begin{cases} \Phi_{k,\varepsilon}:]0,1[\times\Sigma_{k,\varepsilon}\to \overset{\circ}{Q}_{k,\varepsilon} \text{ is a diffeomorphism for any sufficiently small } \varepsilon, \\ \Phi_{k,\varepsilon}(]0,1[\times\Sigma_{k,\varepsilon})\subset \overset{\circ}{Y}\setminus\overline{Y_k\cup Y_{k+1}}, \\ \Phi_{k,\varepsilon}(\{0\}\times\Sigma_{k,\varepsilon})\subset\partial Y_k \text{ and } \Phi_{k,\varepsilon}(\{1\}\times\Sigma_{k,\varepsilon})\subset\partial Y_{k+1}, \\ \max(|\nu|,(d-2)|\beta|)\le c\,\alpha_k(\varepsilon)^{2-d/2}. \end{cases}$$

The third condition of (H4) implies that the neighbourhood of $F_{k,\varepsilon}(0)$ in ∂Y_k and $F_{k,\varepsilon}(1)$ in ∂Y_{k+1} are portions of planes which are perpendicular to the tangent vectors $F'_{k,\varepsilon}(0)$ and $F'_{k,\varepsilon}(1)$ respectively, as shown in Figure 2.

EXAMPLE 2.1. – 1) If $\Sigma_{k,\varepsilon}$ is a rectangle of length l and of width $\alpha_k(\varepsilon)/l$, $Q_{k,\varepsilon}$ is a curved plate of thickness $\alpha_k(\varepsilon)/l$.

2) If $\Sigma_{k,\varepsilon}$ is a disk of radius $\sqrt{\alpha_k(\varepsilon)/\pi}$, $Q_{k,\varepsilon}$ is a tube of cross section area $\alpha_k(\varepsilon)$.

The following result yields sufficient conditions carrying on the curve $F_{k,\varepsilon}$ in order to obtain hypothesis (H4).

LEMMA 2.2. – *Assume that the curve $F_{k,\varepsilon}$ satisfies the condition*

$$\forall(s,t)\in[0,1],\qquad |F_{k,\varepsilon}(s)-F_{k,\varepsilon}(t)| > c(\varepsilon)\,|s-t|$$

where $c(\varepsilon)$ is such that

$$\frac{\alpha_k(\varepsilon)^{2-d/2}}{c(\varepsilon)^2}\|F''_{k,\varepsilon}\|_\infty + \frac{\alpha_k(\varepsilon)^{2-d/2}}{c(\varepsilon)}(\|\varrho_{k,\varepsilon}F'_{k,\varepsilon}\|_\infty + \|\tau_{k,\varepsilon}F'_{k,\varepsilon}\|_\infty) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where $\varrho_{k,\varepsilon}$ is the curvature and $\tau_{k,\varepsilon}$ the torsion of $F_{k,\varepsilon}$.
Then, the function $\Phi_{k,\varepsilon}$ is a diffeomorphism for any sufficiently small ε.

Lemma 2.2 is proved in Subsection 4.3.

2.2. *The problem.* – Our aim is to study the asymptotic behaviour of the spectrum $(\lambda_k(\varepsilon))_{k\in\mathbb{N}}$ where

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \ldots$$

of the domain Ω_ε related to the spectral Neumann problem in Ω_ε

$$\text{(2.7)} \qquad \begin{cases} -\Delta v_\varepsilon = \lambda(\varepsilon)\, v_\varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \displaystyle\fint_{\Omega_\varepsilon} v_\varepsilon^2 = 1\,, \end{cases}$$

where ν denotes the outer unit normal to $\partial\Omega$ and $\fint_Y$ the mean value $1/|Y| \int_Y$.

2.3. *The result.* – We will state a convergence result for the spectrum of problem (2.7) which depends on the limiting behaviour of the characteristic sizes of the bridges $Q_{k,\varepsilon}$ (2.4), defined by

$$\text{(2.8)} \qquad \delta_k(\varepsilon) := \frac{\alpha_k(\varepsilon)}{\varepsilon^2 l_k(\varepsilon)}, \qquad 1 \leq k \leq n-1\,,$$

where $\alpha_k(\varepsilon)$ denotes the area of the cross section of the bridge $Q_{k,\varepsilon}$ and $l_k(\varepsilon)$ its length, *i.e.*

$$l_k(\varepsilon) = \int_0^1 |F'_{k,\varepsilon}(t)|\, dt\,.$$

We will also need the following notations

NOTATION 2.3. – Let A_k, $1 \leq k \leq n$, be n positive definite 3×3 matrices, let m be an integer $1 \leq m \leq n$, and let δ_k, $1 \leq k \leq n-1$, be $(n-1)$ reals such that $\delta_1 = \ldots = \delta_{m-1} = 0$ and $\delta_k > 0$ for any $m \leq k \leq n-1$. We denote by A the $3n \times 3n$ matrix

$$\text{(2.9)} \qquad A := \begin{pmatrix} \dfrac{A_1}{|Y_1|} & & 0 \\ & \ddots & \\ 0 & & \dfrac{A_n}{|Y_n|} \end{pmatrix}$$

and by J the $n \times n$ matrix

$$J := \begin{bmatrix} \frac{\delta_1}{|Y_1|} & -\frac{\delta_1}{|Y_1|} & & & 0 \\ \ddots & \ddots & & & \\ & -\frac{\delta_{k-1}}{|Y_k|} & \frac{\delta_{k-1}+\delta_k}{|Y_k|} & -\frac{\delta_k}{|Y_k|} & \\ & & \ddots & \ddots & \\ 0 & & & -\frac{\delta_{n-1}}{|Y_n|} & \frac{\delta_{n-1}}{|Y_n|} \end{bmatrix}. \tag{2.10}$$

The system

$$\begin{cases} -\operatorname{div}(A_1 \nabla v_1) + \delta_1(v_1 - v_2) = |Y_1| f_1 & \text{in } \Omega, \\ -\operatorname{div}(A_k \nabla v_k) + \delta_{k-1}(v_k - v_{k-1}) + \delta_k(v_k - v_{k+1}) = |Y_k| f_k & \text{in } \Omega, \\ -\operatorname{div}(A_n \nabla v_n) + \delta_{n-1}(v_n - v_{n-1}) = |Y_n| f_n & \text{in } \Omega, \\ A_k \nabla v_k \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

can be written

$$\begin{cases} -\operatorname{div}(A \nabla \vec{v}) + J\vec{v} = \vec{f} & \text{in } \Omega, \\ (A \nabla \vec{v})\,\nu = \vec{0} & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

where $\vec{v} := (v_1, \ldots, v_n)$, $\vec{f} := (f_1, \ldots, f_n)$, A is defined by (2.9) and J by (2.10).

REMARK 2.4. – Since $\delta_1 = \ldots = \delta_{m-1} = 0$, system (2.12) is elliptic in the Hilbert space

$$\vec{H} := \left\{ v \in H^1(\Omega)^n ;\ \int_\Omega v_1 = \ldots = \int_\Omega v_{m-1} = \sum_{k=m}^{n} |Y_k| \int_\Omega v_k = 0 \right\} \tag{2.13}$$

provided with the scalar product and the norm in $L^2(\Omega)^n$

$$(\vec{u}, \vec{v})_\Omega := \sum_{i=1}^{n} ⨍_\Omega \theta_i u_i v_i \quad \text{and} \quad \|\vec{u}\|_\Omega := (\vec{u}, \vec{u})_\Omega^{1/2}, \qquad \theta_i := \frac{|Y_i|}{\sum_{h \le n} |Y_h|}. \tag{2.14}$$

We can now state the main result of the paper.

THEOREM 2.5. – *Assume that the geometrical conditions* (H1)-(H4) *are satisfied. Also assume that the cross section area* $a_k(\varepsilon)$ *and the length* $l_k(\varepsilon)$ *of each bridge* $Q_{k,\varepsilon}$,

$1 \leq k \leq n-1$, *satisfy the limit*

$$\frac{\alpha_k(\varepsilon)}{\varepsilon} + \varepsilon l_k(\varepsilon) + \alpha_k(\varepsilon)^{2-d/2}(\varrho_k(\varepsilon) + \tau_k(\varepsilon)) \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.15}$$

where $\varrho_k(\varepsilon)$ *and* $\tau_k(\varepsilon)$ *respectively denote the maximum of curvature and torsion of the curve* $F_{k,\varepsilon}$ *which defines* $Q_{k,\varepsilon}$ *in* (2.4). *Finally assume that the characteristic lengths* (2.8) *of the bridges* $Q_k(\varepsilon)$ *satisfy*

$$\delta_1(\varepsilon) \ll \ldots \ll \delta_{m-1}(\varepsilon) \ll 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \delta_k(\varepsilon) = \delta_k > 0 \text{ for } m \leq k \leq n-1, \tag{2.16}$$

where m *is a given integer of* $\{1, \ldots, n\}$ $(\alpha(\varepsilon) \ll \beta(\varepsilon)$ *means* $(\alpha(\varepsilon)/\beta(\varepsilon)) \to 0)$.

i) *Then, for any* $1 \leq k \leq m-1$, *the eigenvalue* $\lambda_k(\varepsilon)$ *of problem* (2.7) *converges to 0 and more precisely satisfies the following asymptotic behaviour*

$$\lambda_k(\varepsilon) \underset{\varepsilon \to 0}{\sim} \delta_k(\varepsilon)\left(\frac{1}{|Y_k|} + \frac{1}{\sum_{j>k} |Y_j|}\right) \quad \text{for } 1 \leq k \leq m-1, \tag{2.17}$$

where $|Y_k|$ *denotes the Lebesgue measure of the rescaled period* Y_k *of the domain* $\Omega_{k,\varepsilon}$.

Moreover, for any $1 \leq k \leq m-1$, *any eigenvector* $v_{k,\varepsilon}$ *solution of* (2.8) *with* $\lambda_k(\varepsilon)$ *satisfies the convergence*

$$\left\| v_{k,\varepsilon} - c_{k,k} \mathbf{1}_{\Omega_{k,\varepsilon}} - \sum_{j>k} c_{k,k+1} \mathbf{1}_{\Omega_{j,\varepsilon}} \right\|_{H^1(\Omega_\varepsilon)} \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.18}$$

where $c_{k,k}$ *and* $c_{k,k+1}$ *are constants solutions of*

$$\begin{cases} \theta_k c_{k,k} + \left(\sum_{j>k} \theta_j\right) c_{k,k+1} = 0, \\ \theta_k c_{k,k}^2 + \left(\sum_{j>k} \theta_j\right) c_{k,k+1}^2 = 1, \quad \theta_j := \dfrac{|Y_j|}{\sum_{i=1}^{n} |Y_i|}. \end{cases} \tag{2.19}$$

ii) *For any* $k \geq m$, *the eigenvalue* $\lambda_k(\varepsilon)$ *converges, up to a subsequence, to the eigenvalue* λ_k *of problem* (1.5) *which can be written*

$$\begin{cases} -\operatorname{div}(A\nabla \vec{v}) + J\vec{v}_k = \lambda_k \vec{v}_k & \text{in } \Omega, \\ (A\nabla \vec{v}_k)\nu = \vec{0} & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where $\vec{v}_k \in \vec{H}$ *from* (2.13), *the matrix* J *is defined by* (2.10), *and the matrix* A *is defined by formula* (2.9) *in which each matrix* A_j, $1 \leq j \leq n$, *is the homogenized matrix for the perforated domain* $\Omega_{j,\varepsilon}$, *obtained by the classical formula*

$$A_j \lambda := \int_{Y_j} \nabla W_j^\lambda, \qquad \lambda \in \mathbb{R}^3, \tag{2.21}$$

the function W_j^λ *being the solution of the Neumann problem*

$$\begin{cases} \Delta W_j^\lambda = 0 & \text{in } Y_j, \\ \dfrac{\partial W_j^\lambda}{\partial \nu} = 0 & \text{in } \partial Y_j \setminus \partial Y, \\ W_j^\lambda(y) - \lambda \cdot y & Y\text{-periodic}. \end{cases} \tag{2.22}$$

REMARK 2.6. – 1) The convergence $\lambda_k(\varepsilon) \to \lambda_k$ also holds for $0 \leq k \leq m-1$; indeed, the eigenspace of problem (2.20) related to the eigenvalue 0 is of dimension m because of the equalities $\delta_1 = \ldots = \delta_{m-1} = 0$. The first m eigenvalues $\lambda_k(\varepsilon)$, $0 \leq k \leq m-1$, are also simple since assumption (2.16) and result (2.17) imply $0 = \lambda_0(\varepsilon) \ll \lambda_1(\varepsilon) \ll \ldots \ll \ll \lambda_{m-1}(\varepsilon) \ll 1$.

