Estimates for the Gradient of Solutions of Nonlinear Elliptic Equations with L^1 Data (*)(**)

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Abstract. – We prove estimates for the gradient of solutions u of the homogeneous Dirichlet problem for an equation in the form $A(u) = f$, where $A(u)$ is a p -laplacian type operator and f is a summable function. The estimate on $|Du|$ is obtained via an estimate on its rearrangement.

1. – Introduction.

Let us consider the problem:

$$(1.1) \quad \begin{cases} -\operatorname{div}(a(x, u, Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $a(x, \eta, \xi): \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the assumption:

$$(1.2) \quad a(x, \eta, \xi) \xi \geq |\xi|^p, \quad \text{for a.e. } x \in \Omega, \quad \forall \eta \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^n,$$

with $1 < p \leq n$. If f belongs to $L^1(\Omega)$, the question of existence of a solution of (1.1) has been studied by many authors. Under further assumptions on $a(x, \eta, \xi)$ it has been proved that an entropy solution of (1.1) exists (see, for example, [6] and the references in it). We recall that a measurable function u is an entropy solution of (1.1) if, for any $k > 0$, $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ and

$$(1.3) \quad \int_{\Omega} a(x, u, Du) D(T_k(u - \varphi)) dx \leq \int_{\Omega} f T_k(u - \varphi) dx, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

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where, as usual, we denote by $T_k(\eta)$, $\eta \in \mathbb{R}$, $k > 0$, the truncation function:

$$(1.4) \quad T_k(\eta) = \begin{cases} \eta & \text{if } |\eta| \leq k, \\ k \operatorname{sign}(\eta) & \text{if } |\eta| > k. \end{cases}$$

It is well known that the definition (1.3) reduces to the usual definition of weak solution when $f \in W^{-1, p'}(\Omega)$. We also observe that, when f belongs to $L^1(\Omega)$, an entropy solution u of (1.1) can be proven to belong to $W_0^{1,1}(\Omega)$ under the additional hypothesis $p > 2 - 1/n$, (see, for example, [6] Th. 6.1). It is important to remark that for problem (1.1) various definitions of solutions can be found in the literature. However, even though their formulation is different, in many cases such definitions give the same solution. We will use the definition of entropy solution only because it allows us to skip some technicalities.

In this paper we are concerned with the regularity of the gradient of an entropy solution u of (1.1). More precisely we are interested in studying how the regularity of Du increases when the regularity of the known term f increases. It is well known that if $f \in L^q(\Omega)$, $1 < q < np/(np - n + p)$, then $|Du|$ belongs to $L^{q^*(p-1)}(\Omega)$, where $q^* = nq/(n - q)$ is the Sobolev conjugate of q (see, for example, [21], [22], [9], [11], [6]). Also some limit and intermediate cases have been analyzed. For example it has been proved that if $f \in L^1(\Omega)$, then $|Du|$ belongs to the Marcinkiewicz space M^q with $q = n(p - 1)/(n - 1)$ (see [6] for the case $1 < p < n$, [12] for the case $p = n$).

We will consider the case where f belongs to a Lorentz space and we will prove that if $f \in L(q, m)$, then $|Du| \in L(q^*(p - 1), m(p - 1))$ (see Section 2 for the definition of Lorentz spaces). Our results contain and extend most of the results listed above, with the exception of some limit cases (see Section 4).

In view of the definition, a way to prove that a function belongs to a Lorentz space is to estimate its decreasing rearrangement. This has been successfully made for an entropy solution u of (1.1) (see, for example, [21], [22], [2], [8]) by means of symmetrization methods. Our main result consists in proving (see Section 3) that also the decreasing rearrangement of $|Du|$ can be estimated in a similar way. The difficulty in finding such a result consists in the fact that symmetrization methods permit to estimate the following derivative:

$$(1.5) \quad \frac{d}{dt} \int_{|u| \leq t} |Du|^p dx.$$

The above function is closely related to the notion of pseudo-rearrangement or relative rearrangement of the gradient (see, for example, [3], [18], [20], [14], [13]). By a suitable approximation lemma we have been able to transfer the estimate on (1.5) into a point-wise estimate on the rearrangement of $|Du|$.

It is well known that the estimate on the rearrangement of u is sharp in the sense that it holds as an equality in some cases. The estimate on $|Du|$ that we find is not sharp in the same sense. However, it allows us to obtain the best regularity of the gradient in the scale of Lorentz spaces (see Section 4).

