Scalar Products on Clifford Modules and Pseudo- H -type Lie Algebras (*).

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Abstract. – In 1980 A. Kaplan introduced the so called generalised Heisenberg algebras, which are two step nilpotent algebras endowed with an inner product satisfying a compatibility condition with the Lie algebra structure. In this paper we generalize the definition of A. Kaplan to the case of a nonpositive definite scalar product. In the non-positive definite case the proof of the existence and the classification raise entirely new problems. The natural setting to solve them is that of the theory of Clifford modules.

0. – Introduction.

Since their introduction by A. Kaplan [Ka1], the generalised Heisenberg groups, also known as groups of Heisenberg type or H -type groups, have provided a framework in which to construct interesting examples in geometry and analysis (see for instance, [DR], [R], [Ka2], [Ko], [TV], [BTV]). For example, since the classical nilpotent Lie groups which arise in the Iwasawa decomposition of rank one simple Lie groups are H -type, the H -type groups has been extensively used to study in a unified way the properties of the Iwasawa N -groups (see, for instance, [CDKR1] and [CDKR2]).

The H -type algebras which are the Lie algebras associated to the H -type Lie groups, are Lie algebras endowed with a definite scalar product which satisfies a compatibility condition (see [Ka1]). In this paper we introduce the pseudo- H -type algebras which are Lie algebras provided with a non-definite scalar product.

In the first section we define the H -type Lie algebras with non-definite scalar product. As in the case of definite scalar products, these Lie algebras correspond to representations of Clifford algebras. However, not all Clifford modules give rise to an H -type algebra. Therefore we introduce the notion of admissible Clifford modules, which are just the modules associated to pseudo- H -algebras. We conclude the section proving that a pseudo- H -type Lie algebra exists if and only if there exists the corresponding admissible module.

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In the second section we show that for all pairs of non-negative integers (p, q) there is an admissible $\mathcal{C}(p, q)$ -module. Then we give a general prescription to construct H -type-algebras and determine the signature of the scalar product of an admissible module.

In the third and last section we study the admissibility of an irreducible $\mathcal{C}(p, q)$ -module $\mathfrak{v}$, showing that depending on (p, q) the admissible Clifford module of minimal dimension can be $\mathfrak{v}$ itself, or $\mathfrak{v} \oplus \mathfrak{v}$.

The subject of this paper was suggested to me by professor Fulvio Ricci. I wish to express my gratitude to him for his interest and valuable advice.

For the sake of completeness we spend now some words about notations and definitions concerning scalar products, Clifford algebras, and complex and quaternionic structures on real vector spaces. For more details the reader is, for instance, referred to [O'N] in connection with scalar products, and to [DDS] or to [BT] in connection with Clifford algebras.

Let $\mathfrak{u}$ be a real vector space of dimension k . We will call *scalar product* a symmetric, non-degenerate, (in general non-definite), bilinear form

$$\langle, \rangle: \mathfrak{u} \times \mathfrak{u} \rightarrow \mathbb{R}.$$

We will say that $\langle, \rangle$ has *signature* (p, q) with p and q non-negative integers such that $p + q = k$, if there exists a basis $\{U_1, \dots, U_k\}$ of vectors of $\mathfrak{u}$ which satisfy

$$\langle U_i, U_j \rangle = \varepsilon_i(p, q) \delta_{ij},$$

where

$$\varepsilon_i(p, q) = \begin{cases} 1, & \text{for } i = 1, \dots, p, \\ -1, & \text{for } i = p + 1, \dots, p + q, \end{cases}$$

and δ_{ij} the Kronecker delta.

In particular when the signature of $\langle, \rangle$ is $(k, 0)$, $\langle, \rangle$ is also called *inner product*, while $\langle, \rangle$ is said to be *neutral* when the signature is (p, p) .

We will say that $U \in \mathfrak{u}$ is *normalized* if

$$|\langle U, U \rangle| = 1,$$

and we will say that $V \in \mathfrak{u}$ is *positive*, *negative*, or *neutral* respectively if

$$\langle V, V \rangle > 0, \quad \langle V, V \rangle < 0, \quad \langle V, V \rangle = 0.$$

Let $\mathfrak{w}$ be a subspace of $\mathfrak{u}$. The *orthogonal complement* of $\mathfrak{w}$ is

$$\mathfrak{w}^\perp = \{U \in \mathfrak{u} : \langle W, U \rangle = 0 \quad \forall W \in \mathfrak{w}\}.$$

A subspace $\mathfrak{w}$ of $\mathfrak{u}$ is called *non-degenerate* if the restriction of $\langle, \rangle$ to $\mathfrak{w}$ is non-degenerate.

A subspace $\mathfrak{w}$ of $\mathfrak{u}$ is non-degenerate if and only if $\mathfrak{u}$ is the direct sum of $\mathfrak{w}$ and $\mathfrak{w}^\perp$,

$$(0.1) \quad \mathfrak{u} = \mathfrak{w} \oplus \mathfrak{w}^\perp.$$

We will denote by $\mathcal{C}(p, q)$ or by $\mathcal{C}(\mathfrak{z}, \langle, \rangle)$ the Clifford algebra built on $\mathfrak{z} = \mathbb{R}^{p+q}$ associated to a scalar product $\langle, \rangle$ of signature (p, q) . An orthogonal basis $(Z_1, \dots, Z_{p+q})$ of normalized vectors of $\mathfrak{z}$ will be called a *set of Clifford generators of $\mathcal{C}(p, q)$* . They satisfy the *fundamental relations of $\mathcal{C}(p, q)$* , i.e.

$$Z_i Z_j + z_j Z_i = \varepsilon_i(p, q) \delta_{ij}.$$

We will say that $\mathfrak{J} \in \text{End}(\mathfrak{z})$ is a *complex structure* on $\mathfrak{z}$ if

$$(0.2) \quad \mathfrak{J}^2 = -I_{\mathfrak{z}}.$$

Complex structures exist only if the dimension of $\mathfrak{z}$ is even. Then $A \in \text{End}(\mathfrak{z})$ will be called a *$\mathfrak{J}$ -complex endomorphism* if

$$(0.3a) \quad \mathfrak{J}A = A\mathfrak{J},$$

and will be called an *anti- $\mathfrak{J}$ -complex endomorphism* if

$$(0.3b) \quad \mathfrak{J}A = -A\mathfrak{J}.$$

Every $A \in \text{End}(\mathfrak{z})$ can be uniquely decomposed into the sum

$$(0.4) \quad A = A_0 + A_1,$$

with $A_0, A_1 \in \text{End}(\mathfrak{z})$, A_0 $\mathfrak{J}$ -complex, and A_1 anti- $\mathfrak{J}$ -complex, taking

$$A_0 = \frac{1}{2}(A + \mathfrak{J}A\mathfrak{J}^{-1}) = \frac{1}{2}(A - \mathfrak{J}A\mathfrak{J}),$$

$$A_1 = \frac{1}{2}(A - \mathfrak{J}A\mathfrak{J}^{-1}) = \frac{1}{2}(A + \mathfrak{J}A\mathfrak{J}).$$

A pair $(\mathfrak{J}, \mathfrak{K}) \in \text{End}(\mathfrak{z})$ is called a *quaternionic structure* if

$$(0.5a) \quad \mathfrak{J}^2 = \mathfrak{K}^2 = -I_{\mathfrak{z}}$$

and

$$(0.5b) \quad \mathfrak{J}\mathfrak{K} + \mathfrak{K}\mathfrak{J} = 0.$$

Quaternionic structures on $\mathfrak{z}$ exist only if $\dim \mathfrak{z} = 4r$ with $r = 1, 2, \dots$. Correspondingly $A \in \text{End}(\mathfrak{z})$ is said to be a *$(\mathfrak{J}, \mathfrak{K})$ -quaternionic endomorphism* if

$$(0.6) \quad \mathfrak{J}A = A\mathfrak{J} \quad \text{and} \quad \mathfrak{K}A = A\mathfrak{K}.$$

Every $A \in \text{End}(\mathfrak{z})$ can be uniquely decomposed into the sum

$$(0.7) \quad A = A_0 + A_{\mathfrak{J}} + A_{\mathfrak{K}} + A_{\mathfrak{J}\mathfrak{K}},$$

where $A_0, A_{\mathfrak{J}}, A_{\mathfrak{K}}, A_{\mathfrak{J}\mathfrak{K}} \in \text{End}(\mathfrak{z})$, A_0 is $(\mathfrak{J}, \mathfrak{K})$ -quaternionic, while $A_{\mathfrak{J}_a}$ commutes with $\mathfrak{J}_a$ and anti-commutes with $\mathfrak{J}_b$ when $b \neq a = 1, 2, 3$, where we have put $\mathfrak{J}_1 = \mathfrak{J}$, $\mathfrak{J}_2 = \mathfrak{K}$ and

$\mathfrak{J}_3 = \mathfrak{J}\mathfrak{K}$. We have

$$A_0 = \frac{1}{4}(A - \mathfrak{J}A\mathfrak{J} - \mathfrak{K}A\mathfrak{K} + \mathfrak{J}\mathfrak{K}A\mathfrak{K}\mathfrak{J}),$$

$$A_{\mathfrak{J}} = \frac{1}{4}(A - \mathfrak{J}A\mathfrak{J} + \mathfrak{K}A\mathfrak{K} - \mathfrak{J}\mathfrak{K}A\mathfrak{K}\mathfrak{J}),$$

$$A_{\mathfrak{K}} = \frac{1}{4}(A + \mathfrak{J}A\mathfrak{J} - \mathfrak{K}A\mathfrak{K} - \mathfrak{J}\mathfrak{K}A\mathfrak{K}\mathfrak{J}),$$

$$A_{\mathfrak{J}\mathfrak{K}} = \frac{1}{4}(A + \mathfrak{J}A\mathfrak{J} + \mathfrak{K}A\mathfrak{K} + \mathfrak{J}\mathfrak{K}A\mathfrak{K}\mathfrak{J}).$$

In the following we will often need the Pauli matrices, so we recall here their definition:

$$(0.8) \quad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. – Pseudo- H -type Lie algebras heading.