2) Theorem 2.5 has a non-trivial consequence concerning the estimate of the Poincaré-Wirtinger constant of the domain Ω_ε. Indeed, if $m = 1$ or equivalently $\delta_1(\varepsilon) \to \delta_1 > 0$, no convergent subsequence of $\lambda_1(\varepsilon)$ tends to 0, and hence the Poincaré-Wirtinger constant $C(\Omega_\varepsilon)$ of Ω_ε, which is equal to $\lambda_1(\varepsilon)^{-2}$, is less than a fixed constant. We have just proved that there exists a constant $C > 0$, independent of ε, such that

$$\forall u \in H^1(\Omega_\varepsilon), \left\| u - \fint_{\Omega_\varepsilon} u \right\|_{L^2(\Omega_\varepsilon)} \leq C \|\nabla u\|_{L^2(\Omega_\varepsilon)}. \tag{2.23}$$

Although the weakly connected domain Ω_ε contains very thin bridges, the constant $C(\Omega_\varepsilon)$ does not blow up with respect to ε which was not *a priori* evident.

3) If $\delta_j(\varepsilon) \to +\infty$ for some $m \leq j \leq n-1$, we have to replace in system (2.20) satisfied by $\vec{v}_k = (v_{k,1}, \ldots, v_{k,n})$ equations

$$\begin{cases} -\operatorname{div}(A_j \nabla v_{k,j}) + \delta_{j-1}(v_{k,j} - v_{k,j-1}) + \delta_j(v_{k,j} - v_{k,j+1}) = \lambda_k |Y_j| v_{k,j} & \Omega, \\ -\operatorname{div}(A_{j+1} \nabla v_{k,j+1}) + \delta_j(v_{k,j+1} - v_{k,j}) + \delta_{j+1}(v_{k,j+1} - v_{k,j+2}) = \lambda_k |Y_{j+1}| v_{k,j+1} & \Omega, \\ A_j \nabla v_{k,j} \cdot \nu = A_{j+1} \nabla v_{k,j+1} \cdot \nu = 0 & \partial\Omega, \end{cases}$$

by the new ones

$$\begin{cases} -\operatorname{div}((A_j + A_{j+1}) \nabla v_{k,j}) + \delta_{j-1}(v_{k,j} - v_{k,j-1}) + \delta_{j+1}(v_{k,j} - v_{k,j+2}) = \lambda_k |Y_j| v_{k,j} & \Omega, \\ v_{k,j} - v_{k,j+1} = 0 & \Omega, \\ (A_j + A_{j+1}) \nabla v_{k,j} \cdot \nu = 0 & \partial\Omega. \end{cases}$$

That exactly corresponds to consider the first system as δ_j tends to $+\infty$. For the sake of simplicity, we will assume in the following that $\delta_j < +\infty$ for any $1 \leq j \leq n-1$.

Theorem 2.5 is partially based on a homogenization result.

2.4. *A homogenization result.* – The proof is divided in several steps. The first of them is to obtain the limiting behaviour of the Neumann problem

$$
\text{(2.24)} \qquad \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}
$$

where u_ε is a bounded sequence from $H^1(\Omega_\varepsilon)$ and f_ε is a bounded sequence from $L^2(\Omega)$. The result of this homogenization problem was first obtained by E. Ya. Hruslov [5] when $f_\varepsilon = f$ is a given function of $L^2(\Omega)$ and completely proved in [3] using in particular the maximum principle. Here, we prove the following result for a general weakly convergent sequence f_ε from $L^2(\Omega)$, but with more restrictive geometrical conditions on the bridges since we cannot use the maximum principle.

THEOREM 2.7. – *Assume that hypotheses* (H1)-(H4) *are satisfied as well as conditions* (2.15) *on the size parameters of the bridges* $Q_{k,\varepsilon}$, $1 \leq k \leq n-1$. *Also assume as in Theorem* 2.5 *that there exists an integer* $m \in \{1, \ldots, n\}$ *such that the characteristic sizes* (2.8) *of the bridges satisfy the condition*

$$
\text{(2.25)} \qquad \begin{cases} \lim\limits_{\varepsilon \to 0} \delta_k(\varepsilon) = \delta_k = 0 & \textit{for } 1 \leq k \leq m-1, \\ \lim\limits_{\varepsilon \to 0} \delta_k(\varepsilon) = \delta_k > 0 & \textit{for } m \leq k \leq n-1. \end{cases}
$$

Let Ω'_ε *be the subset defined by*

$$
\text{(2.26)} \qquad \Omega'_\varepsilon := \left(\bigcup_{k=m}^{n} \Omega_{k,\varepsilon} \right) \cup \left(\bigcup_{k=m}^{n-1} \omega_{k,\varepsilon} \right).
$$

Let u_ε *be a bounded sequence of* $H^1(\Omega_\varepsilon)$ *and let* f_ε *be a bounded sequence of* $L^2(\Omega_\varepsilon)$ *of zero* Ω_ε*-mean which are solutions of problem* (2.24) *and such that*

$$
\text{(2.27)} \qquad \mathbf{1}_{\Omega_{k,\varepsilon}} f_\varepsilon \rightharpoonup |Y_k| f_k \quad \textit{weakly in } L^2(\Omega) \textit{ for } 1 \leq k \leq n.
$$

Then, the following convergences hold true

$$
\text{(2.28)} \qquad \begin{cases} \mathbf{1}_{\Omega_{k,\varepsilon}} \left(u_\varepsilon - \displaystyle\fint_{\Omega_{k,\varepsilon}} u_\varepsilon \right) \rightharpoonup |Y_k| u_k & \textit{weakly in } L^2(\Omega) \quad \textit{for } 1 \leq k \leq m-1, \\ \mathbf{1}_{\Omega_{k,\varepsilon}} \left(u_\varepsilon - \displaystyle\fint_{\Omega'_\varepsilon} u_\varepsilon \right) \rightharpoonup |Y_k| u_k & \textit{weakly in } L^2(\Omega) \quad \textit{for } m \leq k \leq n, \end{cases}
$$

where $\vec{u} = (u_1, \ldots, u_n)$ *belongs to the space* $\vec{H}$ *from* (2.13) *and is solution of system* (2.12), *i.e.*

$$
\text{(2.29)} \qquad \begin{cases} -\operatorname{div}(A \nabla \vec{u}) + J \vec{u} = \vec{f} & \textit{in } \Omega, \\ (A \nabla \vec{u})\, \nu = \vec{0} & \textit{on } \Omega. \end{cases}
$$

REMARK 2.8. – 1) The result of Theorem 2.7 also holds true if we only assume that u_ε is bounded in $L^2(\Omega'_\varepsilon)$ and ∇u_ε is bounded in $L^2(\Omega_\varepsilon)^3$.

2) Remark 2.6 3) can be also applied to the case of Theorem 2.7.

3. – Proof of the spectral result.

The proof of Theorem 2.5 is divided in several steps which are detailed in the following subsections. In the first subsection, we make precise the convergence of eigenvectors $v_{k,\varepsilon}$ associated to the $(m-1)$ first non-zero eigenvalues $\lambda_k(\varepsilon)$, $1 \leq k \leq m-1$. In the second one, we give an estimate of the first non-zero eigenvalue $\lambda_1(\varepsilon)$, then in the third one, we yield an estimate of $\lambda_k(\varepsilon)$ for $1 \leq k \leq m-1$. In the fourth and last subsection, we study the convergence of the rest of the spectrum $(\lambda_k(\varepsilon))_{k\geq m}$.

3.1. *Convergence of the* $(m-1)$ *first eigenvectors.* – We consider a family $(v_k(\varepsilon))_{k\in\mathbb{N}}$ of eigenvectors associated to the spectrum $(\lambda_k(\varepsilon))_{k\in\mathbb{N}}$, *i.e.* $v_0(\varepsilon)=1$ and

$$\begin{cases} -\Delta v_{k,\varepsilon} = \lambda_k(\varepsilon)\, v_{k,\varepsilon} & \text{in } \Omega_\varepsilon \\ \dfrac{\partial v_{k,\varepsilon}}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \qquad \text{for } k \geq 1,$$

which is orthonormal with respect to the scalar product of $L^2(\Omega_\varepsilon)$

$$(u, v)_{\Omega_\varepsilon} := \fint_{\Omega_\varepsilon} uv, \qquad \|u\|_{\Omega_\varepsilon} := (u, v)^{1/2}_{\Omega_\varepsilon}.$$

Following the same argument as Vanninathan [8], it is easy to check that each sequence $(\lambda_k(\varepsilon))_{\varepsilon>0}$ for $k \geq 1$, is bounded with respect to ε by using the Courant-Fisher formula

$$\lambda_k(\varepsilon) = \min_{\substack{V\subset H^1(\Omega_\varepsilon)\\ \dim V = k+1}} \ \max_{v\in V\setminus\{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2}{\int_{\Omega_\varepsilon} v^2} \qquad (\lambda_0(\varepsilon)=0)$$

and by taking for V the space spanned by the $(k+1)$ first eigenvectors of the Neumann problem in the domain Ω.

Assume now that $\lambda_k(\varepsilon) \to 0$ up to a subsequence of ε and for some $k \geq 1$. Then by the homogenization result of Theorem 2.7, we have for any $1 \leq j \leq k$

$$\mathbf{1}_{\Omega_{i,\varepsilon}} v_{j,\varepsilon} \rightharpoonup |Y_i|\, v_{j,i} \qquad \text{weakly in } L^2(\Omega) \text{ for } 1 \leq i \leq n,$$

where $\vec{v}_j = (v_{j,1}, \ldots, v_{j,n})$ is solution of the system

$$\begin{cases} -\operatorname{div}(A\nabla \vec{v}_j) + J\vec{v}_j = \vec{0} & \text{in } \Omega, \\ (A\nabla \vec{v}_j)\,\nu = \vec{0} & \text{on } \partial\Omega. \end{cases}$$

By putting $\vec{v}_j$ as test function in the latter equation and summing over $m \leqslant i \leqslant n$, we obtain the equality

$$\sum_{i=1}^{n} \int_{\Omega} A_i \nabla v_{j,i} \cdot \nabla v_{j,i} + \sum_{i=m}^{n} \int_{\Omega} \delta_i (v_{j,i} - v_{j,i+1})^2 = 0,$$

which implies, since A_i is positive definite, that $v_{j,i}$ are constant $c_{j,i}$ satisfying

(3.1) $$\mathbf{1}_{\Omega_{i,\varepsilon}} v_{j,\varepsilon} \rightharpoonup |Y_i|\, c_{j,i} \quad \text{weakly in } L^2(\Omega) \text{ for } 1 \leqslant i \leqslant n,$$

and

(3.2) $$c_{j,i} = c_{j,m} \quad \text{for } m \leqslant i \leqslant n,$$

since $\delta_i > 0$ for $m \leqslant i \leqslant n-1$.

Let us compute the constants $c_{j,i}$. We have

$$\fint_{\Omega_\varepsilon} v_{j,\varepsilon} = 0,$$

then after passing to the limit thanks to (3.1) and by noting that $\mathbf{1}_{\omega_{i,\varepsilon}} v_{j,\varepsilon} \rightharpoonup 0$ weakly in $L^2(\Omega)$ since $|\omega_{i,\varepsilon}| \to 0$, we obtain

(3.3) $$\sum_{i=1}^{n} \theta_i c_{j,i} = 0, \qquad \theta_i := \frac{|Y_i|}{\sum_{h \leqslant n} |Y_h|}.$$

We also have

$$\fint_{\Omega_\varepsilon} v_{j,\varepsilon}^2 = 1$$

which would imply similarly to (3.3)

(3.4) $$\sum_{i=1}^{n} \theta_i c_{j,i}^2 = 1.$$

However, the passing to the limit (3.4) is more delicate since we need strong convergence. On the first hand, by an extension result for periodic domains due to Acerbi, Chiado Piat, Dal Maso, Percivale [1] and thanks to assumptions (H1)-(H2), there exists for each $1 \leqslant i \leqslant n$, a bounded extension operator $P_{i,\varepsilon}$ from $H^1(\Omega_{i,\varepsilon})$ into $H^1(\Omega)$. The fact that Ω is composed of entire cells $\varepsilon Y + \varepsilon\kappa$, $\kappa \in \mathbb{Z}^3$, (assumption (H1)) combined with the connectedness and the regularity of the period cell Y_i (assumption (H2)) allows us to extend functions from $\Omega_{i,\varepsilon}$ to the whole domain Ω. On the contrary, in [1] quite less restrictive assumptions on the periodic domain E_i of (2.1) only provide a local extension in Ω. Since $P_{i,\varepsilon} v_{j,\varepsilon}$ is bounded in $H^1(\Omega)$ it strongly converges in $L^2(\Omega)$ up to a subsequence, which implies

$$1_{\Omega_{i,\varepsilon}}(v_{j,\varepsilon} - c_{j,i}) \to 0 \quad \text{strongly in } L^2(\Omega) \text{ for } 1 \leqslant i \leqslant n. \tag{3.5}$$

On the other hand, we have to estimate the contribution of the quadratic terms $1_{\omega_{i,\varepsilon}} v_{j,\varepsilon}^2$. It is given by the following result which is proved in the last section.