2. - Preliminary results.

In this section we will remind the definition of decreasing rearrangement of a function and the definition of Lorentz spaces. Furthermore we will state some integral inequalities which will be useful in the next sections.

Let $\varphi(x)$ be a measurable function in an open bounded set $\Omega \subset \mathbb{R}^n$, we denote by $\mu_\varphi(t)$ the distribution function of φ :

$$\mu_\varphi(t) = |\{x \in \Omega: |\varphi(x)| > t\}|^{(1)}, \quad t \geq 0.$$

The decreasing rearrangement $\varphi^*(s)$ of φ is defined by

$$\varphi^*(s) = \sup \{t > 0: \mu_\varphi(t) > s\}, \quad s \in (0, |\Omega|),$$

and the increasing rearrangement $\varphi_*(s)$ of φ by

$$\varphi_*(s) = \varphi^*(|\Omega| - s), \quad s \in (0, |\Omega|),$$

For an exhaustive treatment of rearrangements we refer for example to [5], [17], [1]. We only recall the well known Hardy-Littlewood inequality:

$$\int_0^{|\Omega|} \varphi_*(s) \psi^*(s) ds \leq \int_\Omega |\varphi(x) \psi(x)| dx \leq \int_0^{|\Omega|} \varphi^*(s) \psi^*(s) ds,$$

for all measurable functions $\varphi(x), \psi(x)$.

Let us put

$$\overline{\varphi}(s) = \frac{1}{s} \int_0^s \varphi^*(t) dt, \quad s \in (0, +\infty),$$

where, as usual, we extend φ to zero outside of $(0, |\Omega|)$. We say (see [16], [19]) that φ belongs to the Lorentz space $L(p, q)$, $1 < p < \infty$, $0 < q \leq \infty$, if the quantity

$$(2.1) \quad \|\varphi\|_{p,q} = \begin{cases} \left(\int_0^{+\infty} [\overline{\varphi}(t) t^{1/p}]^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t>0} \overline{\varphi}(t) t^{1/p} & \text{if } q = \infty \end{cases}$$

⁽¹⁾ We denote by $|A|$ the measure of a measurable set $A \subset \mathbb{R}^n$.

is finite. The above quantity is equivalent to the quantity

$$\|\varphi\|_{p,q}^* = \begin{cases} \left(\int_0^{|\Omega|} [\varphi^*(t) t^{1/p}]^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t \in (0, |\Omega|)} \varphi^*(t) t^{1/p} & \text{if } q = \infty, \end{cases}$$

in the sense that (see [19]) there exists a constant $c = c(p, q)$ such that:

$$(2.2) \quad \|\varphi\|_{p,q}^* \leq \|\varphi\|_{p,q} \leq c \|\varphi\|_{p,q}^*.$$

We emphasize that $L(p, p) = L^p(\Omega)$, $L(p, \infty) = M^p$ (the Marcinkiewicz space) and, for $0 < q < p < r < \infty$, the following inclusions hold:

$$(2.3) \quad L^r \subset L(p, q) \subset L(p, p) = L^p \subset L(p, r) \subset L(p, \infty) \subset L^q.$$

We finally recall an integral inequality.

LEMMA 2.1. — *Let $\varphi: (0, +\infty) \rightarrow (0, +\infty)$ be a decreasing function. For $\beta \geq 0$ and $\gamma \neq 1$, let $F_\gamma(t)$ be defined as follows:*

$$F_\gamma(t) = \begin{cases} \int_0^t r^\beta \varphi(r) dr & \text{if } \gamma < 1, \\ \int_t^{+\infty} r^\beta \varphi(r) dr & \text{if } \gamma > 1. \end{cases}$$

If $\lambda > 0$, then:

$$(2.4) \quad \int_0^{+\infty} \left(\frac{F_\gamma(t)}{t} \right)^\lambda t^{\lambda\gamma} \frac{dt}{t} \leq c \int_0^{+\infty} (\varphi(t))^\lambda t^{\lambda(\beta+\gamma)} \frac{dt}{t},$$

where c is a constant which depends only on β , γ and λ .