Let $\mathfrak{n}$ be a nilpotent two-step real Lie algebra endowed with a scalar product $\langle, \rangle$. Assume that the restriction $\langle, \rangle_{\mathfrak{z}}$ of the scalar product to the center $\mathfrak{z}$ of $\mathfrak{n}$ be non-degenerate, which by (0.1) is equivalent to say that the sum between $\mathfrak{z}$ and $\mathfrak{v} = \mathfrak{z}^\perp$ is direct and equal to $\mathfrak{n}$.

Define for all $Z \in \mathfrak{z}$ a map $J_Z: \mathfrak{v} \rightarrow \mathfrak{v}$ by

$$(1.1) \quad \langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle$$

for $X, Y \in \mathfrak{v}$.

DEFINITION 1.1. – We will call *pseudo- H -type Lie algebra* if

$$(1.2) \quad \langle J_Z X, J_Z X \rangle = - \langle Z, Z \rangle \langle X, X \rangle$$

for all $Z \in \mathfrak{z}$ and for all $X \in \mathfrak{v}$. In particular we will say that $\mathfrak{n}$ is a (p, q) - H -type algebra or a (p, q) -algebra if $\langle, \rangle_{\mathfrak{z}}$ has signature (p, q) .

REMARK (1.1). – From (1.1) it follows immediately that for all $Z \in \mathfrak{z}$ the operator J_Z is skew-adjoint.

REMARK (1.2). – By polarization (1.2) implies that

$$(1.3a) \quad \langle J_Z X, J_{Z'} X \rangle = - \langle Z, Z' \rangle \langle X, X \rangle$$

and that

$$(1.3b) \quad \langle J_Z X, J_Z X' \rangle = - \langle Z, Z \rangle \langle X, X' \rangle$$

for all $Z, Z' \in \mathfrak{z}$ and all $X, X' \in \mathfrak{v}$.

The skew-adjointness of J_Z and (1.3b) together imply that

$$(1.4) \quad J_Z^2 = \langle Z, Z \rangle I_{\mathfrak{v}},$$

where $I_{\mathfrak{v}}$ is the identity operator on $\mathfrak{v}$.

REMARK (1.3). – From (1.3a) it follows that

$$\langle J_Z(X + X'), J_{Z'}(X + X') \rangle = - \langle Z, Z' \rangle \langle (X + X'), (X + X') \rangle$$

which, using (1.3a), implies that

$$-2 \langle Z, Z' \rangle \langle X, X' \rangle = \langle J_Z X, J_{Z'} X' \rangle + \langle J_{Z'} X, J_Z X' \rangle = - \langle (J_Z J_{Z'} + J_{Z'} J_Z) X, X' \rangle,$$

for all $Z, Z' \in \mathfrak{z}$ and all $X, X' \in \mathfrak{v}$. Hence

$$J_Z J_{Z'} + J_{Z'} J_Z = 2 \langle Z, Z' \rangle I_{\mathfrak{v}}$$

for all $Z, Z' \in \mathfrak{z}$, which is equivalent to say that $J : \mathcal{C}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \rightarrow \text{End}(\mathfrak{v})$ defines a representation of the Clifford algebra $\mathcal{C}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$.

Given a Clifford algebra $\mathcal{C}(p, q)$, a $\mathcal{C}(p, q)$ -module is a vector space which is the carrier space for a representation of $\mathcal{C}(p, q)$. In the following we shall find modules which are canonically identified with $\mathbb{C}^n$ or $\mathbb{H}^n$. We shall always consider these as real vector spaces (resp. $\mathbb{R}^{2n}$ and $\mathbb{R}^{4n}$).

Remark (1.1) motivates the following definition.

DEFINITION 1.2. – A pair $(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$, where $\mathfrak{v}$ is a $\mathcal{C}(p, q)$ -module and $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ is a scalar product on $\mathfrak{v}$, will be said to be an *admissible* (p, q) -module or an *admissible module* if, denoted by $J : \gamma \rightarrow J_{\gamma}$ the representation carried by $\mathfrak{v}$, the operators J_Z with $Z \in \mathfrak{z}$ are skew-adjoint, i.e.

$$(1.5) \quad \langle J_Z X, Y \rangle_{\mathfrak{v}} = - \langle X, J_Z Y \rangle_{\mathfrak{v}}$$

for all $Z \in \mathfrak{z}$ and all $X, Y \in \mathfrak{v}$.

We remark that not all $\mathcal{C}(p, q)$ -modules are admissible. In Section 3 we will give examples of families of non-admissible modules.

We finish this section with the following lemma.

LEMMA 1.1. – Let $\mathfrak{z}$ be a real vector space of dimension k endowed with a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{z}}$ of signature (p, q) ($p + q = k$), and let $\mathfrak{v}$ be a $\mathcal{C}(p, q)$ -module. Then $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ can be supplied with a (p, q) -algebra structure if and only if there exists a scalar product $\langle \cdot, \cdot \rangle : \mathfrak{v} \times \mathfrak{v} \rightarrow \mathbb{R}$ such that $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ is admissible.

PROOF. – Suppose that $(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ is an admissible module and let $J : \gamma \rightarrow J_{\gamma}$ be the corresponding representation of $\mathcal{C}(p, q)$. Endow $\mathfrak{n}$ with the scalar product $\langle \cdot, \cdot \rangle$ consisting

of the orthogonal sum of $\langle, \rangle_{\mathfrak{z}}$ and $\langle, \rangle_{\mathfrak{v}}$. Then define a bracket

$$[,]: \mathfrak{n} \times \mathfrak{n} \rightarrow \mathbb{R},$$

by

$$[Z, V] = 0$$

for all $Z \in \mathfrak{z}$ and all $V \in \mathfrak{n}$, and by

$$\langle Z, [X, Y] \rangle = \langle J_Z X, Y \rangle$$

for all $Z \in \mathfrak{z}$ and all $X, Y \in \mathfrak{v}$.

It is easy now to verify that $\mathfrak{u}$ equipped with $\langle, \rangle$ and $[\cdot, \cdot]$, is a (p, q) -algebra.

For the converse, assume that $\mathfrak{u}$ is a (p, q) -algebra. Choose a basis of the center $\mathfrak{z}$ of $\mathfrak{n}$, $\{Z_1, \dots, Z_k\}$, consisting of orthogonal normalized vectors. Define $J_i = J_{Z_i}$ for $i = 1, \dots, k$ by (1.1). Then, noticing that $\{Z_1, \dots, Z_k\}$ is a set of generators of $\mathcal{C}(p, q)$, extend the definition of J to all the Clifford algebra by

$$J_{Z_{i(1)} \dots Z_{i(l)}} = J_{Z_{i(1)}} \dots J_{Z_{i(l)}}$$

for $1 \leq l \leq k$ and $i(1), \dots, i(l) = 1, \dots, k$, and by linearity. Since the restriction of $\langle, \rangle$ to $\mathfrak{z}$ is non degenerate, the sum between $\mathfrak{z}$ and its orthogonal $\mathfrak{v}$ is direct and equal to $\mathfrak{n}$ by (0.1). Then it follows from Remark (1.1) that $(\mathfrak{v}, \langle, \rangle_{\mathfrak{v}})$ is an admissible module. ■

2. – Existence of pseudo- H -algebras.

We discuss now the existence of pseudo- H -type algebras. By Lemma 1.1 this is equivalent to the existence of admissible modules. For all pairs of non-negative integers (p, q) we will show that given a $\mathcal{C}(p, q)$ -module $\mathfrak{v}$, then $\mathfrak{v}$ itself, or $\mathfrak{v} \oplus \mathfrak{v}$ can be equipped with a scalar product which satisfies (1.5). To reach the goal we need the following lemma, see [Hu]:

LEMMA 2.1. – *For any representation $J: \mathcal{C}(p, q) \rightarrow \text{End}(\mathfrak{v})$ on a real vector space $\mathfrak{v}$ there exists an inner product $(\cdot, \cdot)$, with respect to which J_{Z_i} is an orthogonal transformation for all Clifford generators $\{Z_1, \dots, Z_{p+q}\}$ of $\mathcal{C}(p, q)$, i.e.*

$$(2.1) \quad (J_{Z_i} X, J_{Z_i} Y) = (X, Y)$$

for all $i = 1, \dots, p + q$ and all $X, Y \in \mathfrak{v}$. If $\mathfrak{v}$ is also provided with a complex structure $\mathfrak{J}$ (or with a quaternionic structure $(\mathfrak{J}, \mathfrak{K})$) commuting with J_{Z_i} for $i = 1, \dots, p + q$, the inner product can be defined in such a way that $\mathfrak{J}$ (correspondingly $\mathfrak{J}$ and $\mathfrak{K}$) be an orthogonal transformation.