LEMMA 3.1. – *Assume that conditions* (H3) *and* (H4) *are satisfied. Let* $1 \leqslant k \leqslant n-1$, *then there exists a constant* $c > 0$ *such that for any function* $V \in H^1(Y_k \cup Q_{k,\varepsilon})$

$$\|V\|_{L^1(Q_{k,\varepsilon})} \leqslant c l_k(\varepsilon)(\alpha_k(\varepsilon)\|V\|_{L^1(Y_k)} + \|\nabla V\|_{L^1(Y_k \cup Q_{k,\varepsilon})}). \tag{3.6}$$

By rescaling estimate (3.6) in each cell $\varepsilon\kappa + \varepsilon Y$ with function $V(y) := v_{j,\varepsilon}^2(\varepsilon y + \varepsilon\kappa)$ and by summing over $\kappa \in \mathbb{Z}^3$, we obtain the following $L^2(\omega_{i,\varepsilon})$-estimate

$$\|v_{j,\varepsilon}\|^2_{L^2(\omega_{i,\varepsilon})} \leqslant c\alpha_i(\varepsilon)\, l_i(\varepsilon)\|v_{j,\varepsilon}\|^2_{L^2(\Omega_{i,\varepsilon})} + c\varepsilon l_i(\varepsilon)\|v_{j,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\|\nabla v_{j,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant$$

$$\leqslant c'(\alpha_i(\varepsilon)\, l_i(\varepsilon) + \varepsilon l_i(\varepsilon)) \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ by (2.15)},$$

i.e. $1_{\omega_{i,\varepsilon}} v_{j,\varepsilon} \to 0$ strongly in $L^2(\Omega)$. The latter and (3.5) imply the desired limit (3.4).

We have thus obtained three equalities (3.2), (3.3), (3.4) satisfied by the constants $c_{j,i}$ for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant n$. The next step consists in proving by induction on $1 \leqslant k \leqslant m-1$ that these constant satisfy

$$c_{k,j} = 0 \quad \text{for } 1 \leqslant j \leqslant k-1, \tag{3.7}$$

$$c_{k,j} = c_{k,k+1} \quad \text{for } j \geqslant k+1, \tag{3.8}$$

and that the eigenvalue $\lambda_k(\varepsilon)$ satisfies equivalence (2.17).

The following subsection is devoted to the proof of (3.7), (3.8) and (2.17) for $k = 1$.

3.2. *Estimate of the first non-zero eigenvalue.*

3.2.1. A capacitary result carrying on the bridges. – Let us give a capacitary estimate of each bridge $Q_{k,\varepsilon}$, $1 \leqslant k \leqslant n-1$, which is an adaptation of a similar one in [3] and a key ingredient for obtaining the limiting behaviour of the spectrum.

LEMMA 3.2. – *Let* $1 \leqslant k \leqslant n-1$ *and let* $\widehat{V}_{k,\varepsilon}$ *be the function defined by*

$$
\text{(3.9)}\qquad
\begin{cases}
\widehat{V}_{k,\varepsilon}(y) = 1 & \text{if } y \in \left(\bigcup_{j=1}^{k} Y_j\right) \cup \left(\bigcup_{j=1}^{k-1} Q_{j,\varepsilon}\right), \\
\widehat{V}_{k,\varepsilon}(y) = 0 & \text{if } y \in \left(\bigcup_{j=k+1}^{n} Y_j\right) \cup \left(\bigcup_{j=k+1}^{n-1} Q_{j,\varepsilon}\right), \\
\widehat{V}_{k,\varepsilon}(y) = \dfrac{\sigma_{k,\varepsilon}(t)}{l_k(\varepsilon)} & \text{if } y = \Phi_{k,\varepsilon}(t,\nu,\beta) \in Q_{k,\varepsilon},
\end{cases}
$$

where Y_ε *is the period of the weakly connected domain defined by* (2.2), *the function* $\Phi_{k,\varepsilon}$ *is the diffeomorphism from* (2.4) *and* $\sigma_{k,\varepsilon}$ *denotes the curvilinear coordinate along the curve* $F_{k,\varepsilon}$, *i.e.*

$$
\sigma_{k,\varepsilon}(t) := \int_0^t |F'_{k,\varepsilon}(s)|\,ds\,.
$$

Assume that conditions (H3)-(H4) *and* (2.15) *hold true.*

Then, the bridge $Q_{k,\varepsilon}$ *satisfies the equivalence*

$$
\text{(3.10)}\qquad |Q_{k,\varepsilon}| \underset{\varepsilon\to 0}{\sim} \alpha_k(\varepsilon)\, l_k(\varepsilon)\,.
$$

Moreover the function $\widehat{V}_{k,\varepsilon}$ *verifies the estimates*

$$
\text{(3.11)}\qquad |\nabla \widehat{V}_{k,\varepsilon}| = \frac{1_{Q_{k,\varepsilon}} + o_\varepsilon(1)}{l_k(\varepsilon)} \quad \text{and} \quad \|\nabla \widehat{V}_{k,\varepsilon}\|_{L^2(Q_{k,\varepsilon})} \underset{\varepsilon\to 0}{\sim} \frac{\alpha_k(\varepsilon)}{l_k(\varepsilon)}\,,
$$

where $o_\varepsilon(\alpha(\varepsilon))$ *means* $\ll \alpha(\varepsilon)$, *and for any* $V \in H^1(Y_\varepsilon)$

$$
\text{(3.12)}\qquad \left| \frac{1}{\varepsilon^2} \int_{Q_{k,\varepsilon}} \nabla V \cdot \nabla \widehat{V}_{k,\varepsilon} - \widehat{\delta}_k(\varepsilon) \left(\fint_{Y_k} V - \fint_{Y_{k+1}} V \right) \right| \leqslant o_\varepsilon\!\left(\frac{\delta_k(\varepsilon)^{1/2}}{\varepsilon} \right) \|\nabla V\|_{L^2(Y_\varepsilon)},
$$

where $\widehat{\delta}_k(\varepsilon) \underset{\varepsilon\to 0}{\sim} \delta_k(\varepsilon)$ *the characteristic size* (2.8) *of the bridge* $Q_{k,\varepsilon}$.

This lemma is proved in the next section after the proof of Theorem 2.7.

By rescaling estimate (3.12) with the function

$$
\text{(3.13)}\qquad \widehat{v}_{k,\varepsilon}(x) := \widehat{V}_{k,\varepsilon}\left(\frac{x}{\varepsilon}\right),
$$

which is equal to zero outside $\omega_{k,\varepsilon}$, we obtain for any function $v \in H^1(\Omega_\varepsilon)$

$$
\text{(3.14)}\qquad \left| \int_{\Omega_\varepsilon} \nabla v \cdot \nabla \widehat{v}_{k,\varepsilon} - \widehat{\delta}_k(\varepsilon) \int_{\Omega_\varepsilon} \left(\frac{1_{\Omega_{k,\varepsilon}}}{|Y_k|} - \frac{1_{\Omega_{k+1,\varepsilon}}}{|Y_{k+1}|} \right) v \right| \leqslant o_\varepsilon(\delta_k(\varepsilon)^{1/2}) \|\nabla v\|_{L^2(\Omega_\varepsilon)}.
$$

3.2.2. First estimate of $\lambda_1(\varepsilon)$. – The Courant-Fisher formulas yield

$$\lambda_1(\varepsilon) = \min_{\substack{v \in H^1(\Omega_\varepsilon)\setminus\{0\} \\ v \perp 1}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2}{\int_{\Omega_\varepsilon} v^2}.$$

We then put the function $v_\varepsilon := \widehat{v}_{1,\varepsilon} - \fint_{\Omega_\varepsilon} \widehat{v}_{1,\varepsilon}$ in the previous infimum. Since $\fint_{\Omega_\varepsilon} \widehat{v}^2_{1,\varepsilon} \to \theta_1$ by (3.3) and (3.9), we have $\fint_{\Omega_\varepsilon} v_\varepsilon^2 \to \theta_1(1-\theta_1) > 0$ which implies that

$$\lambda_1(\varepsilon) \leqslant c \int_{\Omega_\varepsilon} |\nabla \widehat{v}_{1,\varepsilon}|^2 \leqslant \frac{c'}{\varepsilon^2} \int_{\Omega_\varepsilon} |\nabla \widehat{V}_{1,\varepsilon}|^2.$$

It follows thanks to (3.11) the first estimate

$$\lambda_1(\varepsilon) \leqslant c\delta_1(\varepsilon). \tag{3.15}$$

3.2.3. Proof of $c_{1,j} = c_{1,2}$ for $j \geqslant 2$. – Let $2 \leqslant j \leqslant n-1$. We put $v := v_{1,\varepsilon}$ the eigenvector associated to $\lambda_1(\varepsilon)$ and $k = j$ in estimate (3.14), which yields

$$\left| \lambda_1(\varepsilon) \int_{\Omega_\varepsilon} v_{1,\varepsilon} \widehat{v}_{j,\varepsilon} - \widehat{\delta}_j(\varepsilon) \int_{\Omega_\varepsilon} \left(\frac{\mathbf{1}_{\Omega_{j,\varepsilon}}}{|Y_j|} - \frac{\mathbf{1}_{\Omega_{j+1,\varepsilon}}}{|Y_{j+1}|} \right) v_{1,\varepsilon} \right| \leqslant o_\varepsilon(\delta_j(\varepsilon)),$$

since $\|\nabla v_{1,\varepsilon}\|_{L^2(\Omega_\varepsilon)} = (|\Omega_\varepsilon| \delta_1(\varepsilon))^{1/2} = o_\varepsilon(\delta_j(\varepsilon))^{1/2}$ by (2.16). However, estimate (3.15) implies that $\lambda_1(\varepsilon) = o_\varepsilon(\delta_j(\varepsilon))$. Then by dividing the previous estimate by $\delta_j(\varepsilon) \underset{\varepsilon\to 0}{\sim} \widehat{\delta}_j(\varepsilon)$, the following limit holds

$$\int_{\Omega_\varepsilon} \left(\frac{\mathbf{1}_{\Omega_{j,\varepsilon}}}{|Y_j|} - \frac{\mathbf{1}_{\Omega_{j+1,\varepsilon}}}{|Y_{j+1}|} \right) v_{1,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

and consequently $c_{1,j} = c_{1,j+1}$ since $\mathbf{1}_{\Omega_{j,\varepsilon}} v_{1,\varepsilon} \rightharpoonup |Y_i| c_{1,j}$ by (3.1). We thus obtain $c_{1,j} = c_{1,2}$ for $j \geqslant 2$, *i.e.* (3.8) for $k = 1$.

3.2.4. Proof of the equivalence $\lambda_1(\varepsilon) \sim c_1 \delta_1(\varepsilon)$. – We now put $v := v_{1,\varepsilon}$ and $k = 1$ in estimate (3.14) divided by $\delta_1(\varepsilon)$ to obtain

$$\left| \frac{\lambda_1(\varepsilon)}{\delta_1(\varepsilon)} \int_{\Omega_\varepsilon} v_{1,\varepsilon} \widehat{v}_{1,\varepsilon} - \frac{\widehat{\delta}_1(\varepsilon)}{\delta_1(\varepsilon)} \int_{\Omega_\varepsilon} \left(\frac{\mathbf{1}_{\Omega_{1,\varepsilon}}}{|Y_1|} - \frac{\mathbf{1}_{\Omega_{2,\varepsilon}}}{|Y_2|} \right) v_{1,\varepsilon} \right| \leqslant o_\varepsilon(1)$$

which implies the convergence

$$\frac{\lambda_1(\varepsilon)}{\delta_1(\varepsilon)} \int_{\Omega_\varepsilon} v_{1,\varepsilon} \widehat{v}_{1,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} |\Omega|(c_{1,1} - c_{1,2}).$$

We have also

$$\int_{\Omega_\varepsilon} v_{1,\varepsilon}\widehat{v}_{1,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} |\Omega|\,|Y_1|\,c_{1,1}$$

in which $c_{1,1} \neq 0$ since (3.3), (3.4) and (3.8) for $k = 1$ give

$$\theta_1 c_{1,1} + \Big(\sum_{j>1}\theta_j\Big) c_{1,2} = 0 \quad \text{and} \quad \theta_1 c_{1,1}^2 + \Big(\sum_{j>1}\theta_j\Big) c_{1,2}^2 = 1$$

i.e. (2.19) for $k = 1$. Both previous limits thus imply that

$$\frac{\lambda_1(\varepsilon)}{\delta_1(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} \frac{c_{1,1} - c_{1,2}}{c_{1,1}\,|Y_1|} = \frac{1}{|Y_1|} + \frac{1}{\sum_{j>1}|Y_j|},$$

which is the desired equivalence (2.17).

3.3. *Estimate of the k-th non-zero eigenvalue.* – We proceed as in Subsection 3.2 by giving a first estimate of $\lambda_k(\varepsilon)$, then by proving equalities (3.8). From this point, the proof is quite different of the case $\lambda_1(\varepsilon)$ since it is not evident that

$$c_{k,k} \neq c_{k,k+1}. \tag{3.16}$$

Assuming (3.16), we yield a second estimate of $\lambda_k(\varepsilon)$ which allows us to prove (3.7) and then to obtain equivalence (2.17). We conclude this section with the proof of inequality (3.16) by using in an essential way the induction hypothesis, *i.e.* equalities (3.7) and (3.8) for the $(k-1)$ first non-zero eigenvalues.