PROOF. — Inequality (2.4) is an immediate consequence of Hardy's inequality, when $\lambda \geq 1$ (see [15]). We remark that in such a case no monotonicity assumption on φ is needed. So we will limit ourselves to the case $0 < \lambda < 1$. The proof is similar to one contained in [19]. If $\gamma < 1$ we observe that we can write:

$$\int_0^{+\infty} \left(\frac{1}{t} \int_0^t r^\beta \varphi(r) dr \right)^\lambda t^{\lambda\gamma} \frac{dt}{t} = \int_0^{+\infty} t^{-\lambda+\lambda\gamma} \left(\sum_{j=0}^{+\infty} \int_{t/2^{j+1}}^{t/2^j} r^\beta \varphi(r) dr \right)^\lambda \frac{dt}{t}.$$

It follows that:

$$\begin{aligned} \int_0^{+\infty} \left(\frac{1}{t} \int_0^t r^\beta \varphi(r) dr \right)^\lambda t^{\lambda\gamma} \frac{dt}{t} &\leq c_1 \sum_{j=0}^{+\infty} \frac{1}{2^{j\lambda(\beta+1)}} \int_0^{+\infty} t^{\lambda(\beta+\gamma)} (\varphi(t/2^j))^\lambda \frac{dt}{t} = \\ &= c_2 \int_0^{+\infty} (\varphi(t))^\lambda t^{\lambda(\beta+\gamma)} \frac{dt}{t}, \end{aligned}$$

where c_1, c_2 are constants which depend on β, γ, λ only.

If $\gamma > 1$ we observe that we can write:

$$\int_0^{+\infty} \left(\frac{1}{t} \int_t^{+\infty} r^\beta \varphi(r) dr \right)^\lambda t^{\lambda\gamma} \frac{dt}{t} = \int_0^{+\infty} t^{-\lambda+\lambda\gamma} \left(\sum_{j=0}^{+\infty} \int_{2^j t}^{2^{j+1}t} r^\beta \varphi(r) dr \right)^\lambda \frac{dt}{t}.$$

It follows that:

$$\begin{aligned} \int_0^{+\infty} \left(\frac{1}{t} \int_t^{+\infty} r^\beta \varphi(r) dr \right)^\lambda t^{\lambda\gamma} \frac{dt}{t} &\leq c_1 \sum_{j=0}^{+\infty} 2^{j\lambda(\beta+1)} \int_0^{+\infty} t^{\lambda(\beta+\gamma)} (\varphi(2^j t))^\lambda \frac{dt}{t} = \\ &= c_2 \int_0^{+\infty} (\varphi(t))^\lambda t^{\lambda(\beta+\gamma)} \frac{dt}{t}, \end{aligned}$$

where c_1, c_2 are constants which depend on β, γ, λ only. ■

3. - Main results.

In this section we will assume that an entropy solution u of (1.1) exists and that it belongs to $W_0^{1,1}(\Omega)$. We can use the function $\varphi = T_t(u)$, $t > 0$, as test function in (1.3), obtaining:

$$(3.1) \quad \int_{\Omega} a(x, u, Du) D(T_k(u - T_t(u))) dx \leq \int_{\Omega} f T_k(u - T_t(u)) dx.$$

It is clear that:

$$T_k(u - T_t(u)) = \begin{cases} k \operatorname{sign}(u) & \text{if } |u| > t + k \\ (|u| - t) \operatorname{sign}(u) & \text{if } t < |u| \leq t + k \\ 0 & \text{if } |u| \leq t. \end{cases}$$

So (3.1) and (1.2) give:

$$(3.2) \quad \frac{1}{k} \int_{t < |u| \leq t+k} |Du|^p dx \leq \frac{1}{k} \int_{t < |u| \leq t+k} |f|(|u| - t) dx + \int_{|u| > t+k} |f| dx.$$

At this point we can use a standard procedure which has been used by many authors (see, for example, [21], [22], [2], [8]). Using Hölder inequality we get from (3.2):

$$\begin{aligned} \left(\frac{1}{k} \int_{t < |u| \leq t+k} |Du| dx \right)^p &\leq \left(\frac{1}{k} \int_{t < |u| \leq t+k} |Du|^p dx \right) \left(\frac{|\{t < |u| \leq t+k\}|}{k} \right)^{p-1} \leq \\ &\leq \left(\int_{|u| > t} |f| dx \right) \left(\frac{|\{t < |u| \leq t+k\}|}{k} \right)^{p-1}. \end{aligned}$$

By Hardy-Littlewood inequality, letting k go to zero, we have, for a.e. $t > 0$,

$$\left(-\frac{d}{dt} \int_{|u| > t} |Du| dx \right)^p \leq \left(\int_0^{\mu_u(t)} f^*(\sigma) d\sigma \right) (-\mu'_u(t))^{p-1}.$$