PROOF. – Suppose that $\mathfrak{v}$ is provided with a quaternionic structure $(\mathfrak{J}, \mathfrak{K})$ (when $\mathfrak{v}$ has only a complex structure, or no extra-structure the proof is similar and simpler).

Let Γ be the finite group generated by $\pm J_{Z_i} = \pm J_i$ for $i = 1, \dots, p + q$, $\mathfrak{J}$ and $\mathfrak{K}$.

Let $(,)_1$ be any inner product on $\mathfrak{v}$. We define a new inner product by

$$(U, V) = \frac{1}{\text{card}(\Gamma)} \sum_{\sigma \in \Gamma} (\sigma U, \sigma V)_1$$

for $U, V \in \mathfrak{v}$. Since $Z_i \in \Gamma$ and $\mathfrak{J}^2 = \mathfrak{K}^2 = -I_{\mathfrak{v}}$, we have

$$(J_i U, J_i V) = (U, V)$$

for all $i = 1, \dots, p + q$, and

$$(\mathfrak{J}U, \mathfrak{J}V) = (\mathfrak{K}U, \mathfrak{K}V) = (U, V). \quad \blacksquare$$

REMARK (2.1). – If $\mathfrak{v}$ is an irreducible $\mathcal{C}(p, q)$ -module a scalar product with the above property is unique up to a multiplicative constant, by Schur's lemma.

Now let $(,)$ an inner product which satisfies (2.1). For any invertible $A \in \text{End}(\mathfrak{v})$ which is self-adjoint with respect to $(,)$, we define a scalar product by

$$\langle X, Y \rangle_A = (X, AY),$$

for $X, Y \in \mathfrak{v}$. The operator J_Z will be skew-adjoint with respect to $\langle, \rangle_A$ if and only if

$$(2.2) \quad {}^t J_Z A = -A J_Z$$

where t denotes the transposition with respect to $(,)$.

Since every J_i is an orthogonal transformation for $(,)$,

$$(X, Y) = (J_i X, J_i Y) = (X, {}^t J_i J_i Y)$$

for all $X, Y \in \mathfrak{v}$, that is ${}^t J_i J_i = I_{\mathfrak{v}}$. Since $J_i^2 = \varepsilon_i(p, q) I_{\mathfrak{v}}$ it follows that

$${}^t J_i = \varepsilon_i(p, q) J_i = \begin{cases} J_i, & \text{for } i = 1, \dots, p \\ -J_i, & \text{for } i = p + 1, \dots, p + q, \end{cases}$$

where $\varepsilon_i(p, q)$ has been defined in Section 0.

We define

$$(2.3a) \quad P = \begin{cases} J_1 \dots J_p, & \text{for } p \neq 0, \\ I, & \text{for } p = 0, \end{cases}$$

and

$$(2.3b) \quad Q = \begin{cases} J_{p+1} \dots J_{p+q}, & \text{for } q \neq 0, \\ I, & \text{for } q = 0. \end{cases}$$

Then straightforward computations show that the following identities hold

$$(2.4a) \quad {}^t J_i P = J_i P = (-1)^{p-1} P J_i, \quad \text{for } i = 1, \dots, p$$

and

$$(2.4b) \quad {}^t J_i Q = -J_i Q = (-1)^q Q J_i, \quad \text{for } i = p+1, \dots, p+q.$$

Moreover we have

$$(2.5a) \quad {}^t P = (-1)^{p(p-1)/2} P,$$

and

$$(2.5b) \quad {}^t Q = (-1)^{q(q-1)/2} Q.$$

We distinguish two cases: (i) when p is even or q is odd, and (ii) when p is odd and q is even.

Case i: p even or q odd

It follows from (2.4a) that when p is even P satisfies (2.2), while (2.4b) implies that when q is odd Q satisfies (2.2).

However the operators P and Q can be directly used to get a scalar product only when they are symmetric with respect to $(,)$, i.e. when ${}^t P = P$ or ${}^t Q = Q$.

From (2.5a) and (2.5b) it follows respectively that

$$(2.6a) \quad {}^t P = P \quad \text{for } p \equiv 0 \pmod{4}$$

and

$$(2.6b) \quad {}^t Q = Q \quad \text{for } q \equiv 3 \pmod{4}.$$

This proves that when $p \equiv 0$ or $q \equiv 3 \pmod{4}$, there exists an admissible module.

To deal with the remaining cases $p \equiv 2 \pmod{4}$ and $q \equiv 1 \pmod{4}$, let

$$(2.7) \quad \mathfrak{v}' = \mathfrak{v} \oplus \mathfrak{v}.$$

Then the tensor products

$$A_P = \tau_2 \otimes P \quad \text{and} \quad A_Q = \tau_2 \otimes Q$$

define endomorphisms of $\mathfrak{v}'$ which are symmetric with respect to the inner product $(,)'$ consisting of the orthogonal sum of two copies of $(,)$

$$(2.8) \quad (,)' = (,) \oplus (,).$$

Moreover the map

$$J' = \tau_0 \otimes J: \mathbb{C}(p, q) \rightarrow \text{End}(\mathfrak{v}')$$

is a representation satisfying (2.2) with $A = A_P$ for $p \equiv 2 \pmod{4}$ and with $A = A_Q$ for $q \equiv 1 \pmod{4}$.

Case ii: p odd and q even

We distinguish two subcases:

Subcase ii-1:

$$(p \equiv 1 \pmod{4}, q \equiv 0 \pmod{4}),$$

$$(p \equiv 3 \pmod{4}, q \equiv 0 \pmod{4}),$$

$$(p \equiv 1 \pmod{4}, q \equiv 2 \pmod{4}).$$

In this case we consider the Clifford algebra $\mathcal{C}(p, q-1)$ and a representation on a real vector space $\mathfrak{v}$ which is endowed with the inner product $(\cdot, \cdot)$ provided by Lemma 2.1, with respect to which the generators K_i with $i = 1, \dots, p+q-1$ are orthogonal.

We put

$$\varepsilon = K_1 \dots K_{p+q-1}$$

and

$$\vartheta = K_{p+1} \dots K_{p+q-1}.$$

If $\nu = q + p$, we have

$$\varepsilon^2 = (-1)^{(q-1)}(-1)^{(p+q-1)(p+q-2)/2} I_{\mathfrak{v}} = (-1)^{(\nu-1)\nu/2} I_{\mathfrak{v}} = \begin{cases} I_{\mathfrak{v}}, & \text{se } \nu \equiv 1 \pmod{4}, \\ -I_{\mathfrak{v}}, & \text{se } \nu \equiv 3 \pmod{4}, \end{cases}$$

which implies that

$$(2.9) \quad {}^t\varepsilon = \begin{cases} \varepsilon, & \text{for } \nu \equiv 1 \pmod{4}, \\ -\varepsilon, & \text{for } \nu \equiv 3 \pmod{4}. \end{cases}$$

Analogously we find

$$(2.10a) \quad \vartheta^2 = \begin{cases} I_{\mathfrak{v}}, & \text{for } q \equiv 0 \pmod{4}, \\ -I_{\mathfrak{v}}, & \text{for } q \equiv 2 \pmod{4}, \end{cases}$$

and

$$(2.10b) \quad {}^t\vartheta = \begin{cases} \vartheta, & \text{for } q \equiv 0 \pmod{4}, \\ -\vartheta, & \text{for } q \equiv 2 \pmod{4}. \end{cases}$$

We have

$$(2.11a) \quad K_i \varepsilon = (-1)^{(p+q-2)} \varepsilon K_i = -\varepsilon K_i,$$

which implies that

$$(2.11b) \quad \varepsilon \vartheta = (-1)^{(p+q-2)(q-1)} \vartheta \varepsilon = -\vartheta \varepsilon.$$

From the discussion made for (i) it follows that

$$(2.11c) \quad {}^t K_i \vartheta + \vartheta K_i = 0.$$

For $(p \equiv 1 \pmod{4}, q \equiv 0 \pmod{4})$ we put

$$\begin{aligned} J_i &= \tau_0 \otimes K_i & \text{for } i = 1, \dots, p, \\ J_i &= \tau_1 \otimes K_i & \text{for } i = p+1, \dots, p+q-1, \end{aligned}$$

and

$$J_{p+q} = \tau_2 \otimes \vartheta.$$

Then $\{J_1, \dots, J_{p+q}\}$ is a set of linear endomorphisms of the space $\mathfrak{v}'$ defined by (2.7), which generates a representation of $\mathcal{C}(p, q)$, i.e.

$$J_i J_j + J_j J_i = \varepsilon_i(p, q) \delta_{ij},$$

where $\eta_{ij}(p, q)$ has been defined at the end, of Section 0.