3.3.1. First estimate of $\lambda_k(\varepsilon)$. – Let us prove that

$$\lambda_k(\varepsilon) \leqslant c\delta_k(\varepsilon). \tag{3.17}$$

Let us define the fonction

$$v_\varepsilon := \widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon} - \sum_{j=0}^{k-1} (\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}, v_{j,\varepsilon})_{\Omega_\varepsilon} v_{j,\varepsilon},$$

where $\widehat{v}_{k,\varepsilon}$ is defined by (3.9) and (3.13). The function v_ε is perpendicular to the projection in the vector space spanned by the k first eigenvectors $v_{j,\varepsilon}$ of the function $\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}$ which is a smooth approximation of $\mathbf{1}_{\Omega_{k,\varepsilon}}$ by definition (3.9). Then by the Courant-Fisher formulas, we have

$$\lambda_k(\varepsilon) \leqslant \frac{\|\nabla v_\varepsilon\|^2_{\Omega_\varepsilon}}{\|v_\varepsilon\|^2_{\Omega_\varepsilon}}$$

if $v_\varepsilon \neq 0$. We have by estimate (3.11) satisfied by $\widehat{V}_{k,\varepsilon}$

$$\|\nabla \widehat{v}_{k,\varepsilon}\|^2_{\Omega_\varepsilon} \leqslant \frac{c}{\varepsilon^2}\|\nabla \widehat{V}_{k,\varepsilon}\|^2_{Q_\varepsilon} \leqslant c'\,\delta_k(\varepsilon),$$

which implies

$$\|\nabla v_\varepsilon\|^2_{\Omega_\varepsilon} \leqslant c_1\left(\|\nabla \widehat{v}_{k,\varepsilon}\|^2_{\Omega_\varepsilon} + \|\nabla \widehat{v}_{k-1,\varepsilon}\|^2_{\Omega_\varepsilon} + \sum_{j=0}^{k-1}\|\nabla v_{j,\varepsilon}\|^2_{\Omega_\varepsilon}\right) \leqslant$$

$$\leqslant c_2\left(\delta_k(\varepsilon) + \delta_{k-1}(\varepsilon) + \sum_{j=0}^{k-1}\lambda_j(\varepsilon)\right) \leqslant c_3\sum_{j=0}^{k-1}\delta_j(\varepsilon) \leqslant c_4\delta_k(\varepsilon)$$

since by induction hypothesis $\lambda_j(\varepsilon) \leqslant c\delta_j(\varepsilon)$ for any $0 \leqslant j \leqslant k-1$.

It remains to prove that $\|v_\varepsilon\|_{\Omega_\varepsilon} \geqslant c > 0$ to obtain estimate (3.17). The orthonormality of the family of eigenvectors $(v_{j,\varepsilon})_{j \geqslant 1}$ yields

$$\|v_\varepsilon\|^2_{\Omega_\varepsilon} = \|\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}\|^2_{\Omega_\varepsilon} - \sum_{j=0}^{k-1}(\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}, v_{j,\varepsilon})^2_{\Omega_\varepsilon},$$

which tends to

$$\beta := \theta_k - \sum_{j=0}^{k-1}\theta_k^2 c_{j,k}^2$$

by convergences (3.5) and since $\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon} - \mathbf{1}_{\Omega_{k,\varepsilon}} \to 0$ strongly in $L^2(\Omega)$. Moreover, we have by (3.3), (3.4) and by the induction hypotheses (3.7), (3.8) for $1 \leqslant j \leqslant n-1$,

$$c_{0,k}^2 = 1 \quad \text{and} \quad c_{j,k}^2 = c_{j,j+1}^2 = \frac{\theta_j}{\sum_{i>j}\theta_i \sum_{i \geqslant j}\theta_i} = \frac{1}{\sum_{i>j}\theta_i} - \frac{1}{\sum_{i \geqslant j}\theta_i}.$$

The limit β of $\|v_\varepsilon\|^2_{\Omega_\varepsilon}$ thus satisfies

$$\beta = \theta_k - \theta_k^2 - \theta_k^2\sum_{j=1}^{k-1}\left(\frac{1}{\sum_{i>j}\theta_i} - \frac{1}{\sum_{i \geqslant j}\theta_i}\right) =$$

$$= \theta_k - \theta_k^2 - \theta_k^2\left(\frac{1}{\sum_{i \geqslant k}\theta_i} - 1\right) = \theta_k\left(1 - \frac{\theta_k}{\sum_{i \geqslant k}\theta_i}\right) > 0 \quad \text{since } k < n,$$

which implies (3.17).

3.3.2. Proof of $c_{k,j} = c_{k,k+1}$ for $j \geqslant k+1$. – We repeat the same argument as for $\lambda_1(\varepsilon)$ by putting $v := v_{k,\varepsilon}$ and $\widehat{v}_{j,\varepsilon}$ for $k+1 \leqslant j \leqslant m$ in estimate (3.14), which yields

$$\left|\frac{\lambda_k(\varepsilon)}{\delta_j(\varepsilon)}\int_{\Omega_\varepsilon} v_{k,\varepsilon}\widehat{v}_{j,\varepsilon} - \frac{\widehat{\delta}_j(\varepsilon)}{\delta_j(\varepsilon)}\int_{\Omega_\varepsilon}\left(\frac{\mathbf{1}_{\Omega_{j,\varepsilon}}}{|Y_j|} - \frac{\mathbf{1}_{\Omega_{j+1,\varepsilon}}}{|Y_{j+1}|}\right)v_{k,\varepsilon}\right| \leqslant o_\varepsilon\left(\frac{\lambda_k(\varepsilon)}{\delta_j(\varepsilon)}\right)^{1/2} = o_\varepsilon(1),$$

since by (3.17) and (2.16) $\lambda_k(\varepsilon) \leqslant c\delta_k(\varepsilon) = o_\varepsilon(\delta_j(\varepsilon))$ for $j > k$. By passing to the limit in the latter, we thus obtain thanks to convergence (3.1)

$$\int_{\Omega_\varepsilon}\left(\frac{\mathbf{1}_{\Omega_{j,\varepsilon}}}{|Y_j|}-\frac{\mathbf{1}_{\Omega_{j+1,\varepsilon}}}{|Y_{j+1}|}\right)v_{k,\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow}|\Omega|(c_{k,j}-c_{k,j+1})=0\qquad i.e.\ (3.8).$$

3.3.3. Proof of the equivalence $\lambda_k(\varepsilon)\sim c_k\delta_k(\varepsilon)$. – We also proceed as for $\lambda_1(\varepsilon)$ by using (3.14) with functions $v:=v_{k,\varepsilon}$ and $\widehat{v}_{k,\varepsilon}$, which implies thanks to (3.17)

$$\left|\frac{\lambda_k(\varepsilon)}{\delta_k(\varepsilon)}\int_{\Omega_\varepsilon}v_{k,\varepsilon}\widehat{v}_{k,\varepsilon}-\frac{\widehat{\delta}_k(\varepsilon)}{\delta_k(\varepsilon)}\int_{\Omega_\varepsilon}\left(\frac{\mathbf{1}_{\Omega_{k,\varepsilon}}}{|Y_k|}-\frac{\mathbf{1}_{\Omega_{k+1,\varepsilon}}}{|Y_{k+1}|}\right)v_{k,\varepsilon}\right|\leqslant o_\varepsilon\left(\frac{\lambda_k(\varepsilon)}{\delta_k(\varepsilon)}\right)^{1/2}=o_\varepsilon(1).$$

We also have

$$\int_{\Omega_\varepsilon}\left(\frac{\mathbf{1}_{\Omega_{k,\varepsilon}}}{|Y_k|}-\frac{\mathbf{1}_{\Omega_{k+1,\varepsilon}}}{|Y_{k+1}|}\right)v_{k,\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow}|\Omega|(c_{k,k}-c_{k,k+1}).$$

Let us assume for the moment inequality (3.16). Both last estimates combined with (3.17) then imply that

$$0\neq\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}v_{k,\varepsilon}\widehat{v}_{k,\varepsilon}=|\Omega|\sum_{j=1}^{k}|Y_j|c_{k,j}$$

and therefore the equivalence

$$\frac{\lambda_k(\varepsilon)}{\delta_k(\varepsilon)}\underset{\varepsilon\to 0}{\sim}\frac{c_{k,k}-c_{k,k+1}}{\sum_{j\leqslant k}|Y_j|c_{k,j}}\tag{3.18}$$

which is not exactly (2.17) but it is useful to prove (3.7). Once (3.7) will be proved, we will deduce (2.17) from (3.7) and (3.18).

3.3.4. Proof of $c_{k,j}=0$ for $1\leqslant j\leqslant k-1$. – Estimate (3.14) with $v:=v_{k,\varepsilon}$ and $\widehat{v}_{j,\varepsilon}$ for $1\leqslant j\leqslant k-1$, divided by $\lambda_k(\varepsilon)$ and combined with equivalence (3.18) and (2.16), yields

$$\left|\int_{\Omega_\varepsilon}v_{k,\varepsilon}\widehat{v}_{j,\varepsilon}-\frac{\widehat{\delta}_j(\varepsilon)}{\lambda_k(\varepsilon)}\int_{\Omega_\varepsilon}\left(\frac{\mathbf{1}_{\Omega_{j,\varepsilon}}}{|Y_j|}-\frac{\mathbf{1}_{\Omega_{j+1,\varepsilon}}}{|Y_{j+1}|}\right)v_{k,\varepsilon}\right|\leqslant o_\varepsilon\left(\frac{\delta_j(\varepsilon)}{\lambda_k(\varepsilon)}\right)^{1/2}=o_\varepsilon\left(\frac{\delta_j(\varepsilon)}{\delta_k(\varepsilon)}\right)^{1/2}=o_\varepsilon(1),$$

which implies the limit

$$\int_{\Omega_\varepsilon}v_{k,\varepsilon}\widehat{v}_{j,\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow}0=\sum_{i=1}^{j}|Y_i|c_{k,i}\qquad\text{for } 1\leqslant j\leqslant k-1,$$

and therefore (3.7).

Now equivalence (2.17) can be easily deduced from (3.2), (3.7), (3.8) and (3.18). It thus remains to prove inequality (3.16) we assumed before to conclude the induction proof.

3.3.5. Proof of $c_{k,k} \neq c_{k,k+1}$. – Let us prove it by contradiction. The equalities $c_{k,j} = c_{k,k+1} = c_{k,k}$ hold true since the proof of (3.8) only uses first estimate (3.17) and not equivalence (3.18). Then by the orthogonality of the family $(v_{j,\varepsilon})_{j \in \mathbb{N}}$ in $L^2(\Omega_\varepsilon)$, we have for any $1 \leq j \leq k-1$

$$(v_{j,\varepsilon}, v_{k,\varepsilon})_{\Omega_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 0 = \sum_{i=1}^{n} \theta_i c_{j,i} c_{k,i} = \sum_{i=1}^{k-1} \theta_i c_{j,i} c_{k,i} + \left(\sum_{i=k}^{n} \theta_i c_{j,i} \right) c_{k,k},$$

which combined with equality (3.2) shows that the vector

$$C_k := \begin{pmatrix} c_{1,k} \\ \vdots \\ c_{k,k} \end{pmatrix} \in \mathbb{R}^{k \times 1} \tag{3.19}$$

and the matrix

$$T_k := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{k-1} & \sum_{i \geq k} \theta_i \\ \theta_1 c_{1,1} & \theta_2 c_{1,2} & \cdots & \theta_{k-1} c_{1,k-1} & \sum_{i \geq k} \theta_i c_{1,i} \\ \theta_1 c_{2,1} & \theta_2 c_{2,2} & \cdots & \theta_{k-1} c_{2,k-1} & \sum_{i \geq k} \theta_i c_{2,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_1 c_{k-1,1} & \theta_1 c_{k-1,2} & \cdots & \theta_{k-1} c_{k-1,k-1} & \sum_{i \geq k} \theta_i c_{k-1,i} \end{bmatrix} \in \mathbb{R}^{k \times k} \tag{3.20}$$

satisfy the linear system $T_k C_k = 0$. However, by the induction hypothesis the coefficients $c_{j,i}$ satisfy (3.7) for $1 \leq j \leq k-1$, which imply the equality

$$T_k = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{k-1} & \sum_{i \geq k} \theta_i \\ \theta_1 c_{1,1} & \theta_2 c_{1,2} & \cdots & \theta_{k-1} c_{1,k-1} & \sum_{i \geq k} \theta_i c_{1,i} \\ & \theta_2 c_{2,2} & \cdots & \theta_{k-1} c_{2,k-1} & \sum_{i \geq k} \theta_i c_{2,i} \\ & & \ddots & \vdots & \vdots \\ 0 & & & \theta_{k-1} c_{k-1,k-1} & \sum_{i \geq k} \theta_i c_{k-1,i} \end{bmatrix}.$$

Then by replacing the k-th column $T_k(k)$ of T_k by $T_k(k) + \sum_{j<k} T_k(j)$ in the determinant

of T_k, we obtain thanks to equalities (3.3) satisfied by the coefficients $c_{j,i}$

$$\det T_k = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{k-1} & 1 \\ \theta_1 c_{1,1} & \theta_2 c_{1,2} & \cdots & \theta_{k-1} c_{1,k-1} & 0 \\ & \theta_2 c_{2,2} & \cdots & \theta_{k-1} c_{2,k-1} & 0 \\ & & \ddots & \vdots & \vdots \\ 0 & & & \theta_{k-1} c_{k-1,k-1} & 0 \end{bmatrix} = (-1)^{k-1} \prod_{j=1}^{k-1} \theta_j c_{j,j} \neq 0$$

since the coefficients $c_{j,j}$ satisfy (2.19) by the induction hypothesis. The matrix T_k defined by (3.20) is thus invertible and consequently the vector C_k defined by (3.19) and solution of $T_k C_k = 0$ is equal to 0. Finally by (3.8), $c_{k,j} = 0$ for any $1 \leqslant j \leqslant n$ which contradicts (3.4).