The coarea formula and the isoperimetric inequality (see [10]) give:

$$(3.3) \quad (nC_n^{1/n} \mu_u(t)^{1-1/n})^p \leq \left(\int_0^{\mu_u(t)} f^*(\sigma) d\sigma \right) (-\mu'_u(t))^{p-1}.$$

In a standard way (3.3) implies:

$$(3.4) \quad (-u^*(s))' \leq \frac{1}{(nC_n^{1/n})^{p'} s^{p'(1-1/n)}} \left(\int_0^s f^*(\sigma) d\sigma \right)^{1/(p-1)}, \quad \text{a.e. } s \in (0, |\Omega|),$$

where $p' = p/(p-1)$.

An immediate consequence of (3.4) is for example the following result (see also [21], [22], [2], [8]).

PROPOSITION 3.1. — *Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). Then:*

$$(3.5) \quad u^*(s) \leq \frac{1}{(nC_n^{1/n})^{p'}} \int_s^{|\Omega|} \frac{1}{r^{p'(1-1/n)}} \left(\int_0^r f^*(\sigma) d\sigma \right)^{1/(p-1)} dr, \quad s \in (0, |\Omega|).$$

Inequality (3.5) can be used as a starting point to obtain sharp estimates of some norms of u in terms of suitable norms of f . Such estimates are easily obtained not only

in Lebesgue spaces, but also in Lorentz space. For example, if $2 - 1/n < p < n$, one has:

$$f \in L(q, m), \quad 1 < q < \frac{n}{p}, \quad 0 < m < \infty \Rightarrow u \in L\left(\frac{nq}{n-pq}(p-1), m(p-1)\right),$$

and a similar statement holds when $f \in L^1(\Omega)$. Inequalities (3.2) and (3.4) are also the starting point (see [21], [22], [2]) in order to obtain an estimate of $\|Du\|_q$ for $1 \leq q \leq p < n$, in terms of the norm of f . If one wants to obtain estimates of the norm of $|Du|$ in Lorentz spaces, it is necessary to prove an inequality of the same type as (3.5) which involves the rearrangement of $|Du|$. Such a result is not known in the literature and it is the main result of this paper.

THEOREM 3.2. — *Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). Then, for every $\alpha > (p'/n') - 1$, we have:*

$$(3.6) \quad (|Du|^*(s))^p \leq \frac{\alpha+1}{(nC_n^{1/n})^{p'}} \left[\frac{1}{s^{\alpha+1}} \int_0^s r^{\alpha-p'(1-1/n)} \left(\int_0^r f^*(\sigma) d\sigma \right)^{p'} dr + \right. \\ \left. + \frac{1}{s} \int_s^{|\Omega|} \frac{1}{r^{p'(1-1/n)}} \left(\int_0^r f^*(\sigma) d\sigma \right)^{p'} dr \right], \quad s \in (0, |\Omega|).$$

The proof of Theorem 3.2 is based on the properties of the function

$$(3.7) \quad \Psi(s) = \int_{|u| \leq u^*(s)} |Du|^p dx, \quad s \in (0, |\Omega|),$$

where $u \in W_0^{1,1}(\Omega)$ is an entropy solution of (1.1). By (3.2) we have that the function

$$(3.8) \quad \Phi(t) = \int_{|u| \leq t} |Du|^p dx, \quad t \in (0, +\infty),$$

is an increasing, Lipschitz continuous function. Indeed, for $0 < t_1 < t_2$, we have:

$$\Phi(t_2) - \Phi(t_1) = \int_{t_1 < |u| \leq t_2} |Du|^p dx \leq (t_2 - t_1) \|f\|_1.$$

Furthermore, as a consequence of (3.2), we have:

$$(3.9) \quad \frac{d}{dt} \Phi(t) \leq \int_{|u| > t} |f| dx, \quad \text{a.e. } t \in (0, +\infty).$$

We recall that $u^*(s)$ is absolutely continuous on $[a, |\Omega|]$ for every $0 < a < |\Omega|$. Thus the definitions (3.7), (3.8) and inequality (3.9) imply that $\Psi(s)$ is absolutely continuous

on $[a, |\Omega|]$ for every $0 < a < |\Omega|$ and

$$(3.10) \quad \Psi'(s) = \Phi'(u^*(s))(u^*(s))', \quad \text{a.e. } s \in (0, |\Omega|).$$

Using (3.4), (3.9) and (3.10) we have:

$$(3.11) \quad -\Psi'(s) \leq \frac{1}{(nC_n^{1/n})^{p'} s^{p'(1-1/n)}} \left(\int_0^s f^*(\sigma) d\sigma \right)^{p'}, \quad \text{a.e. } s \in (0, |\Omega|).$$

We can prove the following result.