In fact, we have

$$J_i^2 = \tau_0 \otimes K_i^2 = \eta_{ii}(p, q-1) \tau_0 \otimes I_{\mathfrak{v}} \quad \text{for } i = 1, \dots, p+q-1,$$

and

$$J_i J_j + J_j J_i = \tau_1 \otimes (K_i K_j + K_j K_i) = 0 \quad \text{for } i \neq j.$$

Moreover, in force of (2.11c) we have

$$J_i J_{p+q} = \tau_0 \tau_2 \otimes K_i \vartheta = -\tau_2 \tau_0 \otimes \vartheta K_i = -J_{p+q} J_i \quad \text{for } i = 1, \dots, p,$$

and also

$$J_i J_{p+q} = \tau_1 \tau_2 \otimes K_i \vartheta = -\tau_2 \tau_1 \otimes \vartheta K_i = -J_{p+q} J_i \quad \text{for } i = p+1, \dots, p+q.$$

Finally from (2.10a) it follows that

$$J_{p+q}^2 = -\tau_0 \otimes J_{\mathfrak{v}}.$$

Now we endow $\mathfrak{v}'$ with the scalar product $(,)'$ which has been defined in (2.8). Then we define the following invertible endomorphism of $\mathfrak{v}'$

$$(2.12) \quad A = \tau_3 \otimes \varepsilon,$$

which, as follows from (2.9), is self-adjoint. We have,

$$(2.13) \quad {}^t J_i A + A J_i = 0$$

for all $i = 1, \dots, p+q$, where t denotes the trasposition with respect to $(,)'$. In fact, for $i = 1, \dots, p$ from (2.11a) it follows that

$${}^t J_i A = J_i A = \tau_0 \tau_3 \otimes K_i \varepsilon = -\tau_3 \tau_0 \otimes \varepsilon K_i = -A J_i,$$

for $i = p + 1, \dots, p + q - 1$ from (2.11a) again it follows that

$${}^t J_i A = -J_i A = -\tau_1 \tau_3 \otimes K_i \varepsilon = -\tau_3 \tau_1 \otimes \varepsilon K_i = -A J_i,$$

and finally (2.11b) implies that

$${}^t J_{p+q} A = -J_{p+q} A = -\tau_2 \tau_3 \otimes \vartheta \varepsilon = -\tau_3 \tau_2 \otimes \varepsilon \vartheta = -A J_{p+q}.$$

For $(p \equiv 3 \pmod{4}, q \equiv 0 \pmod{4})$ and $(p \equiv 1 \pmod{4}, q \equiv 2 \pmod{4})$, we take

$$J_i = \tau_0 \otimes K_i \quad \text{for } i = 1, \dots, p + q - 1,$$

$$J_{p+q} = \tau_1 \otimes \varepsilon,$$

and

$$A = \begin{cases} \tau_3 \otimes \vartheta & \text{for } (p \equiv 3 \pmod{4}, q \equiv 0 \pmod{4}), \\ \tau_2 \otimes \vartheta & \text{for } (p \equiv 1 \pmod{4}, q \equiv 2 \pmod{4}). \end{cases}$$

An argument analogous to that described above for $(p \equiv 1 \pmod{4}, q \equiv 0 \pmod{4})$, shows that the set of operators $\{J_1, \dots, J_{p+q}\}$ defines a representation of $\mathcal{C}(p, q)$ on $\mathfrak{v}$. Then, endowed $\mathfrak{v}$ with $(\cdot, \cdot)'$, we obtain

$${}^t J_i A + A J_i = 0$$

for all $i = 1, \dots, p + q$.

Subcase ii-2:

$$(p \equiv 3 \pmod{4}, q \equiv 2 \pmod{4})$$

To deal with this case we consider the Clifford algebra $\mathcal{C}(p - 1, q)$ and a representation on a real vector space $\mathfrak{v}$ which is endowed with an inner product $(\cdot, \cdot)$ with respect to which the generators L_i with $i = 1, \dots, p + q - 1$ are orthogonal.

We put

$$\varphi = L_1 \dots L_{p+q-1}$$

and

$$\varrho = L_1 \dots L_{p-1}.$$

Hence,

$$\varphi^2 = (-1)^q (-1)^{(p+q-1)(p+q-2)/2} I_{\mathfrak{v}} = I_{\mathfrak{v}},$$

and

$${}^t \varphi = \varphi.$$

Analogously we find

$$\varrho^2 = -I_{\mathfrak{v}},$$

and

$${}^t\varrho = -\varrho.$$

We also have

$$\varphi\varrho = (-1)^{(p+q-2)(p-1)}\varrho\varphi = \varrho\varphi,$$

and from the discussion of (i)

$${}^tL_i\varrho + \varrho L_i = 0.$$

We put

$$J_i = \tau_0 \otimes L_i \quad \text{for } i = 1, \dots, p-1,$$

$$J_p = \tau_2 \otimes \varrho,$$

and

$$J_i = \tau_1 \otimes L_{i-1} \quad \text{for } i = p+1, \dots, p+q.$$

Now an argument similar to those described above shows that $\{J_1, \dots, J_{p+q}\}$ is a set of linear endomorphisms of the space $\mathfrak{v}'$ defined by (2.7), which generates a representation of $\mathcal{C}(p, q)$.

Once $\mathfrak{v}'$ has been endowed with the scalar product $(,)'$ defined by (2.8), the operators J_i satisfy (2.13) with

$$A = \tau_3 \otimes \varphi.$$

We have therefore proved the following theorem:

THEOREM 2.1. – *For all (p, q) there exists at least an admissible $\mathcal{C}(p, q)$ -module.*

REMARK (2.2). – We give now a prescription to obtain H -type algebras of bigger dimension from those provided by Theorem 2.1.

a) Take a module of the form

$$\mathfrak{v}' = \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m,$$

where $\mathfrak{v}_1, \dots, \mathfrak{v}_m$ are admissible $\mathcal{C}(\beta, \langle, \rangle)$ -modules.

b) Endow $\mathfrak{v}'$ with the scalar product given by the *signed* orthogonal sum of the scalar products $\langle, \rangle_i$ of $\mathfrak{v}_i$ which satisfies (1.5),

$$(2.14) \quad \langle, \rangle' = (\pm \langle, \rangle_1) \oplus \dots \oplus (\pm \langle, \rangle_m).$$

c) Define

$$J' = J_1 \oplus \dots \oplus J_m,$$

where $J_i: \mathcal{C}(\beta, \langle, \rangle) \rightarrow \text{End}(\mathfrak{v}_i)$ is the representation associated to the module $\mathfrak{v}_i$.

d) Then $u' = v' \oplus z$ with the bracket defined as in the proof of Lemma 1.1 and the scalar product given by the orthogonal sum of $\langle, \rangle_{\mathfrak{z}}$ and $\langle, \rangle'$ is a pseudo- H -type Lie algebra with center $\mathfrak{z}$.

REMARK (2.3). – Let $\mathfrak{n}$ and $\mathfrak{v}$ be as in Definition 1.1. Our aim now will be to determine the signature of $\langle, \rangle_{\mathfrak{v}}$.

We distinguish two cases: (i) when $p = 0$, (ii) when $p > 0$.

(i) Here in the construction described in the proof of Theorem 2.1 we can take $A = \pm I_{\mathfrak{v}}$, since

$${}^t J_i = -J_i$$

for all $i = 1, \dots, q$.

In particular the *classical* case corresponds to $p = 0$ with the choice $A = -I_{\mathfrak{v}}$, (see A. Kaplan [Ka]).

We observe also that if $p = 0$ the construction described in Remark 2.1 can give mixed signature on $\mathfrak{v}$.

(ii) Let $Z_1 \in \mathfrak{z}$ be such that

$$\langle Z_1, Z_1 \rangle = 1,$$

and set $J_1 = J_{Z_1}$. From (1.2) it follows for all $X \in \mathfrak{v}$

$$(2.15) \quad \langle J_1 X, J_1 X \rangle = -\langle X, X \rangle.$$

There always exists a vector $X_1 \in \mathfrak{v}$, such that

$$\langle X_1, X_1 \rangle = \pm 1,$$

since $\langle, \rangle_{\mathfrak{v}}$ is non-degenerate. From (1.1) it follows that

$$\langle J_1 X, X \rangle = 0,$$

and from (2.15) it follows that

$$\langle J_1 X_1, J_1 X_1 \rangle = \mp 1.$$

Thus the restriction of $\langle, \rangle$ to $\mathfrak{v}_1 = \text{span}_{\mathbb{R}}\{X_1, J_1 X_1\}$ is non-degenerate and neutral. Hence by (0.1) $\mathfrak{u} = \mathfrak{v}_1 \oplus \mathfrak{v}_1^\perp$. Furthermore the restriction of $\langle, \rangle$ to $\mathfrak{v}_1^\perp$ is also non-degenerate, so there exists $X_2 \in \mathfrak{v}_1^\perp$ such that

$$\langle X_2, X_2 \rangle = \pm 1,$$

and the process can be started over again.