3.4. *Convergence of the rest of the spectrum.* – We proceed in two steps. In the first step, we prove that $\lambda_k(\varepsilon)$ converges to a non-zero eigenvalue of operator (2.20) for any $k \geqslant m$. In the second one, following the same argument as Vanninathan [8] for a strongly connected domain, we prove that any eigenvalue μ of elliptic problem (2.20) is the limit of a subsequence $(\lambda_k(\varepsilon))_{\varepsilon > 0}$ for a suitable $k \geqslant m$. As Vanninathan noted in [8], the previous result implies the convergence of the spectrum $(\lambda_k(\varepsilon))_{k \geqslant m}$ to the non-zero part of the spectrum from (2.20). Also note that this convergence still holds true for the zero eigenvalue from (2.20) whose multiplicity is equal to m since $\lambda_k(\varepsilon) \to 0$ for $0 \leqslant k \leqslant m-1$.

3.4.1. First step. – By using a diagonal extraction, we can assume that $\lambda_k(\varepsilon) \to \lambda_k$ for any $k \geqslant m$ up to a sequence of ε. Theorem 2.7 applied to the functions $u_\varepsilon := v_{k,\varepsilon}$ and $f_\varepsilon := \lambda_k(\varepsilon)\, v_{k,\varepsilon}$ implies that λ_k for $k \geqslant m$ is an eigenvalue from (2.20). It thus remains to prove that $\lambda_k > 0$ for any $k \geqslant m$ or equivalently $\lambda_m > 0$ since λ_m is the smallest. We proceed by contradiction assuming that $\lambda_m = 0$. Then by the convergence results of Subsection 3.1, the normalized eigenvector $v_{m,\varepsilon}$ associated to the eigenvalue $\lambda_m(\varepsilon)$ satisfies

$$1_{\Omega_{i,\varepsilon}} v_{m,\varepsilon} \rightharpoonup |Y_i|\, c_{m,i} \quad \text{weakly in } L^2(\Omega) \text{ for } 1 \leqslant i \leqslant n,$$

where $c_{m,i}$ are constant such that

$$\sum_{i=1}^{n} \theta_i c_{m,i} = 0 \quad \text{and} \quad \sum_{i=1}^{n} \theta_i c_{m,i}^2 = 1 .$$

Moreover by proceeding as in Subsection 3.3, estimate (3.14) combined with $\lambda_i(\varepsilon) \sim \delta_i(\varepsilon) = o_\varepsilon(\delta_m(\varepsilon))$ (since $\delta_m(\varepsilon) \to \delta_m > 0$) imply $c_{m,i} = 0$ for any $1 \leqslant i \leqslant m-1$, and estimate (3.14) combined with $\lambda_m(\varepsilon) \to 0$, and thus $\lambda_m(\varepsilon) = o_\varepsilon(\delta_i(\varepsilon))$, imply the equality $c_{m,i} = c_{m,m}$ for any $m \leqslant i \leqslant n$. Consequently, the vector $C_m \in \mathbb{R}^{m \times 1}$ from (3.19) in Subsection 3.3.5, is solution of the linear system $T_m C_m = 0$ in which the matrix $T_m \in \mathbb{R}^{m \times m}$ from (3.20) is invertible. It thus follows that $c_{m,i} = 0$ for any $1 \leqslant i \leqslant m$, which implies the contradiction

$$1=\sum_{i=1}^{n}\theta_i c_{m,i}^2=\left(\sum_{i=m}^{n}\theta_i\right)c_{m,m}=0\,.$$

3.4.2. Second step. – We denote by $\vec{v}_k:=(v_{k,1},\,\dots,\,v_{k,n})$ the limit of each eigenvector $v_{k,\varepsilon}$, $k\in\mathbb{N}$, according to the homogenization Theorem 2.7, *i.e.*

$$\mathbf{1}_{\Omega_{i,\varepsilon}}v_{k,\varepsilon}\rightharpoonup|Y_i|\,v_{k,i}\qquad\text{weakly in } L^2(\Omega)\ \text{ for } 1\leqslant i\leqslant n\,.$$

Note that the family $(\vec{v}_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)^n$ provided with the norm $\|\cdot\|_\Omega$ defined by (2.14), since the extension result of Subsection 3.1 applied to each domain $\Omega_{i,\varepsilon}$ implies the strong convergence

$$\mathbf{1}_{\Omega_{i,\varepsilon}}(v_{k,\varepsilon}-v_{k,i})\to0\qquad\text{strongly in } L^2(\Omega)\ \text{ for } 1\leqslant i\leqslant n\,.$$

Let us now prove the correspondence between the limit of the spectrum $(\lambda_k(\varepsilon))_{k\geqslant m}$ and the non-zero part of the spectrum of problem (2.20). We proceed by contradiction as in [8] by assuming that there exists an eigenvalue $\mu>0$ and an eigenvector $\vec{u}=(u_1,\,\dots,\,u_n)\in\vec{H}$ from (2.13) such that

$$\text{(3.21)}\qquad\begin{cases}-\operatorname{div}(A\nabla\vec{u})+J\vec{u}=\mu\vec{u} & \text{in } \Omega\,,\\ (A\nabla\vec{u})\,\nu=\vec{0} & \text{on } \partial\Omega\,,\\ \|\vec{v}\|_\Omega=1 & \\ (\vec{u},\,\vec{v}_k)_\Omega=0 & \text{for any } k\in\mathbb{N}\,,\\ \lim\limits_{\varepsilon\to0}\lambda_k(\varepsilon)\neq\mu & \text{for any } k\geqslant m\,.\end{cases}$$

Let $k\in\mathbb{N}$ such that $2\mu<\lambda_{k+1}$ and let u_ε be the solution of the Neumann problem

$$\text{(3.22)}\qquad\begin{cases}-\Delta u_\varepsilon=f_\varepsilon & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial u_\varepsilon}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \displaystyle⨍_{\Omega'_\varepsilon}u_\varepsilon=0 & \end{cases}$$

where the domain Ω'_ε is defined by (2.26), the zero Ω_ε-mean function f_ε by

$$f_\varepsilon:=\mu\sum_{i=1}^{m-1}\mathbf{1}_{\Omega_{i,\varepsilon}}\left(u_i-⨍_{\Omega_{i,\varepsilon}}u_i\right)+\mu\sum_{i=m}^{n}\mathbf{1}_{\Omega_{i,\varepsilon}}u_i-\mu c_\varepsilon\mathbf{1}_{\Omega'_\varepsilon}$$

and the constant c_ε by the mean value

$$c_\varepsilon := \sum_{i=m}^{n} \fint_{\Omega'_\varepsilon} \mathbf{1}_{\Omega_{i,\varepsilon}} u_i .$$

Contrary to the case of [8] where Ω_ε is a strongly connected domain, *i.e.* Ω_ε is reduced to one of the domains $\Omega_{i,\varepsilon}$, the sequence u_ε from (3.22) is not necessarily bounded in $H^1(\Omega_\varepsilon)$. However, the part *i*) of Theorem 2.5 applied with $m=1$ shows that the first non-zero eigenvalue of each domain $\Omega_{i,\varepsilon}$, $1 \leq i \leq n$, as well as the domain Ω'_ε, related to the Neumann problem are greater than a positive constant and therefore the Poincaré-Wirtinger constant (see (2.23)) of these domains is bounded with respect to ε. Then by putting u_ε in (3.22), we have

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \mu \sum_{i=1}^{m-1} \int_{\Omega_{i,\varepsilon}} \left(u_i - \fint_{\Omega_{i,\varepsilon}} u_i\right)\left(u_\varepsilon - \fint_{\Omega_{i,\varepsilon}} u_\varepsilon\right) +$$

$$+ \mu \int_{\Omega'_\varepsilon} \left(\sum_{i=m}^{n} \mathbf{1}_{\Omega_{i,\varepsilon}} u_i - c_\varepsilon\right)\left(u_\varepsilon - \fint_{\Omega'_\varepsilon} u_\varepsilon\right) \leq c \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

which implies the boundness of ∇u_ε in $L^2(\Omega_\varepsilon)^3$ as well as the boundness of u_ε in $L^2(\Omega'_\varepsilon)$ by the uniform Poincaré-Wirtinger inequality in Ω'_ε since u_ε has a zero Ω'_ε-mean. We thus deduce from Theorem 2.7

$$\begin{cases} \mathbf{1}_{\Omega_{i,\varepsilon}} \left(u_\varepsilon - \fint_{\Omega_{i,\varepsilon}} u_\varepsilon\right) \rightharpoonup |Y_i| u'_i & \text{for } 1 \leq i \leq m-1 , \\ \mathbf{1}_{\Omega_{i,\varepsilon}} u_\varepsilon \rightharpoonup |Y_i| u'_i & \text{for } m \leq i \leq n , \end{cases}$$

where $\vec{u}' := (u'_1, \ldots, u'_n)$ is an eigenvector from (3.21). Furthermore, since

$$\int_\Omega u'_i = 0 \text{ for } 1 \leq i \leq m-1 \quad \text{and} \quad \sum_{i=m}^{n} \int_\Omega |Y_i| u'_i = \lim_{\varepsilon \to 0} \int_{\Omega'_\varepsilon} u_\varepsilon = 0 ,$$

the function $\vec{u}'$ belongs to the space $\vec{H}$ (2.13) in which problem (2.12) has a unique solution, whence it follows the equality $\vec{u}' = \vec{u}$. Then, by putting u_ε in (3.22) we obtain the limit

$$(3.23) \quad \fint_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \fint_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon = \sum_{i=1}^{m-1} \frac{\mu}{|\Omega_\varepsilon|} \int_{\Omega_{i,\varepsilon}} \left(u_i - \fint_{\Omega_{i,\varepsilon}} u_i\right)\left(u_\varepsilon - \fint_{\Omega_{i,\varepsilon}} u_\varepsilon\right) +$$

$$+ \sum_{i=m}^{n} \frac{\mu}{|\Omega_\varepsilon|} \int_{\Omega_{i,\varepsilon}} u_i u_\varepsilon - 0 \underset{\varepsilon \to 0}{\longrightarrow} \mu \sum_{i=1}^{n} \fint_\Omega \theta_i u_i^2 = \mu \|\vec{v}\|^2_\Omega = \mu .$$

The next step now consists in substracting to the function u_ε its projection in the space spanned by the $(k+1)$ first eigenvectors $v_{j,\varepsilon}$, $0 \leq j \leq k$. However, since u_ε is not necessarily bounded in $L^2(\Omega_\varepsilon)$, we have to modify it. Let us then consider the functions u_ε^p and v_ε^p defined for any $p \in \mathbb{N}$ by

$$\begin{cases} u_\varepsilon^p := \sum_{i=1}^{m-1} (\widehat{v}_{i,\varepsilon} - \widehat{v}_{i-1,\varepsilon})\, T^p(\widetilde{u}_{i,\varepsilon}) + (1 - \widehat{v}_{m-1,\varepsilon})\, T^p(u_\varepsilon) \quad (\widehat{v}_{0,\varepsilon} := 0), \\ v_\varepsilon^p := u_\varepsilon^p - \sum_{j=0}^{k} (u_\varepsilon^p, v_{j,\varepsilon})_{\Omega_\varepsilon} v_{j,\varepsilon}, \end{cases}$$

where $\widehat{v}_{i,\varepsilon}$ is defined by (3.13), $\widetilde{u}_{i,\varepsilon}$ is a bounded extension of $\left(u_\varepsilon - \fint_{\Omega_{i,\varepsilon}} u_\varepsilon \right)$ according to the extension result of Subsection 3.1, and $T^p(t) := \min(p, \max(-p, t))$ denotes the truncature at the size p. By definition of the functions $\widehat{v}_{i,\varepsilon}$, we have

$$\begin{cases} \widehat{v}_{i,\varepsilon} - \widehat{v}_{i-1,\varepsilon} - 1_{\Omega_{i,\varepsilon}} \to 0 & \text{strongly in } L^2(\Omega) \quad \text{for } 1 \leq i \leq n, \\ 1 - \widehat{v}_{m-1,\varepsilon} - \sum_{i=m}^{n} 1_{\Omega_{i,\varepsilon}} \to 0 & \text{strongly in } L^2(\Omega). \end{cases}$$