LEMMA 3.3. — *Let $\Psi(s)$ be defined as in (3.7). There exists a sequence $\{G_j(s)\}_{j \in \mathbb{N}}$ of functions such that $G_j^*(s) = |Du|^*(s)$, $s \in (0, |\Omega|)$, and*

$$(3.12) \quad \lim_{j \rightarrow +\infty} \int_0^{|\Omega|} G_j^p(s) v(s) ds = \int_0^{|\Omega|} (-\Psi'(s)) v(s) ds,$$

for every $v \in \text{BV}([0, |\Omega|])$, compactly supported in $(0, |\Omega|]$.

PROOF The techniques we use here are similar to those used in a similar lemma contained in [3]. Let $v \in \text{BV}([0, |\Omega|])$, be supported in $[a, |\Omega|]$, where $0 < a < |\Omega|$, and let us consider a decomposition of $[0, |\Omega|]$, $0 = s_0 < s_1 < \dots < s_j = |\Omega|$, where $j \in \mathbb{N}$ and $s_i - s_{i-1} = |\Omega|/j$, for $i = 1, \dots, j$. For every interval $(s_{i-1}, s_i]$, $i = 1, \dots, j$, we denote by $|Du|_i(x)$ the function which coincides with $|Du|(x)$ on the set $E_i = \{x \in \Omega: u^*(s_i) < u(x) \leq u^*(s_{i-1})\}$ and vanishes outside. Then we define $G_j(s)$ as the function which on every interval $(s_{i-1}, s_i]$, $i = 1, \dots, j$, coincides with the decreasing rearrangement of $|Du|_i$. In other words we put:

$$G_j(s) = |Du|_i^*(s - s_{i-1}), \quad \text{if } s \in (s_{i-1}, s_i], \quad i = 1, \dots, j.$$

We explicitly observe that in general $|E_i| \neq s_i - s_{i-1}$, but the fact that $|Du| = 0$ a.e. on the set $\{x \in \Omega: |u(x)| = t\}$ for every $t \geq 0$, implies that:

$$G_j^*(s) = |Du|^*(s), \quad \forall s \in (0, |\Omega|).$$

Furthermore, for $j > i$,

$$\int_{s_i}^{|\Omega|} G_j^p(r) dr = \sum_{h=i}^{j-1} \int_{s_h}^{s_{h+1}} G_j^p(r) dr = \sum_{h=i}^{j-1} \int_{E_{h+1}} |Du|^p dx = \int_{|u| \leq u^*(s_i)} |Du|^p dx = \Psi(s_i).$$

Thus, if $s_{i-1} < s \leq s_i$, we have:

$$\lambda(s) \equiv \int_s^{|\Omega|} G_j^p(r) dr - \Psi(s) = \int_s^{s_i} G_j^p(r) dr - \int_{u^*(s_i) < |u(x)| \leq u^*(s)} |Du|^p dx.$$

This implies:

$$-\int_{E_i} |Du|^p dx \leq \lambda(s) \leq \int_s^{s_i} G_j^p(r) dr \leq \int_{E_i} |Du|^p dx,$$

that is,

$$|\lambda(s)| \leq |\Psi(s_{i-1}) - \Psi(s_i)|.$$

Because of the absolute continuity of Ψ in $[a, |\Omega|]$, if j is big enough, we have:

$$\max_{s \in [a, |\Omega|]} |\lambda(s)| \leq \varepsilon,$$

where ε is a fixed positive number.

Finally:

$$\begin{aligned} & \left| \int_0^{|\Omega|} G_j^p(s) v(s) ds - \int_0^{|\Omega|} (-\Psi'(s)) v(s) ds \right| = \\ & = \left| - \int_0^{|\Omega|} \left(\frac{d}{ds} \int_s^{|\Omega|} G_j^p(r) dr - \Psi'(s) \right) v(s) ds \right| = \left| \int_a^{|\Omega|} \left(\int_s^{|\Omega|} G_j^p(r) dr - \Psi(s) \right) dv(s) \right| \leq \\ & \leq \|v\|_{BV} \max_{s \in [a, |\Omega|]} |\lambda(s)| \leq \varepsilon \|v\|_{BV}, \end{aligned}$$

and (3.12) is proven. ■

PROOF OF THEOREM 3.2. - Let us fix $s \in (0, |\Omega|)$. For $\varepsilon \in (0, s)$ and $\alpha > (p'/n') - 1$ we put:

$$v_\alpha(r) = \begin{cases} 0 & \text{if } 0 \leq r < \varepsilon, \\ s \left(\frac{r - \varepsilon}{s - \varepsilon} \right)^\alpha & \text{if } \varepsilon \leq r < s, \\ s & \text{if } s \leq r \leq |\Omega|. \end{cases}$$