We have so proved the following proposition:

PROPOSITION 2.2. – Let $\mathfrak{n}$ be a (p, q) -algebra. If $p \geq 1$, the restriction of $\langle, \rangle$ to $\mathfrak{v}$ is neutral.

3. – Admissibility of irreducible $\mathcal{C}(p, q)$ -modules.

We begin this section recalling that the Clifford algebras $\mathcal{C}(p, q)$ are isomorphic, as associative algebras with unit, to the matrix algebras listed below,

$$p - q \equiv 0, 6 \pmod{8} \quad \mathcal{C}(p, q) \cong \mathbb{R}(2^r) \quad \text{with } r = \frac{p+q}{2},$$

$$p - q \equiv 2, 4 \pmod{8} \quad \mathcal{C}(p, q) \cong \mathbb{H}(2^r) \quad \text{with } r = \frac{p+q}{2} - 1,$$

$$p - q \equiv 1, 5 \pmod{8} \quad \mathcal{C}(p, q) \cong \mathbb{C}(2^r) \quad \text{with } r = \frac{p+q}{2} - \frac{1}{2},$$

$$p - q \equiv 3 \pmod{8} \quad \mathcal{C}(p, q) \cong \mathbb{H}(2^r) \oplus \mathbb{H}(2^r) \quad \text{with } r = \frac{p+q}{2} - \frac{3}{2},$$

$$p - q \equiv 7 \pmod{8} \quad \mathcal{C}(p, q) \cong \mathbb{R}(2^r) \oplus \mathbb{R}(2^r) \quad \text{with } r = \frac{p+q}{2} - \frac{1}{2}.$$

In the first three cases $\mathcal{C}(p, q)$ has up to equivalence only one irreducible module v , of real dimension 2^r , 2^{r+2} and 2^{r+1} respectively. The dimension is related to (p, q) by the fact that $\mathcal{C}(p, q)$ has dimension 2^{p+q} as real vector space. In the last two cases $\mathcal{C}(p, q)$ has two irreducible inequivalent modules v_1, v_2 of dimension 2^r when $p - q \equiv 7 \pmod{8}$ and 2^{r+2} when $p - q \equiv 3 \pmod{8}$. The two modules are distinguished by the action of $\varepsilon_{(p,q)} = J_1 \dots J_{p+q}$, i.e. the action of $\varepsilon_{(p,q)}$ is equal to the identity on v_1 and it is equal to minus the identity on v_2 .

LEMMA 3.1. – *Let v be an irreducible $\mathcal{C}(p, q)$ -module and J the associated representation. Assume that J is faithful. Let $A \in \text{End}(v)$ belong to $J(\mathcal{C}(p, q))$ and satisfies (1.7), i.e.*

$$(3.1a) \quad J_i A = -A J_i \quad \text{for } i = 1, \dots, p$$

and

$$(3.1b) \quad J_i A = A J_i \quad \text{for } i = p+1, \dots, p+q.$$

Then

$$A = \begin{cases} \xi P, & \text{for } p \text{ even and } q \text{ even,} \\ \zeta Q, & \text{for } p \text{ odd and } q \text{ odd,} \\ \xi P + \zeta Q, & \text{for } p \text{ even and } q \text{ odd,} \\ 0, & \text{for } p \text{ odd and } q \text{ even,} \end{cases}$$

where P and Q are defined respectively by (2.3a) and (2.3b) and $\xi, \zeta \in \mathbb{R}$.

PROOF. – Let $J: \mathcal{C}(p, q) \rightarrow \text{End}(v)$ be an irreducible representation. Since $A \in J((\mathcal{C}(p, q)))$ there exists at least an element γ of $\mathcal{C}(p, q)$, such that $A = J_\gamma$. Therefore, denoting by $\{Z_1, \dots, Z_{p+q}\}$ a set of Clifford generators of $\mathcal{C}(p, q)$, we have

$$(3.3) \quad A = \alpha^{(0)} I_v + \sum_i \alpha_i^{(1)} J_i + \sum_{i < j} \alpha_{ij}^{(2)} J_i J_j + \dots + \alpha^{(p+q)} J_1 \dots J_{p+q},$$

with $\alpha^{(0)}, \alpha_i^{(1)}, \dots, \alpha^{(p+q)} \in \mathbb{R}$.

Assume $p \neq 0$ and $q \neq 0$. By (3.1a) $\alpha^{(0)}$ can be different from zero only if $p = 0$.

(i) Let r be an even positive integer.

Suppose first $r < p$. Then for all sets $\{i_1, \dots, i_r\}$ of indexes with $1 \leq i_1 < \dots < i_r \leq p + q$, there exists an index l with $1 \leq l \leq p$ which does not belong to $\{i_1, \dots, i_r\}$, so that

$$J_l J_{i_1} \dots J_{i_r} = J_{i_1} \dots J_{i_r} J_l.$$

It follows that (3.1a) is satisfied only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0$$

for all r even with $0 \leq r < p$.

Suppose now $r > p$. Then for all sets $\{i_1, \dots, i_r\}$ of indexes with $1 \leq i_1 < \dots < i_r \leq p + q$, there exists an index l with $p + 1 \leq l \leq p + q$ such that $l \in \{i_1, \dots, i_r\}$. Hence,

$$J_l J_{i_1} \dots J_{i_r} = -J_{i_1} \dots J_{i_r} J_l.$$

Therefore (3.1b) is satisfied only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0$$

for all r even with $p < r \leq p + q$.

When p is even, there is also the case $r = p$. If at least one of the indexes $\{i_1, \dots, i_p\}$, say l , is strictly greater than p ($p + 1 \leq l \leq p + q$), we have

$$J_l J_{i_1} \dots J_{i_p} = -J_{i_1} \dots J_{i_p} J_l.$$

Therefore (3.1b) is satisfied only if

$$\alpha_{i_1, \dots, i_p}^{(p)} = 0$$

for all $\{i_1, \dots, i_p\}$ with $i_p \geq p + 1$.

Thus when p is even and not zero, only the coefficient of P in the set $\{\alpha_{i_1, \dots, i_p}^{(p)}\}$ can be different from zero.

(ii) Let now r be an odd positive integer.

Suppose first $r < q$. Then for all sets $\{i_1, \dots, i_r\}$ of indexes there exists an index l with $p + 1 \leq l \leq p + q$ such that $l \notin \{i_1, \dots, i_r\}$. Hence,

$$J_l J_{i_1} \dots J_{i_r} = -J_{i_1} \dots J_{i_r} J_l.$$

It follows that (3.1b) is satisfied only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0$$

for all odd integers r with $0 \leq r < q$.

Now suppose $r > q$. Then for all sets $\{i_1, \dots, i_r\}$ there exists an index l with $1 \leq l \leq p$ such that $l \in \{i_1, \dots, i_r\}$. Hence,

$$J_l J_{i_1} \dots J_{i_r} = J_{i_1} \dots J_{i_r} J_l.$$

It follows that (3.1a) is satisfied only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0$$

for all odd integers r with $q < r \leq p + q$.

When q is odd, we have also the case $r = q$. If there exists l which belongs to $\{i_1, \dots, i_q\}$ such that, $1 \leq l \leq p$, it follows

$$J_l J_{i_1} \dots J_{i_p} = J_{i_1} \dots J_{i_p} J_l.$$

Thus (3.1a) is satisfied only if

$$\alpha_{i_1, \dots, i_q}^{(q)} = 0,$$

for all $\{i_1, \dots, i_q\}$ with $i_1 \leq p$. Therefore when q is odd, only the coefficient of Q in the set $|\alpha_{i_1, \dots, i_q}^{(q)}|$ can be different from zero.

(iii) Assume now $p \neq 0$ and $q = 0$. The argument developed in (i), holding also in the present case, implies that all terms with an even number of factors in the left hand side of (3.3) must have coefficient zero, except for P when p is even. Let r be an odd integer such that $1 \leq r \leq p$ and let $l \in \{i_1, \dots, i_r\}$ then

$$J_l J_{i_1} \dots J_{i_r} = J_{i_1} \dots J_{i_r} J_l.$$

Therefore (3.1a) is satisfied if and only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0,$$

for all $\{i_1, \dots, i_r\}$ with $0 \leq r \leq p$, when p is odd, and with $1 \leq r < p$, when p is even.

(iv) Assume now $p = 0$ and $q \neq 0$. In this case $\alpha^{(0)}$ may be different from zero and A may contain a term proportional to I_b . The argument developed in (ii), holding also here, implies that all terms with an odd number of factors must disappear from the left hand side of (3.3), except for Q when q is odd. Moreover if r is an even integer such that $1 \leq r \leq q$ and $l \in \{i_1, \dots, i_r\}$ then

$$J_l J_{i_1} \dots J_{i_r} = -J_{i_1} \dots J_{i_r} J_l.$$

Therefore (3.1b) is satisfied if and only if

$$\alpha_{i_1, \dots, i_r}^{(r)} = 0,$$

for all $\{i_1, \dots, i_r\}$ with $1 \leq r \leq q$ when q is even, and with $1 \leq r < q$ when q is odd. ■

Let (k, l) be a pair of non-negative integers, assume that $\mathcal{C}(k+4, l)$ is isomorphic to a matrix subalgebra $\mathcal{A}$ of $R(2^r)$. Consequently the irreducible $\mathcal{C}(k+4, l)$ -module is $\mathfrak{v} = R^{2^r}$. Let $J': \mathcal{C}(p, q) \rightarrow \text{End}(\mathfrak{v})$ be the associated representation.