Thanks to the Rellich Theorem, the extension result in each domain $\Omega_{i,\varepsilon}$ implies the strong compactness in $L^2(\Omega_{i,\varepsilon})$ which combined with the Lipschitz property of the truncature gives

$$\begin{cases} T^p(\widetilde{u}_{i,\varepsilon}) \to T^p(u_i) & \text{strongly in } L^2(\Omega) \quad \text{for } 1 \leq i \leq m-1, \\ 1_{\Omega_{i,\varepsilon}}(T^p(u_\varepsilon) - T^p(u_i)) \to 0 & \text{strongly in } L^2(\Omega) \quad \text{for } m \leq i \leq n. \end{cases}$$

Moreover, the norm of $T^p(\widetilde{u}_{i,\varepsilon})$ as well as $T^p(u_\varepsilon)$ over each bridge $\omega_{i,\varepsilon}$ for $1 \leq i \leq n-$ -1, tends to 0 since T^p is bounded by p and $|\omega_{i,\varepsilon}| \to 0$. Combinning the previous limits then yields

$$\|u_\varepsilon^p\|_{\Omega_\varepsilon}^2 \underset{\varepsilon \to 0}{\longrightarrow} \fint_\Omega \sum_{i=1}^{n} \theta_j T^p(u_j)^2 = \|\vec{u}\|_\Omega^2 + o_p(1) = 1 + o_p(1),$$

where $o_p(1) \to 0$ when $p \to +\infty$, and similarly for any $0 \leq j \leq k$

$$(3.24) \qquad (u_\varepsilon^p, v_{j,\varepsilon})_{\Omega_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} \fint_\Omega \sum_{i=1}^{n} \theta_j T^p(u_j)\, v_{j,i} = (\vec{u}, \vec{v}_j)_\Omega + o_p(1) = o_p(1),$$

which imply

$$(3.25) \qquad \|v_\varepsilon^p\|_{\Omega_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 1 + o_p(1).$$

On the other hand, we have by estimate (3.11)

$$\begin{cases} |\hat{v}_{i,\varepsilon} - \hat{v}_{i-1,\varepsilon}| \leq \mathbf{1}_{\Omega_{i,\varepsilon} \cup \omega_{i,\varepsilon} \cup \omega_{i-1,\varepsilon}} & \text{for } 1 \leq i \leq n\,, \\ |1 - \hat{v}_{m-1,\varepsilon}| \leq \sum_{i=m}^{n-1} \mathbf{1}_{\Omega_{i,\varepsilon} \cup \omega_{i,\varepsilon}} + \mathbf{1}_{\Omega_{n,\varepsilon}} & \\ \|\nabla \hat{v}_{i,\varepsilon}\|^2_{\Omega_\varepsilon} \leq c\delta_i(\varepsilon) \to 0 & \text{for } 1 \leq i \leq m-1\,. \end{cases}$$

Then since $|\nabla T^p(u)| \leq |\nabla u|$, we deduce from the previous estimates

$$\overline{\lim_{\varepsilon \to 0}} \|\nabla u^p_\varepsilon\|^2_{\Omega_\varepsilon} \leq 2 \overline{\lim_{\varepsilon \to 0}} \|\nabla u_\varepsilon\|^2_{\Omega_\varepsilon} = 2\mu \qquad \text{by (3.23)}$$

(the factor 2 is due to the fact that the integral over the sets $\omega_{i,\varepsilon}$ is counted twice by summing over the sets $\Omega_{i,\varepsilon} \cup \omega_{i,\varepsilon} \cup \omega_{i-1,\varepsilon}$ for $1 \leq i \leq m-1$), which combined with (3.24) implies that

$$\overline{\lim_{\varepsilon \to 0}} \|\nabla v^p_\varepsilon\|_{\Omega_\varepsilon} \leq 2\mu + o_p(1)\,. \tag{3.26}$$

Finally, since the function v^p_ε is orthogonal to the functions $v_{j,\varepsilon}$, $1 \leq j \leq k$, the Courant-Fisher formulas imply that

$$\lambda_{k+1} = \lim_{\varepsilon \to 0} \lambda_{k+1}(\varepsilon) \leq \overline{\lim_{\varepsilon \to 0}} \frac{\|\nabla v^p_\varepsilon\|^2_{\Omega_\varepsilon}}{\|v^p_\varepsilon\|^2_{\Omega_\varepsilon}} \leq 2\mu + o_p(1) \qquad \text{by (3.25) and (3.26)}$$

and hence $\lambda_{k+1} \leq 2\mu$ which yields the contradiction.

4. – Proof of the auxiliary results.

The first subsection is devoted to the proof of the homogenization result Theorem 2.7. The second subsection consists in proving the results concerning the bridge $Q_{k,\varepsilon}$: the capacitary result of Lemma 3.2 and the $L^1(Q_{k,\varepsilon})$-estimate of Lemma 3.1. In the third subsection, we prove technical results related to the geometry of the bridges as Lemma 2.2.

4.1. *Proof of the homogenization Theorem.* – The proof is quite similar to that of [3] for $n = 2$ weakly connected materials except that the maximum principle cannot be used in our context since the right hand side f_ε of the Neumann problem (2.24) is only a bounded sequence of $L^2(\Omega_\varepsilon)$. We then replace the maximum principle by estimate (3.6) over each bridge $Q_{j,\varepsilon}$, $1 \leq j \leq n-1$.

For the reader convenience, we recall the main points of the proof which is divided in two steps. The first step consists in getting the limit of each sequence $\mathbf{1}_{\Omega_{k,\varepsilon}} \nabla u_\varepsilon$, $1 \leq k \leq n$, and the second one in obtaining the coupling terms $J\vec{u}$ from (2.29).

4.1.1. Limits of $\mathbf{1}_{\Omega_{k,\varepsilon}} \nabla u_\varepsilon$, $1 \leq k \leq n$. By using the extension operator $P_{k,\varepsilon}$, $1 \leq k \leq n$, from $H^1(\Omega_{k,\varepsilon})$ into $H^1(\Omega)$, introduced in Subsection 3.1, we obtain up to a subsequence

$$(4.1)\quad \begin{cases} \tilde{u}_{k,\varepsilon} := P_{k,\varepsilon}\left(u_\varepsilon - \fint_{\Omega_{k,\varepsilon}} u_\varepsilon\right) \rightharpoonup u_k & \text{weakly in } H^1(\Omega) \quad \text{for } 1 \leqslant k \leqslant m-1\,, \\ \tilde{u}_{k,\varepsilon} := P_{k,\varepsilon}\left(u_\varepsilon - \fint_{\Omega'_\varepsilon} u_\varepsilon\right) \rightharpoonup u_k & \text{weakly in } H^1(\Omega) \quad \text{for } m \leqslant k \leqslant n\,, \end{cases}$$

where Ω'_ε is defined by (2.26) and $u \in \vec{H}$ defined by (2.13). As in [3], let us define the test-function $w^\lambda_{k,\varepsilon}$ for any $1 \leqslant k \leqslant n$ and $\lambda \in \mathbb{R}^3$ by

$$w^\lambda_{k,\varepsilon}(x) := \lambda \cdot x - \varepsilon \psi_k\left(\frac{x}{\varepsilon}\right) P_{k,\varepsilon}\left[(W^\lambda_k(y) - \lambda \cdot y)\left(\frac{x}{\varepsilon}\right)\right],$$

where W^λ_k is defined by (2.22) and ψ_k is a smooth Y-periodic function such that $\psi_k = 1$ in Y_k and $\psi_k = 0$ in Y_j for $j \neq k$. One can prove that $w^\lambda_{k,\varepsilon}$ satisfies (see [3] for more details)

$$(4.2)\quad \begin{cases} w^\lambda_{k,\varepsilon}(x) \rightharpoonup \lambda \cdot x & \text{weakly in } W^{1,p}(\Omega), \\ w^\lambda_{k,\varepsilon}(x) = \lambda \cdot x & \text{in } \Omega_{j,\varepsilon} \text{ for } j \neq k\,, \\ \Delta(\mathbf{1}_{\Omega_{k,\varepsilon}} w^\lambda_{k,\varepsilon}) = 0\,, & \\ \mathbf{1}_{\Omega_{k,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \rightharpoonup A_k \lambda & \text{weakly in } L^2(\Omega)^3, \end{cases}$$

where $p > 2$ and A_k is the matrix defined by (2.21).

Since $\operatorname{div}(\mathbf{1}_{\Omega_\varepsilon} \nabla u_\varepsilon) = \mathbf{1}_{\Omega_\varepsilon} f_\varepsilon$ is a compact sequence of $H^{-1}(\Omega)$, the Murat-Tartar div-curl Lemma [6] yields $\mathbf{1}_{\Omega_\varepsilon} \nabla w^\lambda_{k,\varepsilon} \rightharpoonup \xi \cdot \lambda$ in $\mathcal{D}'(\Omega)$ where $\mathbf{1}_{\Omega_\varepsilon} \nabla u_\varepsilon \rightharpoonup \xi$ in $L^2(\Omega)$. We have $\xi = \sum_{k=1}^n \xi_k$ where $\mathbf{1}_{\Omega_{k,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \rightharpoonup \xi_k$ in $L^2(\Omega)$ since the weak limit in $L^2(\Omega)^3$ of $\mathbf{1}_{\omega_{k,\varepsilon}} \nabla u_\varepsilon$ which strongly converges to 0 in $L^1(\Omega)^3$, is equal to 0. On the other hand, we have for any $\lambda \in \mathbb{R}^3$

$$(4.3)\quad \mathbf{1}_{\Omega_{k,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \cdot \nabla u_\varepsilon = \mathbf{1}_{\omega_{k,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \cdot \nabla \tilde{u}_{k,\varepsilon} + \sum_{j \neq k} \mathbf{1}_{\omega_{j,\varepsilon}} \nabla u_\varepsilon \cdot \lambda + r_{k,\varepsilon},$$

where

$$r_{k,\varepsilon} := \sum_{j=1}^{n-1} \mathbf{1}_{\omega_{j,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \cdot \lambda \to 0 \qquad \text{strongly in } L^1(\Omega),$$

since $\nabla w^\lambda_{k,\varepsilon}$ is bounded in $L^p(\Omega)^3$ for some $p > 2$. Furthermore by the div-curl Lemma combined with the zero-divergence from (4.2) and convergence (4.1), we get

$$\mathbf{1}_{\Omega_{k,\varepsilon}} \nabla w^\lambda_{k,\varepsilon} \cdot \nabla \tilde{u}_{k,\varepsilon} \rightharpoonup A_k \lambda \cdot \nabla u_k \qquad \text{weakly in } \mathcal{D}'(\Omega) \text{ for } 1 \leqslant k \leqslant n\,.$$

Then, passing to the limit in (4.3) yields

$$\xi = \sum_{j=1}^{n} \xi_j = A_k \nabla u_k + \sum_{j \neq k} \xi_j \qquad \text{for any } 1 \leq k \leq n ,$$

which implies that $\xi_k = A_k \nabla u_k$ and therefore

$$\mathbf{1}_{\Omega_{k,\varepsilon}} \nabla u_\varepsilon \rightharpoonup A_k \nabla u_k \qquad \text{weakly in } L^2(\Omega) \text{ for } 1 \leq k \leq n . \tag{4.4}$$

4.1.2. Determination of the coupling terms. – Let us define for any function $\varphi_k \in C^1(\overline{\Omega})$, $1 \leq k \leq n$, the function

$$\varphi_\varepsilon := \sum_{k=1}^{n-1} \varphi_k(\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}) + \varphi_n(1 - \widehat{v}_{n-1,\varepsilon}),$$

where $\widehat{v}_{k,\varepsilon}$ is defined by (3.13) and $\widehat{v}_{0,\varepsilon} := 0$. Putting the function φ_ε in equation (2.24) yields

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi_\varepsilon = \int_{\Omega_\varepsilon} f_\varepsilon \varphi_\varepsilon . \tag{4.5}$$

Since $(\widehat{v}_{k,\varepsilon} - \widehat{v}_{k,\varepsilon} - \mathbf{1}_{\Omega_{k,\varepsilon}}) \to 0$ for $1 \leq k \leq n-1$, and $(1 - \widehat{v}_{n-1,\varepsilon} - \mathbf{1}_{\Omega_{n,\varepsilon}}) \to 0$ strongly in $L^2(\Omega)$, we have

$$\int_{\Omega_\varepsilon} f_\varepsilon \varphi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \int_{\Omega} \sum_{k=1}^{n} |Y_k| f_k \varphi_k . \tag{4.6}$$