Using the notation $G(r) = |Du|^*(r)$, by Hardy-Littlewood inequality we have:

$$\begin{aligned} (3.13) \quad & \int_0^{|\Omega|} G^p(r) v_\alpha(r) dr \geq \int_0^s G^p(r) v_\alpha(r) dr \geq G^p(s) \int_0^s v_\alpha(r) dr = \\ & = G^p(s) \int_\varepsilon^s s \left(\frac{r - \varepsilon}{s - \varepsilon} \right)^\alpha dr = G^p(s) \frac{s(s - \varepsilon)}{\alpha + 1}. \end{aligned}$$

On the other hand, if $\{G_j(s)\}_{j \in \mathbb{N}}$ is the sequence given by Lemma 3.3, Hardy-Little-

wood inequality implies:

$$(3.14) \quad \int_0^{|\Omega|} (-\Psi'(r)) v_\alpha(r) dr = \lim_{j \rightarrow +\infty} \int_0^{|\Omega|} G_j^p(r) v_\alpha(r) dr \geq \int_0^{|\Omega|} G^p(r) v_\alpha(r) dr.$$

Collecting (3.11), (3.13), (3.14), and using the arbitrariness of ε , we have:

$$\begin{aligned} \frac{s^2}{\alpha+1} G^p(s) &\leq \frac{1}{(nC_n^{1/n})^{p'}} \left[\frac{1}{s^{\alpha-1}} \int_0^s r^{\alpha-p'(1-1/n)} \left(\int_0^r f^*(\sigma) d\sigma \right)^{p'} dr + \right. \\ &\quad \left. + s \int_s^{|\Omega|} \frac{1}{r^{p'(1-1/n)}} \left(\int_0^r f^*(\sigma) d\sigma \right)^{p'} dr \right], \quad s \in (0, |\Omega|), \end{aligned}$$

that is (3.6). ■

REMARK 3.1. – As it is clear from the proof of Theorem 3.2, inequality (3.6) can be proven for every $\alpha > 0$, but it can become meaningless. Indeed, if we only suppose $f \in L^1(\Omega)$, the condition $\alpha > (p'/n') - 1$ is necessary in order to guarantee the finiteness of the first integral on the right-hand side of (3.6).

4. – Some applications.

The inequality (3.6) stated in Theorem 3.2 allows us to estimate the norm of the gradient of an entropy solution of (1.1) in Lorentz spaces in terms of the norm of f .

THEOREM 4.1. – Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). If $f \in L(q, m)$, $\max\{1, n/(np - n + 1)\} < q < np/(np - n + p)$, $0 < m \leq \infty$, then $|Du| \in L(q^*(p-1), m(p-1))$.

PROOF. – Taking into account (2.2) it is enough to estimate $\| |Du| \|_{q^*(p-1), m(p-1)}^*$. We explicitly observe that the bound from below on q implies that $q^*(p-1) > 1$.

The case $m = \infty$ is immediate. Indeed, if $f \in L(q, \infty)$ we have:

$$\bar{f}(s) \leq \frac{\|f\|_{q, \infty}}{s^{1/q}}, \quad s \in (0, +\infty).$$

Now, recalling Theorem 3.2, we have:

$$(|Du|^*(s))^p \leq \frac{\alpha+1}{(nC_n^{1/n})^{p'}} \left(\frac{1}{s^{\alpha+1}} \int_0^s r^{\alpha-p'/q^*} dr + \frac{1}{s} \int_s^{|\Omega|} r^{-p'/q^*} dr \right) \|f\|_{q, \infty}^p,$$

for every $\alpha > (p'/n') - 1$. The above inequality gives:

$$|Du|^*(s) \leq \frac{c}{s^{1/q^*(p-1)}},$$

and then $|Du| \in L(q^*(p-1), \infty)$.