Let $(Z_1, \dots, Z_{k+l+4})$ be a set of Clifford generators of $\mathcal{C}(k+4, l)$. We recall that by Lemma 2.1 there exists an inner product $(,)'$ with respect to which $(J'_1, \dots, J'_{k+l+4}) \subset R(2^r)$ are orthogonal, i.e. satisfy (2.1). Let $S \in GL(2^r, R)$ be the matrix associated to the conjugation relating $(,)'$ to the canonical inner product of R^{2^r}

$$(x, y) = \sum_{i=1}^{2^r} x_i y_i.$$

Then the matrix,

$$J_i = S J'_i S^{-1},$$

which generates the conjugate representation, are orthogonal with respect to $(,)$. From now on we will assume that the representations are generated by matrixes $(J_1, \dots, J_{k+l+4})$ which are orthogonal with respect to $(,)$. Then the transposition t with respect to $(,)$ corresponds to the matrix transposition T for the rest of the section.

It is easy to see that the matrixes

$$\widehat{\gamma}_j = J_j, \quad \text{for } j = 1, \dots, k \text{ and } j = k+5, \dots, k+l+4,$$

and

$$\begin{aligned} \widehat{\gamma}_{k+1} &= J_{k+2} J_{k+3} J_{k+4}, & \widehat{\gamma}_{k+2} &= J_{k+3} J_{k+4} J_{k+1}, \\ \widehat{\gamma}_{k+3} &= J_{k+4} J_{k+1} J_{k+2}, & \widehat{\gamma}_{k+4} &= J_{k+1} J_{k+2} J_{k+3}, \end{aligned}$$

satisfy the fundamental relations of a set of Clifford generators of $\mathcal{C}(k, l+4)$, i.e.

$$\widehat{\gamma}_i \widehat{\gamma}_j + \widehat{\gamma}_j \widehat{\gamma}_i = \varepsilon(k, l+4) \delta_{ij},$$

and that they are still orthogonal with respect to $(,)$. Therefore

$$\widehat{J}: \mathcal{C}(k, l+4) \rightarrow \mathcal{A}, \quad \widehat{J}: \widehat{Z}_i \mapsto \widehat{\gamma}_i,$$

where $(\widehat{Z}_1, \dots, \widehat{Z}_{k+l+4})$ is a set of Clifford generators of $\mathcal{C}(k, l+4)$, is a representation.

We omit the proof of the following lemma, since it is trivial.

LEMMA 3.2. – Let $A \in R(2^r)$, we put

$$\widehat{A} = J_{k+1} J_{k+2} J_{k+3} J_{k+4} A.$$

Then:

$${}^T J_i A = -A J_i \text{ if and only if } {}^T \widehat{\gamma}_i \widehat{A} = -\widehat{A} \widehat{\gamma}_i,$$

$${}^T A = \pm A \text{ if and only if } {}^T A = \pm A.$$

If $\mathfrak{J} \in \mathbb{R}(2^r)$ commutes with $(J_1, \dots, J_{k+l+4})$, then $\mathfrak{J}A = \pm A\mathfrak{J}$ if and only if $\mathfrak{J}A = \pm A\mathfrak{J}$.

In the following five lemmas we will study the admissibility of the irreducible $\mathcal{C}(p, q)$ -modules distinguishing among the five cases indicated at the beginning of this section.

We begin by considering the case of Clifford algebras which are isomorphic to real matrix algebras. For future reference we fix some notation on the algebras $\mathcal{C}(r, r) \cong \mathbb{R}(2^r)$ with $r = 0, 1, \dots$, and $\mathcal{C}(r+1, r-1) \pm \mathbb{R}(2^r)$ with $r = 1, 2, \dots$. Let $(\eta_1, \dots, \eta_{2r}) \subset \mathbb{R}(2^r)$ be a set of matrixes which satisfy the fundamental relations of the Clifford algebra $\mathcal{C}(r, r)$, we will say that $(\eta_1, \dots, \eta_{2r})$ is a *set of Clifford generators for a representation of $\mathcal{C}(r, r)$* . We may assume that the matrixes $(\eta_1, \dots, \eta_{2r})$ are orthogonal with respect to the canonical inner product. Set

$$P_\eta = \eta_1 \dots \eta_r, \quad Q_\eta = \eta_{r+1} \dots \eta_{2r}$$

and

$$\varepsilon_\eta = \eta_1 \dots \eta_{2r}.$$

Then

$$(3.4) \quad \varepsilon_\eta^2 = (-1)^r (-1)^{2r(2r-1)/2} I_\eta = I_\eta \Rightarrow {}^T \varepsilon_\eta = \varepsilon_\eta,$$

$$(3.5) \quad \eta_i \varepsilon_\eta = -\varepsilon_\eta \eta_i, \quad i = 1, \dots, 2r,$$

$$(3.6a) \quad {}^T P_\eta = (-1)^{r(r-1)/2} P_\eta = \begin{cases} P_\eta, & \text{for } r \equiv 0, 1 \pmod{4}, \\ -P_\eta, & \text{for } r \equiv 2, 3 \pmod{4}, \end{cases}$$

$$(3.6b) \quad {}^T Q_\eta = (-1)^r (-1)^{r(r-1)/2} Q_\eta = \begin{cases} Q_\eta, & \text{for } r \equiv 0, 3 \pmod{4}, \\ -Q_\eta, & \text{for } r \equiv 1, 2 \pmod{4}, \end{cases}$$

$$(3.7) \quad {}^T \eta_i P_\eta = (-1)^{r+1} P_\eta \eta_i, \quad {}^T \eta_i Q_\eta = (-1)^r Q_\eta \eta_i.$$

Put now

$$(3.8) \quad \begin{cases} \bar{\eta}_i = \eta_i & \text{if } i = 1, \dots, r, \\ \bar{\eta}_{r+1} = \varepsilon_\eta, \\ \bar{\eta}_i = \eta_{i-1} & \text{if } i = r+2, \dots, 2r. \end{cases}$$

Then $(\bar{\eta}_1, \dots, \bar{\eta}_{2r})$ is a set of matrixes which satisfy (2.1) and the fundamental relations of the Clifford algebra $\mathcal{C}(r+1, r-1)$.

Set

$$\bar{P}_\eta = \bar{\eta}_1 \dots \bar{\eta}_{r+1}, \quad \bar{Q}_\eta = \bar{\eta}_{r+2} \dots \bar{\eta}_{2r}$$

and

$$\bar{\varepsilon}_\eta = \bar{\eta}_1 \dots \bar{\eta}_{2r}.$$

Then

$$(3.9a) \quad \bar{\varepsilon}_\eta^2 = (-1)^{r-1} (-1)^{2r(2r-1)/2} I_\eta = -I_\eta,$$

$$(3.9b) \quad \bar{\eta}_i \bar{\varepsilon}_\eta = -\bar{\varepsilon}_\eta \bar{\eta}_i, \quad i = 1, \dots, 2r,$$

$$(3.10a) \quad {}^T \bar{P}_\eta = (-1)^{r(r-1)/2} \bar{P}_\eta = \begin{cases} \bar{P}_\eta, & \text{if } r \equiv 0, 3 \pmod{4}, \\ -\bar{P}_\eta, & \text{if } r \equiv 1, 2 \pmod{4}, \end{cases}$$

$$(3.10b) \quad {}^T \bar{Q}_\eta = (-1)^{r-1} (-1)^{(r-1)(r-2)/2} \bar{Q}_\eta = \begin{cases} \bar{Q}_\eta, & \text{if } r \equiv 0, 1 \pmod{4}, \\ -\bar{Q}_\eta, & \text{if } r \equiv 2, 3 \pmod{4}, \end{cases}$$

$$(3.11) \quad {}^T \bar{\eta}_i \bar{P}_\eta = (-1)^r \bar{P}_\eta \bar{\eta}_i, \quad {}^T \bar{\eta}_i \bar{Q}_\eta = (-1)^r \bar{Q}_\eta \bar{\eta}_i.$$

LEMMA 3.3. – Assume that $\mathcal{C}(p, q)$ is isomorphic to some full real matrix algebra, i.e. $\mathcal{C}(p, q) \cong \mathbb{R}(2^r)$ for some non-negative integer r . Then the real irreducible $\mathcal{C}(p, q)$ -module is admissible if and only if $r \equiv 0 \pmod{4}$ or $r \equiv 3 \pmod{4}$. Indeed we can take

$$A = \begin{cases} \xi P, & \text{for } r \equiv 0 \pmod{4}, \\ \zeta Q, & \text{for } r \equiv 3 \pmod{4}, \end{cases}$$

with $\xi, \zeta \in \mathbb{R}$.