Similarly, we have thanks to convergences (4.4)

$$\int_{\Omega_\varepsilon} \sum_{k=1}^{n-1} \nabla u_\varepsilon \cdot \nabla \varphi_k(\widehat{v}_{k,\varepsilon} - \widehat{v}_{k-1,\varepsilon}) + \nabla u_\varepsilon \cdot \nabla \varphi_k(1 - \widehat{v}_{n-1,\varepsilon}) \underset{\varepsilon \to 0}{\longrightarrow} \int_{\Omega} \sum_{k=1}^{n} A_k \nabla u_k \cdot \nabla \varphi_k . \tag{4.7}$$

Assume for the moment that for any $\varphi \in C^1(\overline{\Omega})$

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \widehat{v}_{k,\varepsilon} \varphi \underset{\varepsilon \to 0}{\longrightarrow} \int_{\Omega} \delta_k(u_k - u_{k+1})\, \varphi \qquad \text{for } 1 \leq k \leq n-1 . \tag{4.8}$$

The latter implies that $(\delta_0(\varepsilon) := 0)$

$$\int_{\Omega_\varepsilon} \sum_{k=1}^{n-1} \nabla u_\varepsilon \cdot (\nabla \widehat{v}_{k,\varepsilon} - \nabla \widehat{v}_{k-1,\varepsilon})\, \varphi_k - \nabla u_\varepsilon \cdot \nabla \widehat{v}_{n-1,\varepsilon} \varphi_n \underset{\varepsilon \to 0}{\longrightarrow} \tag{4.9}$$

$$\underset{\varepsilon \to 0}{\longrightarrow} \int_{\Omega} \sum_{k=1}^{n-1} [\delta_{k-1}(u_k - u_{k-1}) + \delta_k(u_k - u_{k+1})]\, \varphi_k + \delta_{n-1}(u_n - u_{n-1})\, \varphi_{n-1} .$$

Finally by passing to the limit in (4.5) thanks to the limits (4.6), (4.8) and (4.9), we obtain the equality ($\delta_0 = \delta_{n-1} := 0$)

$$\sum_{k=1}^{n} \int_{\Omega} A_k \nabla u_k \cdot \nabla \varphi_k + [\delta_{k-1}(u_k - u_{k-1}) + \delta_k(u_k - u_{k+1})] \varphi_k = \sum_{k=1}^{n} \int_{\Omega} |Y_k| f_k \varphi_k,$$

which is exactly the variational form of (2.29) by Notation 2.3.

It remains to prove (4.8). It is clear for $1 \leq k \leq m-1$ since by (3.11) we have $\|\nabla \hat{v}_{k,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leq c\delta_k(\varepsilon)^{1/2} \to 0$ and $\delta_k = 0$.

Let $m \leq k \leq n-1$. Estimate (3.14) applied to the function $v := (\varphi u_\varepsilon)(\varepsilon y)$ gives

$$\left| \int_{\Omega_\varepsilon} \nabla \hat{v}_{k,\varepsilon} \cdot \nabla(\varphi u_\varepsilon) - \hat{\delta}_k(\varepsilon) - \int_{\Omega_\varepsilon} \left(\frac{\mathbf{1}_{\Omega_{k,\varepsilon}}}{|Y_k|} - \frac{\mathbf{1}_{\Omega_{k+1,\varepsilon}}}{|Y_{k+1}|} \right) \varphi u_\varepsilon \right| \leq o_\varepsilon(1),$$

which combined with convergences (4.1) also yields

$$\int_{\Omega_\varepsilon} \nabla \hat{v}_{k,\varepsilon} \cdot \nabla(\varphi u_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \int_{\Omega} \delta_k(u_k - u_{k+1}) \varphi .$$

Now (4.8) is reduced to

$$\int_{\Omega_\varepsilon} \nabla \hat{v}_{k,\varepsilon} \cdot \nabla \varphi u_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0 . \tag{4.10}$$

In [3], the latter is an immediate consequence of the maximum principle. Here, it is a consequence of Lemma 3.1 about $L^1(Q_{k,\varepsilon})$-estimate. Rescaling estimate (3.6) with function $V(y) := u_\varepsilon^2(\varepsilon y)$ yields

$$\|u_\varepsilon\|^2_{L^2(\omega_{k,\varepsilon})} \leq c l_k(\varepsilon)(\alpha_k(\varepsilon) + \varepsilon)$$

since u_ε is bounded in $H^1(\Omega'_\varepsilon)$. Then by the Cauchy-Schwarz inequality and by estimate (3.11), we get

$$\begin{aligned} \left(\int_{\Omega_\varepsilon} \nabla \hat{v}_{k,\varepsilon} \cdot \nabla \varphi u_\varepsilon \right)^2 &\leq c_1 \frac{|\omega_{k,\varepsilon}|}{\varepsilon^2 l_k(\varepsilon)^2} \|u_\varepsilon\|^2_{L^2(\omega_{k,\varepsilon})} \\ &\leq c_2 \frac{|Q_{k,\varepsilon}|}{\varepsilon^2 l_k(\varepsilon)} (\alpha_k(\varepsilon) + \varepsilon) \\ \text{by (3.10)} &\leq c_3 \frac{\alpha_k(\varepsilon)}{\varepsilon} \left(\frac{\alpha_k(\varepsilon)}{\varepsilon} + 1 \right) \underset{\varepsilon \to 0}{\longrightarrow} 0 . \end{aligned}$$

by condition (2.15), which yields (4.10) and hence (4.8). ∎

4.2. *Proof of the bridge capacitary result.* – For proving Lemma 3.2 we need a few results of differential geometry on each bridge $Q_{k,\varepsilon}$, $1 \leq k \leq n-1$.

LEMMA 4.1. – *Assume that geometrical conditions* (H3)-(H4) *as well as limit* (2.15) *hold true. Then, for any* $1 \leq k \leq n-1$, *the curvilinear coordinate* $\sigma_{k,\varepsilon}$ *and the tangent vector* $T_{k,\varepsilon} := dF_{k,\varepsilon}/d\sigma_{k,\varepsilon}$ *related to the curve* $F_{k,\varepsilon}$ *which defines the bridge* $Q_{k,\varepsilon}$ *by* (2.4), *satisfy*

$$\|\nabla\sigma_{k,\varepsilon} - T_{k,\varepsilon}\|_{L^\infty(Q_{k,\varepsilon})} \underset{\varepsilon\to 0}{\longrightarrow} 0\,, \tag{4.11}$$

$$\operatorname{div} T_{k,\varepsilon} = 0\,. \tag{4.12}$$

Moreover, the gradient of the function $\Phi_{k,\varepsilon}$ *in* (2.4) *with respect to the coordinates* $(\sigma, \nu, \beta) \in]0, l_k(\varepsilon)[\times \Sigma_{k,\varepsilon}$, *verifies*

$$\||\nabla\Phi_{k,\varepsilon}| - 1\|_{L^\infty(]0,\, l_k(\varepsilon)[\times\Sigma_{k,\varepsilon})} \underset{\varepsilon\to 0}{\longrightarrow} 0\,. \tag{4.13}$$

PROOF OF LEMMA 3.2.

Proof of (3.10): By the change of variable $\Phi_{k,\varepsilon}$ of assumption (H3), we have

$$|Q_{k,\varepsilon}| = \int_0^{l_k(\varepsilon)}\int_{\Sigma_{k,\varepsilon}} |\det(\nabla\Phi_{k,\varepsilon})|\, d\sigma\, d\nu\, d\beta \underset{\varepsilon\to 0}{\sim} l_k(\varepsilon)\,|\Sigma_{k,\varepsilon}| = l_k(\varepsilon)\,\alpha_k(\varepsilon) \quad \text{by (4.13)}.$$

Proof of (3.11): It is an immediate consequence of (4.11) and (3.10).

Proof of (3.12): The proof is an adaptation of Lemma 3.1 in [3] by taking into account the geometry of the bridge $Q_{k,\varepsilon}$ thanks to Lemma 4.1. By definition (3.9) and property (4.11), we have for any $V \in H^1(Y_\varepsilon)$

$$\frac{1}{\varepsilon^2}\int_{Q_{k,\varepsilon}} \nabla V\cdot\nabla\widehat{V}_{k,\varepsilon} = \frac{1}{\varepsilon^2 l_k(\varepsilon)}\int_{Q_{k,\varepsilon}} \nabla V\cdot T_{k,\varepsilon} + \frac{1}{\varepsilon^2 l_k(\varepsilon)}\int_{Q_{k,\varepsilon}} \nabla V\cdot R_{k,\varepsilon},$$

where $\|R_{k,\varepsilon}\|_{L^\infty(Q_{k,\varepsilon})} \to 0$. By the Cauchy-Schwarz inequality, the last term is bounded by

$$\frac{|Q_{k,\varepsilon}|^{1/2}}{\varepsilon^2 l_k(\varepsilon)}\|R_{k,\varepsilon}\|_{L^\infty(Q_{k,\varepsilon})}\|\nabla V\|_{L^2(Q_{k,\varepsilon})} = o_\varepsilon\left(\frac{\delta_k(\varepsilon)^{1/2}}{\varepsilon}\right)\|\nabla V\|_{L^2(Q_{k,\varepsilon})} \quad \text{by (3.10)}.$$

We thus obtain the first estimate

$$\frac{1}{\varepsilon^2}\int_{Q_{k,\varepsilon}} \nabla V\cdot\nabla\widehat{V}_{k,\varepsilon} = \frac{1}{\varepsilon^2 l_k(\varepsilon)}\int_{Q_{k,\varepsilon}} \nabla V\cdot T_{k,\varepsilon} + o_\varepsilon\left(\frac{\delta_k(\varepsilon)^{1/2}}{\varepsilon}\right)\|\nabla V\|_{L^2(Q_{k,\varepsilon})}. \tag{4.14}$$

Denoting by $\overline{V}_k$ the mean value of V over Y_k, we have

$$(4.15)\qquad \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{Q_{k,\varepsilon}} \nabla V \cdot T_{k,\varepsilon} - \widehat{\delta}_k(\varepsilon)(\overline{V}_k - \overline{V}_{k+1}) =$$

$$= \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{Q_{k,\varepsilon}} \nabla(V - \widehat{V}_{k,\varepsilon}\overline{V}_k - (1-\widehat{V}_{k,\varepsilon})\overline{V}_{k+1}) \cdot T_{k,\varepsilon},$$

where by (4.11) together with (3.10)

$$\widehat{\delta}_k(\varepsilon) := \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{Q_{k,\varepsilon}} \nabla \widehat{V}_{k,\varepsilon} \cdot T_{k,\varepsilon} \underset{\varepsilon\to 0}{\sim} \frac{|Q_{k,\varepsilon}|}{\varepsilon^2 l_k(\varepsilon)^2} \underset{\varepsilon\to 0}{\sim} \delta_k(\varepsilon).$$

On the other hand, since $T_{k,\varepsilon}$ has a zero divergence by (4.12) an integration by parts in the right hand side of (4.15) yields

$$\frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{Q_{k,\varepsilon}} \nabla(V - \widehat{V}_{k,\varepsilon}\overline{V}_k - (1-\widehat{V}_{k,\varepsilon})\,\overline{V}_{k+1}) \cdot T_{k,\varepsilon} =$$

$$= \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{\Gamma_{k,\varepsilon}} (V - \overline{V}_k)\, T_{k,\varepsilon} \cdot \nu + \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{\Gamma_{k+1,\varepsilon}} (V - \overline{V}_{k+1})\, T_{k,\varepsilon} \cdot \nu,$$

where $\Gamma_{k,\varepsilon}$ denotes the boundary $\partial Y_k \cap \partial Q_{k,\varepsilon}$. By the Sobolev imbedding Theorem in two-dimension, the trace of the function $(V - \overline{V}_k)$ belongs to $L^4(\Gamma_k)$ where $\Gamma_{k,\varepsilon} \subset \Gamma_k$ a fixed open subset of ∂Y_k. Then by the Poincaré-Wirtinger inequality in Y_k, there exists a constant $c > 0$ such that $\|V - \overline{V}_k\|_{L^4(\Gamma_k)} \leq c \|\nabla V\|_{L^2(Y_k)}$, which implies owing to the Hölder inequality in $L^4(\Gamma_{k,\varepsilon})$ combined with the estimate $|\Gamma_{k,\varepsilon}| \leq c\alpha_k(\varepsilon)$

$$\left| \frac{1}{\varepsilon^2 l_k(\varepsilon)} \int_{Q_{k,\varepsilon}} \nabla(V - \widehat{V}_{k,\varepsilon}\overline{V}_k - (1-\widehat{V}_{k,\varepsilon})\,\overline{V}_{k+1}) \cdot T_{k,\varepsilon} \right| \leq$$

$$\leq \frac{\alpha_k(\varepsilon)^{3/4}}{\varepsilon^2 l_k(\varepsilon)} \|\nabla V\|_{L^2(Y_\varepsilon)} = o_\varepsilon\left(\frac{\delta_k(\varepsilon)^{1/2}}{\varepsilon}\right) \|\nabla V\|_{L^2(Y_\varepsilon)}.$$

The latter combined with (4.14) and (4.15) yields desired estimate (3.12), which concludes the proof of Lemma 3.2. ■