When $0 < m < \infty$ we have to use Lemma 2.1. Theorem 3.2 implies that there exists a constant c_1 , which depends on n, m, p and q only, such that:

$$(4.1) \quad I \leq c_1 \left\{ \int_0^{\infty} \left(\frac{1}{s^{\alpha+1}} \int_0^s r^{\alpha+p'/n} (\bar{f}(r))^{p'} dr \right)^{m/p'} s^{m/q^*} \frac{ds}{s} + \right. \\ \left. + \int_0^{\infty} \left(\frac{1}{s} \int_s^{\infty} r^{p'/n} (\bar{f}(r))^{p'} dr \right)^{m/p'} s^{m/q^*} \frac{ds}{s} \right\},$$

where we have put $I = (\| |Du| \|_{q^*(p-1), m(p-1)}^{m(p-1)})^{m(p-1)}$. The integrals on the right-hand side of (4.1) can be estimated by using Lemma 2.1. We observe that $\bar{f}(r)$ is a decreasing function, so, putting $\varphi(r) = (\bar{f}(r))^{p'}$, $\beta = \alpha + p'/n$, $\gamma = p'/q^* - \alpha$, $\lambda = m/p'$ in Lemma 2.1, we have:

$$(4.2) \quad \int_0^{\infty} \left(\frac{1}{s^{\alpha+1}} \int_0^s r^{\alpha+p'/n} (\bar{f}(r))^{p'} dr \right)^{m/p'} s^{m/q^*} \frac{ds}{s} \leq c \int_0^{\infty} (\bar{f}(r) r^{1/q})^m \frac{dr}{r}.$$

We explicitly observe that in view of the assumptions on α we have $\gamma < 1$.

The second integral on the right-hand side of (4.1) can be estimated in a similar way. Putting $\varphi(r) = (\bar{f}(r))^{p'}$, $\beta = p'/n$, $\gamma = p'/q^* > 1$, $\lambda = m/p'$ in Lemma 2.1, we get:

$$(4.3) \quad \int_0^{\infty} \left(\frac{1}{s} \int_s^{\infty} r^{p'/n} (\bar{f}(r))^{p'} dr \right)^{m/p'} s^{m/q^*} \frac{ds}{s} \leq c \int_0^{\infty} (\bar{f}(r) r^{1/q})^m \frac{dr}{r}.$$

The use of (4.1), (4.2) and (4.3) gives the assertion. ■

Theorem 4.1 allows us to give a complete answer about the regularity of $|Du|$ when $\bar{q} = \max \{1, n/(np - n + 1)\} < q < np/(np - n + p)$. When $q = np/(np - n + p)$ one would expect similar results, but, unfortunately, our method is not sharp enough. Indeed, for such a choice of q , we would have $p'/q^* = 1$ and so it is not possible to use Lemma 2.1 with $\gamma = p'/q^*$. The only result we can prove is the following (see also [4]):

THEOREM 4.2. – Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). If $1 < p < n$ and $f \in L(np/(np - n + p), p')$, then $|Du| \in L^p(\Omega)$.

PROOF. – The use of Theorem 3.2 does not give the right answer. We have to use di-

rectly inequality (3.11). Integrating between 0 and $|\Omega|$, we have:

$$\|Du\|_p^p \leq c \int_0^{|\Omega|} (\bar{f}(r) r^{1/n+1/p'})^{p'} \frac{dr}{r},$$

which proves the theorem. ■

REMARK 4.1. – If we suppose $f \in L(np/(np-n+p), m)$, $0 < m < p'$, we do not get more than $|Du| \in L^p(\Omega)$. In particular this is true when $m = np/(np-n+p)$, that is, when $f \in L^{np/(np-n+p)}(\Omega)$. If $f \in L(np/(np-n+p), m)$, $m > p'$, we would expect that $|Du| \in L(p, m(p-1))$, but, as already mentioned, the method used in Theorem 4.1 cannot be used in such a case. For instance, a simple consequence of inequality (3.6) is only that $|Du| \in L^\tau(\Omega)$, $\forall 1 \leq \tau < p$. Similarly, in the limit case $p = n$, the natural condition on f becomes $f \in L^1(\Omega)$ and (3.6) implies that $|Du| \in L^\tau(\Omega)$, $\forall 1 \leq \tau < n$. However, in this last case a solution u of (1.1) such that $|Du| \in L(n, \infty)$ can be proven to exist (see [12]).