PROOF. – We will first consider a realization of the algebra $\mathcal{C}(r, r) \cong \mathbb{R}(2^r)$ with $r = 0, 1, \dots$ and a realization of the algebra $\mathcal{C}(r+1, r-1) \cong \mathbb{R}(2^r)$ with $r = 1, 2, \dots$. Then because of Lemma 3.2, the property will be true for the entire families of algebras $\mathcal{C}(r+4k, r-4k)$ with $k = 0, \pm 1, \dots, \pm[r/4]$ and $\mathcal{C}(r+1-4k, r-1+4k)$ with $k = -[r+1/4], \dots, [r+1/4]$ (see [DSS]), which exhaust the families of Clifford algebras isomorphic to some full algebra of real matrixes.

Since $\mathcal{C}(r, r)$ and $\mathcal{C}(r+1, r-1)$ are isomorphic to a full algebra $\mathbb{R}(2^k)$ of real matrixes, Lemma 3.1 implies that if $A \in \mathbb{R}(2^k)$ satisfies (3.1a) and (3.1b), then A must be given by (3.2). Hence, (2.5a) and (2.5b) imply that

$${}^T A = \begin{cases} A, & \text{for } r \equiv 0, 3 \pmod{4}, \\ -A & \text{for } r \equiv 1, 2 \pmod{4}, \end{cases}$$

from which the lemma follows. ■

LEMMA 3.4. – *The Clifford algebras which are isomorphic to $R(2^r) \oplus R(2^r)$ for some non-negative integer r have irreducible Clifford modules which are admissible if and only if $r \equiv 3 \pmod{4}$.*

PROOF. – By Lemma 3.2 it is enough to prove the statement for the algebras $C(r+1, r) \cong R(2^r) \oplus R(2^r)$ for $r = 0, 1, \dots$, since if $C(p, q) \cong R(2^r) \oplus R(2^r)$, then there exists an integer k such that $p = r+1+4k$ and $q = r-4k$.

We recall that $C(r+1, r)$ has two real irreducible modules v_1 and v_2 . Let now $v = R^{2^r}$ be the irreducible module of $C(r, r) \cong R(2^r)$. As vector spaces both v_1 and v_2 are isomorphic to v . The action of

$$\varepsilon_J = J_1 \dots J_{p+q},$$

distinguishes between them, i.e.

$$\varepsilon_J = I_{2^r}$$

on v_1 , and

$$\varepsilon_J = -I_{2^r}$$

on v_2 .

First of all we prove that if r is even the irreducible Clifford modules are non-admissible. Let $(Z_1, \dots, Z_{2r+1})$ be a set of Clifford generators of $C(r+1, r)$, and let $A \in R(2^r)$. Suppose that

$${}^t J_i A = -A J_i \quad \text{for } i = 1, \dots, 2r+1.$$

Then

$$(3.12) \quad J_i \varepsilon_J = \varepsilon_J J_i, \quad \text{for } i = 1, \dots, 2r+1,$$

and

$$(3.13) \quad \varepsilon_J^2 = (-1)^r (-1)^{r(2r+1)} I_{2^r} = I_{2^r}.$$

It follows from (3.12) and (3.13) that the carrier space of the representation decomposes into the sum of two eigenspaces associated to the eigenvalues ± 1 , that is

$$\varepsilon_J = \pm I_J.$$

On the other hand if ${}^t J_i A = -A J_i$ for $i = 1, \dots, r+1$, then we have

$$\varepsilon_J A = J_1 \dots J_{2r+1} A = (-1)^{r+1} A J_1 \dots J_{2r+1} = (-1)^{r+1} A \varepsilon_J.$$

Therefore r even implies $A = 0$.

We will show that if r is odd the scalar product which makes the irreducible $C(r, r)$ -modules admissible, makes them also admissible as $C(r+1, r)$ -modules. Since for r odd $C(r, r)$ is admissible only if $r \equiv 3 \pmod{4}$, it will follow that a $C(r+1, r)$ -module is

admissible if and only if $r \equiv 3 \pmod{4}$. Set

$$\begin{cases} \widehat{J}_i = \eta_i & \text{if } i = 1, \dots, r, \\ \widehat{J}_{r+1} = \varepsilon_\eta, \\ \widehat{J}_i = \eta_{i-1} & \text{if } i = r+3, \dots, 2r+1, \end{cases}$$

where $(\eta_1, \dots, \eta_{2r})$ have been introduced in (3.4). Then $(\widehat{J}_1, \dots, \widehat{J}_{2r+1}) \subset \mathbb{R}(2^r)$ is a set of matrixes satisfying (2.1) and the fundamental relations of $\mathcal{C}(r+1, r)$. Therefore

$$\widehat{J}: \widehat{Z}_i \mapsto \widehat{J}_i$$

defines a representation of $\mathcal{C}(r+1, r)$. Let $A \in \mathbb{R}(2^r)$. Suppose that ${}^T \widehat{J}_i A = -A \widehat{J}_i$ for $i = 1, \dots, 2r$. Then automatically we obtain

$${}^T \widehat{J}_{r+1} A = \varepsilon_\eta A = (-1)^r A \varepsilon_\eta = (-1)^r A \widehat{J}_{r+1}.$$

This concludes the proof of the lemma. ■

Next we consider the family of Clifford algebras which are isomorphic to a full algebra of complex matrixes.

LEMMA 3.5. – *The Clifford algebras which are isomorphic to $\mathcal{C}(2^r)$ for some $r \in \mathbb{N}$ have irreducible Clifford modules which are admissible if and only if $r \equiv 0, 2, 3 \pmod{4}$. The operator $A \in \mathbb{R}(2^{r+1})$ which is associated to the scalar product making the $\mathcal{C}$ -module admissible is $\mathfrak{J}$ -complex if $r \equiv 0, 2 \pmod{4}$ and anti- $\mathfrak{J}$ -complex if $r \equiv 3 \pmod{4}$.*

PROOF. – We will consider the algebras $\mathcal{C}(r, r+1) \cong \mathcal{C}(2^r)$ with $r = 0, 1, \dots$, and $\mathcal{C}(r+2, r-1) \cong \mathcal{C}(2^r)$ with $r = 1, 2, \dots$, and we will prove that their real irreducible modules are admissible. Then Lemma 3.2 will imply that the same also holds for all the Clifford algebras which are isomorphic to $\mathcal{C}(2^r)$, since for any $\mathcal{C}(p, q) \cong \mathcal{C}(2^r)$ there exists a non-negative integer k such that $\mathcal{C}(p, q) \cong \mathcal{C}(r+4k, r+1-4k)$ or that $\mathcal{C}(p, q) \cong \mathcal{C}(r+2+4k, r-1-4k)$.

Let

$$\mathfrak{J} = \tau_2 \otimes I_{2^r},$$

then

$$\mathfrak{J}^2 = -\tau_0 \otimes I_{2^r}.$$

Thus $\mathfrak{J}$ defines a complex structure on $\mathfrak{v} = \mathbb{R}^{2^{r+1}}$ which is the irreducible Clifford module of $\mathcal{C}(r, r+1)$ and $\mathcal{C}(r+2, r-1)$.

Let $A \in \mathbb{R}(2^{r+1})$, we already know that $A = A_0 + A_1$, where A_0 is $\mathfrak{J}$ -complex and A_1 is anti- $\mathfrak{J}$ -complex (see (0.4)). Moreover we can decompose A into

$$A = \sum_{\alpha=0}^3 \tau_\alpha \otimes \Xi_\alpha,$$

with $\Xi_\alpha \in \mathcal{R}(2^r)$. We notice that $\mathfrak{J}A_0 = A_0\mathfrak{J}$ if and only if

$$(3.14a) \quad A_0 = \tau_0 \otimes \Xi_0 + \tau_2 \otimes \Xi_2,$$

and that $\mathfrak{J}A_1 = -A_1\mathfrak{J}$ if and only if

$$(3.14b) \quad A_1 = \tau_1 \otimes \Xi_1 + \tau_3 \otimes \Xi_3.$$

We set

$$\gamma_i = \tau_0 \otimes \eta_i \quad \text{for } i = 1, \dots, 2r$$

and

$$\gamma_{2r+1} = \tau_2 \otimes \varepsilon_\eta.$$

It follows then from (3.4) and (3.5) that $(\gamma_1, \dots, \gamma_{2r+1})$ satisfy the fundamental relations of the Clifford algebra $\mathcal{C}(r, r+1)$, are orthogonal matrixes and commute with the complex structure. Therefore, if $(Z_1, \dots, Z_{2r+1})$ denotes as usual a set of Clifford generators of $\mathcal{C}(r, r+1)$,

$$J: Z_i \mapsto \gamma_i \quad \text{for } i = 1, \dots, 2r+1,$$

define a representation of $\mathcal{C}(r, r+1)$.