PROOF OF LEMMA 3.1. – Let $V \in C^1(\overline{Y_k \cup Q_{k,\varepsilon}})$. By the change of variables $\Phi_{k,\varepsilon}$ related to definition (2.4) of the bridge $Q_{k,\varepsilon}$, we have with $W(\sigma, \nu, \beta) := |V|(y)$, $y = \Phi_{k,\varepsilon}(\sigma, \nu, \beta)$,

$$\|V\|_{L^1(Q_{k,\varepsilon})} = \int_0^{l_k(\varepsilon)} \int_{\Sigma_{k,\varepsilon}} W(\sigma, \nu, \beta)\, |\det(\nabla \Phi_{k,\varepsilon})|\, d\sigma\, d\nu\, d\beta$$

and thanks to estimate (4.13) there exists a constant $c > 0$ such that

$$\|V\|_{L^1(Q_{k,\varepsilon})} \leq c \int_{\Sigma_{k,\varepsilon}} d\nu\, d\beta \int_0^{l_k(\varepsilon)} W(\sigma, \nu, \beta)\, d\sigma \,. \tag{4.16}$$

Moreover by assumption (H4), we can extend in Y_k the bridge $Q_{k,\varepsilon}$ by a small cylinder of fixed length l_k in such a way that

$$R_{k,\varepsilon} := \{y = F_{k,\varepsilon}(0) + \sigma T_{k,\varepsilon}(0) + \nu N_{k,\varepsilon}(0) + \beta B_{k,\varepsilon}(0);\ \sigma \in]0, l_k[, (\nu, \beta) \in \Sigma_{k,\varepsilon}\} \tag{4.17}$$

is a subset of Y_k. We can then extend in a natural way the diffeomorphism $\Phi_{k,\varepsilon}$ of (2.4) from $]0, l_k(\varepsilon)[\times \Sigma_{k,\varepsilon}$ to $]-l_k, l_k(\varepsilon)[\times \Sigma_{k,\varepsilon}$, which becomes a diffeomorphism from $]-l_k, l_k(\varepsilon)[\times\Sigma_{k,\varepsilon}$ into the interior of $Q_{k,\varepsilon} \cup R_{k,\varepsilon}$. We have for any $\sigma \in]0, l_k(\varepsilon)[$

$$W(\sigma, \nu, \beta) = W\left(-\frac{l_k}{l_k(\varepsilon)}\sigma, \nu, \beta\right) + \int_{-(l_k/l_k(\varepsilon))\sigma}^{\sigma} \frac{\partial W}{\partial s}(s, \nu, \beta)\, ds$$

and by integrating with respect to $\sigma \in]0, l_k(\varepsilon)[$

$$\int_0^{l_k(\varepsilon)} W(\sigma, \nu, \beta)\, d\sigma \leq \frac{l_k(\varepsilon)}{l_k} \int_{-l_k}^{0} W(\sigma, \nu, \beta)\, d\sigma + l_k(\varepsilon) \int_{-l_k}^{l_k(\varepsilon)} \left| \frac{\partial W}{\partial s}(\sigma, \nu, \beta) \right| d\sigma \,.$$

Then, by integrating the latter with respect to $(\nu, \beta) \in \Sigma_{k,\varepsilon}$ and by using the inverse change of variable $\Phi_{k,\varepsilon}^{-1}$ the gradient of which is uniformly bounded by (4.13), we deduce from estimate (4.16) that there exists a constant $c > 0$ such that

$$\|V\|_{L^1(Q_{k,\varepsilon})} \leq c l_k(\varepsilon)(\|V\|_{L^1(R_{k,\varepsilon})} + \|\nabla V\|_{L^1(R_{k,\varepsilon} \cup Q_{k,\varepsilon})}) \,. \tag{4.18}$$

On the other hand, by the Poincaré Wirtinger inequality in Y_k and since the set $R_{k,\varepsilon}$ from (4.17) satisfies $|R_{k,\varepsilon}| \leq c\alpha_k(\varepsilon)$, we have

$$\|V\|_{L^1(R_{k,\varepsilon})} \leq \left\| V - \fint_{Y_k} V \right\|_{L^1(R_{k,\varepsilon})} + |R_{k,\varepsilon}| \left| \fint_{Y_k} V \right| \leq c\|\nabla V\|_{L^1(Y_k)} + c\alpha_k(\varepsilon)\|V\|_{L^1(Y_k)}$$

which combined with estimate (4.18) yields (3.6). ■

4.3. *Proof of the differential geometry lemmas.*

PROOF OF LEMMA 4.1.

Proof of (4.11): We prove the result for the tube $d = 3$, the case of the curved plate $d = 2$ is quite similar. For the sake of simplicity, we omit the indices k and ε. We have for any component $j \in \{1, 2, 3\}$, $y_j = F_j(\sigma) + \nu N_j(\sigma) + \beta B_j(\sigma)$ which yields for any $i, j \in \{1, 2, 3\}$

$$\delta_{i,j} = \frac{\partial \sigma}{\partial y_i}\left(\frac{\partial F_j(\sigma)}{\partial \sigma} + \nu \frac{\partial N_j(\sigma)}{\partial \sigma} + \beta \frac{\partial B_j(\sigma)}{\partial \sigma}\right) + \frac{\partial \nu}{\partial y_i} N_j(\sigma) + \frac{\partial \beta}{\partial y_i} B_j(\sigma) =$$

$$= (1 - \nu \varrho)\frac{\partial \sigma}{\partial y_i} T_j(\sigma) + \left(\frac{\partial \nu}{\partial y_i} + \beta \tau \frac{\partial \sigma}{\partial y_i}\right) N_j(\sigma) + \left(\frac{\partial \beta}{\partial y_i} - \nu \tau \frac{\partial \sigma}{\partial y_i}\right) B_j(\sigma)$$

since the Frenet formulas give

$$\frac{\partial F(\sigma)}{\partial \sigma} = T(\sigma), \quad \frac{\partial T(\sigma)}{\partial \sigma} = \varrho N(\sigma), \quad \frac{\partial N(\sigma)}{\partial \sigma} = -\varrho T(\sigma) - \tau B(\sigma), \quad \frac{\partial B(\sigma)}{\partial \sigma} = \tau N(\sigma), \tag{4.19}$$

where ϱ and τ respectively denote the curvature and the torsion of the curve F. We then deduce from the latter together with the orthonormality of the Frenet basis (T, N, B)

$$T_i = \sum_{j=1}^{3} \delta_{i,j} T_j = (1 - \nu \varrho)\frac{\partial \sigma}{\partial y_i}$$

or equivalently

$$\nabla \sigma = \frac{1}{1 - \nu \varrho} T \tag{4.20}$$

which implies (4.11) thanks to (2.15).

Proof of (4.12): By (4.20) we have

$$\operatorname{div} T = \sum_{i=1}^{3} \frac{\partial T_i}{\partial y_i} = \sum_{i=1}^{3} \frac{\partial \sigma}{\partial y_i}\frac{\partial T_i}{\partial \sigma} = \frac{\varrho}{1 - \nu \varrho} T \cdot N = 0$$

which proves (4.12).

Proof of (4.13): The gradient of the function

$$\Phi(\sigma, \nu, \beta) := F(\sigma) + \nu N(\sigma) + \beta B(\sigma)$$

is given by the 3×3 matrix

$$\nabla \Phi = \left(\frac{\partial \Phi}{\partial \sigma}, \frac{\partial \Phi}{\partial \nu}, \frac{\partial \Phi}{\partial \beta}\right)$$

and thanks to formulas (4.19) and to the orthonormality of (T, N, B)

$$\nabla \Phi = (1 - \nu \varrho)(T, N, B) - \nu \tau (B, N, B) + \beta \tau (N, N, B).$$

The orthonormality of the family (T, N, B) then implies that

$$|\nabla \Phi| = 1 + O(|\nu \varrho| + |\nu \tau| + |\beta \tau|)$$

which yields (4.13) owing to condition (2.15). ■

PROOF OF LEMMA 2.2. – The only difficulty is to prove that $\Phi_{k,\varepsilon}$ is one to one. We prove it for $d = 3$, the case $d = 2$ is quite similar. We also omit the indice k in the following. We proceed by contradiction. Assume that there exist a sequence of positive numbers ε which converge to 0 and vectors $(s_\varepsilon, \nu_\varepsilon, \beta_\varepsilon) \neq (t_\varepsilon, \mu_\varepsilon, \gamma_\varepsilon) \in [0, 1] \times \Sigma_\varepsilon$ such that $\Phi_\varepsilon(s_\varepsilon, \nu_\varepsilon, \beta_\varepsilon) = \Phi_\varepsilon(t_\varepsilon, \mu_\varepsilon, \gamma_\varepsilon)$, which implies $s_\varepsilon \neq t_\varepsilon$ since the vectors $N_\varepsilon(s_\varepsilon)$, $B_\varepsilon(s_\varepsilon)$ are free. Condition (2.5) and estimate $|\nu_\varepsilon| + |\beta_\varepsilon| + |\mu_\varepsilon| + |\gamma_\varepsilon| \leqslant c\alpha(\varepsilon)^{1/2}$ imply

$$c(\varepsilon)^2 |s_\varepsilon - t_\varepsilon|^2 \leqslant |F_\varepsilon(s_\varepsilon) - F_\varepsilon(t_\varepsilon)|^2 \leqslant c\alpha(\varepsilon)$$

and therefore

$$|s_\varepsilon - t_\varepsilon| \leqslant c\frac{\alpha(\varepsilon)^{1/2}}{c(\varepsilon)} . \tag{4.21}$$

The equality $\Phi_\varepsilon(s_\varepsilon, \nu_\varepsilon, \beta_\varepsilon) = \Phi_\varepsilon(t_\varepsilon, \mu_\varepsilon, \gamma_\varepsilon)$ also implies that

$$\frac{F_\varepsilon(t_\varepsilon) - F_\varepsilon(s_\varepsilon)}{t_\varepsilon - s_\varepsilon} \cdot F_\varepsilon'(s_\varepsilon) = \mu_\varepsilon \frac{N_\varepsilon(t_\varepsilon) - N_\varepsilon(s_\varepsilon)}{t_\varepsilon - s_\varepsilon} \cdot F_\varepsilon'(s_\varepsilon) + \gamma_\varepsilon \frac{B_\varepsilon(t_\varepsilon) - B_\varepsilon(s_\varepsilon)}{t_\varepsilon - s_\varepsilon} \cdot F_\varepsilon'(s_\varepsilon), \tag{4.22}$$

since the family $(F'(s_\varepsilon), N(s_\varepsilon), B(s_\varepsilon))$ is orthogonal. On the other hand, the Taylor-Lagrange formula yields

$$\left| \frac{F_\varepsilon(t_\varepsilon) - F_\varepsilon(s_\varepsilon)}{t_\varepsilon - s_\varepsilon} - F_\varepsilon'(s_\varepsilon) \right| \leqslant \frac{1}{2} |t_\varepsilon - s_\varepsilon| \|F_\varepsilon''\|_\infty . \tag{4.23}$$

Moreover, since $N_\varepsilon' = -(\varrho_\varepsilon N_\varepsilon + \tau_\varepsilon B_\varepsilon)|F_\varepsilon'|$ where ϱ_ε denotes the curvature and τ_ε the torsion related to the curve F_ε, we have

$$|N_\varepsilon(t_\varepsilon) - N_\varepsilon(s_\varepsilon)| \leqslant |t_\varepsilon - s_\varepsilon| (\|\varrho_\varepsilon F_\varepsilon'\|_\infty + \|\tau_\varepsilon F_\varepsilon'\|_\infty), \tag{4.24}$$

and similarly, since $B_\varepsilon' = \tau_\varepsilon N_\varepsilon |F_\varepsilon'|$,

$$|B_\varepsilon(t_\varepsilon) - B_\varepsilon(s_\varepsilon)| \leqslant |t_\varepsilon - s_\varepsilon| \|\tau_\varepsilon F_\varepsilon'\|_\infty . \tag{4.25}$$

Now putting estimates (4.23), (4.24) and (4.25) in equality (4.22) yields

$$|F_\varepsilon'(s_\varepsilon)| \leqslant \frac{1}{2} |t_\varepsilon - s_\varepsilon| \|F_\varepsilon''\|_\infty + |\mu_\varepsilon| \|\varrho_\varepsilon F_\varepsilon'\|_\infty + (|\mu_\varepsilon| + |\gamma_\varepsilon|) \|\tau_\varepsilon F_\varepsilon'\|_\infty ,$$

which implies by (2.5) and (4.21), the existence of a constant $c > 0$ such that

$$c(\varepsilon) \leq |F_\varepsilon'(s_\varepsilon)| \leq c\frac{\alpha(\varepsilon)^{1/2}}{c(\varepsilon)}\|F_\varepsilon''\|_\infty + c\alpha(\varepsilon)^{1/2}(\|\varrho_\varepsilon F_\varepsilon'\|_\infty + \|\tau_\varepsilon F_\varepsilon'\|_\infty)$$

and thus contradicts condition (2.6) satisfied by $c(\varepsilon)$. ∎

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