As regards the case where we put $q = \bar{q}$, we observe that the question is different if we have $\bar{q} > 1$ (i.e., $p < 2 - 1/n$) or $\bar{q} = 1$ (i.e., $p \geq 2 - 1/n$). In the first case, using the same argument of Theorem 4.1, we can prove the following result.

THEOREM 4.3. – Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). If $n/(np-n+1) > 1$, $0 < m \leq 1/(p-1)$ and $f \in L(n/(np-n+1), m)$, then $|Du|$ is such that:

$$\int_0^{|\Omega|} (|Du|^*(s) s)^{m(p-1)} \frac{ds}{s} \leq c \|f\|_{n/(np-n+1), m}^m,$$

where c depends on n, m, p only.

REMARK 4.2. – When $f \in L(n/(np-n+1), 1/(p-1))$, $n/(np-n+1) > 1$, Theorem 4.3 only states that $|Du|$ belongs to $L^1(\Omega)$. When $f \in L(n/(np-n+1), m)$, $n/(np-n+1) > 1$, $0 < m < 1/(p-1)$, Theorem 4.3 states that $|Du|$ belongs to a Lorentz space with first order equal to 1. We observe that one can give different definitions of the Lorentz spaces $L(1, m)$, $0 < m \leq \infty$, but we will not exploit this question here (see [7], [19], [16]).

In view of the above remark, the case $\bar{q} = 1$ is complicated by the fact that we can choose f in Lorentz spaces with first index 1 which can be defined in different ways. For the sake of simplicity we will only analyze the case where $f \in L^1(\Omega)$. An immediate application of Theorem 3.2 gives the following result.

THEOREM 4.4. – Let $u \in W_0^{1,1}(\Omega)$ be an entropy solution of (1.1), under assumption (1.2). If $2 - 1/n < p < n$ and $f \in L^1(\Omega)$, then $|Du| \in L(n'(p-1), \infty)$.

REMARK 4.3. – In the case $p = 2 - 1/n$, $f \in L^1(\Omega)$, it is not possible, in general, to say that $|Du|$ belongs to a space smaller of $L^1(\Omega)$. For the case $p = n$ see Remark 4.1.

REMARK 4.4. – The sharpness of Theorems 4.1, 4.2, 4.3, 4.4, can be illustrated by an example. Consider the problem:

$$(4.4) \quad \begin{cases} -\operatorname{div}(|Du|^{p-2} Du) = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is a ball of radius R centered at the origin and $f(x) = f(|x|)$ is a radially decreasing function which belongs to $L^1(\Omega)$. An entropy solution $u(x) = u(|x|)$ of (4.4) is given by:

$$(4.5) \quad u(r) = \int_r^R \frac{1}{\varrho^{(n-1)/(p-1)}} \left(\int_0^\varrho \sigma^{n-1} f(\sigma) d\sigma \right)^{1/(p-1)} d\varrho,$$

where we have put $r = |x|$. We observe that Hardy-Littlewood inequality gives:

$$\overline{|Du|}(s) = \frac{1}{s} \int_0^s |Du|^*(\sigma) d\sigma \geq \frac{1}{s} \int_{B(s)} |Du| dx,$$

where $B(s) = \{x \in \mathbb{R}^n: |x| < (s/C_n)^{1/n}\}$, $s \in (0, |B|)$. Using (4.5) we have:

$$\begin{aligned} \overline{|Du|}(s) &\geq \frac{nC_n}{s} \int_0^{(s/C_n)^{1/n}} \frac{1}{\varrho^{(n-1)/(p-1)}} \left(\int_0^\varrho \sigma^{n-1} f(\sigma) d\sigma \right)^{1/(p-1)} \varrho^{n-1} d\varrho = \\ &= \frac{c_1}{s} \int_0^s (t^{1/n} \bar{f}(t))^{1/(p-1)} dt \geq c_2 (s^{1/n} \bar{f}(s))^{1/(p-1)}, \end{aligned}$$

for some suitable positive constants c_1, c_2 . It implies, for $0 < m < \infty$,

$$\| |Du| \|_{q^*(p-1), m(p-1)} \geq c_2 \|f\|_{q, m}^{1/(p-1)},$$

and, for $m = \infty$,

$$\| |Du| \|_{q^*(p-1), \infty} \geq c_2 \|f\|_{q, \infty}^{1/(p-1)}.$$

This means that for problem (4.4), which satisfies (1.2), if f does not belong to $L(q, m)$, then $|Du|$ does not belong to $L(q^*(p-1), m(p-1))$.

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