Note that A satisfies (3.1) if and only if A_0 and A_1 separately satisfy (3.1). Assume that $A_0 \in \mathcal{R}(2^{r+1})$ is $\mathfrak{J}$ -complex, so that it takes the form given by (3.14a). Then ${}^T J_i A_0 = -A_0 J_i$ for $i = 1, \dots, 2r$ if and only if

$$(3.15) \quad {}^T \eta_i \Xi_0 = -\Xi_0 \eta_i \quad \text{and} \quad {}^T \eta_i \Xi_2 = -\Xi_2 \eta_i$$

for $i = 1, \dots, 2r$. Hence,

$${}^T \varepsilon_\eta \Xi_0 = \varepsilon_\eta \Xi_0 = (-1)^r \Xi_0 \varepsilon_\eta,$$

and

$${}^T \varepsilon_\eta \Xi_2 = (-1)^r \Xi_2 \varepsilon_\eta,$$

so that

$${}^T J_{2r+1} A_0 = -A_0 J_{2r+1} \quad \text{and} \quad {}^T J_{2r+1} A_2 = -A_2 = -A_2 J_{2r+1},$$

if and only if r is even. In this case, since $\mathcal{C}(r, r) \cong \mathcal{R}(2^r)$, using Lemma 3.1, (3.15) implies that Ξ_0 and Ξ_2 must be proportionanl to P_η . Moreover, from (3.6a) it follows that ${}^T A_0 = A_0$ if and only if

$$A_0 = \zeta \tau_0 \otimes P_\eta = \zeta P_\gamma \quad \text{if } r \equiv 0 \pmod{4},$$

$$A_0 = \zeta \tau_2 \otimes P_\eta = \zeta \mathfrak{J} P_\gamma \quad \text{if } r \equiv 2 \pmod{4},$$

with $\zeta \in \mathcal{R}$.

If $A_1 \mathfrak{J} = -\mathfrak{J} A_1$, then ${}^T \gamma_i A_1 = -A_1 \gamma_i$ for $i = 1, \dots, 2r$ if and only if

$$(3.16) \quad {}^T \eta_i \Xi_1 = -\Xi_1 \eta_i \quad \text{and} \quad {}^T \eta_i \Xi_3 = -\Xi_3 \eta_i$$

for $i = 1, \dots, 2r$. Hence,

$${}^T \varepsilon_\eta \Xi_1 = (-1)^r \Xi_1 \varepsilon_\eta \quad \text{and} \quad {}^T \varepsilon_\eta \Xi_3 = (-1)^r \Xi_3 \varepsilon_\eta,$$

so that

$$\begin{aligned} {}^T \gamma_{2r+1} A_1 &= -(\tau_2 \otimes \varepsilon_\eta)(\tau_1 \otimes \Xi_1 + \tau_3 \otimes \Xi_3) = (\tau_1 \tau_2) \otimes (\varepsilon_\eta \Xi_1) + (\tau_3 \tau_2) \otimes (\varepsilon_\eta \Xi_3) = \\ &= -\mathcal{C}_1 \gamma_{2r+1} = -(\tau_1 \tau_2) \otimes (\Xi_1 \varepsilon_\eta) - (\tau_3 \tau_2) \otimes (\Xi_3 \varepsilon_\eta) = -A_1 \gamma_{2r+1}, \end{aligned}$$

if and only if r is odd. In this case, using Lemma 3.1, (3.16) implies that Ξ_1 and Ξ_3 must be proportional to Q_η . Moreover, from (3.6b) it follows that ${}^T A_1 = A_1$ if and only if $r \equiv 3 \pmod{4}$. In this case

$$A_1 = \xi \tau_1 \otimes Q_\eta + \zeta \tau_3 \otimes Q_\eta,$$

with $\xi, \zeta \in \mathbb{R}$. While if $r \equiv 1 \pmod{4}$ the real irreducible $\mathcal{C}(r, r+1)$ -module is non-admissible.

Set now

$$\begin{aligned} \bar{\gamma}_i &= \tau_0 \otimes \bar{\eta}_i \quad \text{for } i = 1, \dots, r, \\ \bar{\gamma}_{2r+1} &= \tau_2 \otimes \bar{\varepsilon}_\eta, \end{aligned}$$

and

$$\bar{\gamma}_i = \tau_0 \otimes \bar{\eta}_{i-1} \quad \text{for } i = r+2, \dots, 2r+1,$$

where the matrix $\bar{\eta}_i$ have been introduced in (3.8).

It follows then from (3.9a) and (3.9b) that the matrixes $(\bar{\gamma}_1, \dots, \bar{\gamma}_{2r+1})$ satisfy the fundamental relations of the Clifford algebra $\mathcal{C}(r+2, r-1)$, are orthogonal, and commute with the complex structure. Therefore, denoting by $(\bar{Z}_1, \dots, \bar{Z}_{2r+1})$ a set of Clifford generators of $\mathcal{C}(r, r+1)$,

$$\bar{J}: \bar{Z}_i \mapsto \bar{\gamma}_i \quad \text{for } i = 1, \dots, 2r+1$$

defines a representation of $\mathcal{C}(r, r+1)$. Assume that $A_0 \in \mathbb{R}(2^{r+1})$ is complex, so that it takes the form given by (3.14a). Then ${}^T \bar{J}_i A_0 = -A_0 \bar{J}_i$ for $i = 1, \dots, 2r$ if and only if Ξ_0 and Ξ_2 satisfy (3.15). Hence

$${}^T \bar{\varepsilon}_\eta \Xi_0 = -(-1)^{r+2} \Xi_0 \bar{\varepsilon}_\eta,$$

so that

$${}^T \bar{J}_{r+1} A_0 = -A_0 \bar{J}_{r+1}$$

if and only if r is even. In this case, since $\mathcal{C}(r+1, r-1) \cong \mathbb{R}(2^r)$, using Lemma 3.1, (3.15) implies that Ξ_0 and Ξ_2 must be proportional to $\bar{Q}_\eta$. Moreover, from (3.10a) it follows that ${}^T A_0 = A_0$ if and only if

$$A_0 = \zeta \tau_0 \otimes \bar{Q}_\eta = \zeta \bar{Q}_\eta \quad \text{if } r \equiv (0 \pmod{4}),$$

$$A_0 = \zeta \tau_2 \otimes \bar{Q}_\eta = \zeta \mathfrak{J} \bar{Q}_\eta \quad \text{if } r \equiv 2 \pmod{4},$$

with $\xi \in \mathbb{R}$. If A_1 is anti- $\mathfrak{J}$ -complex, then ${}^T\bar{\gamma}_i A_1 = -A_1 \bar{\gamma}_i$ for $i = 1, \dots, 2r$ if and only if (3.16) hold. Hence, ${}^T\bar{\gamma}_{2r+1} A_1 = -A_1 \bar{\gamma}_{2r+1}$ if and only if r is odd. In this case, using Lemma 3.1, (3.16) implies that Ξ_1 and Ξ_3 must be proportional to $\bar{P}_\eta$. Moreover, from (3.10b) it follows that ${}^T A_1 = A_1$ if and only if $r \equiv 3 \pmod{4}$. In this case

$$A_1 = \xi \tau_1 \otimes \bar{P}_\eta + \zeta \tau_3 \otimes \bar{P}_\eta,$$

with $\xi, \zeta \in \mathbb{R}$. While if $r \equiv 1 \pmod{4}$ the real irreducible $\mathcal{C}(r+2, r-1)$ -module is non-admissible. This concludes the proof. ■

From the proof of the above lemma it also follows that the scalar product which makes admissible for $r \not\equiv 1 \pmod{4}$ the family of irreducible $\mathcal{C}(r, r+1)$ -modules with $r = 0, 1, \dots$, is given by

$$A = \xi \tau_0 \otimes P_\eta = \xi P_\gamma \quad \text{if } r \equiv 0 \pmod{4},$$

$$A = \xi \tau_2 \otimes P_\eta = \xi \mathfrak{J} P_\gamma \quad \text{if } r \equiv 2 \pmod{4},$$

and

$$A = \xi \tau_1 \otimes Q_\eta + \zeta \tau_3 \otimes Q_\eta \quad \text{if } r \equiv 3 \pmod{4},$$

with $\xi, \zeta \in \mathbb{R}$.

Finally in the next two lemmas we will analyze the irreducible modules of Clifford algebras which are isomorphic to an algebra of quaternionic matrixes.

LEMMA 3.5. – *The Clifford algebras which are isomorphic to $H(2^r)$ with $r = 0, 1, \dots$ have real irreducible modules which are always admissible. The scalar products which make these modules admissible are associated to an endomorphism A of $\mathbb{R}^{2^{r+2}}$ which commutes with the quaternionic structure $(\mathfrak{J}, \mathfrak{K})$ of $\mathbb{R}(2^{r+2})$ if $r \equiv 0, 1 \pmod{4}$, and which anti-commutes with the quaternionic structure if $r \equiv 2, 3 \pmod{4}$.*

PROOF. – We will first consider the algebra $\mathcal{C}(r, r+2) \cong H(2^r)$ with $r = 0, 1, \dots$, and the algebra $\mathcal{C}(r-1, r+3) \cong H(2^{r+1})$ with $r = 1, 2, \dots$, and we will show that their real irreducible modules are admissible. Then Lemma 3.2 will imply that the same also holds for the other Clifford algebras isomorphic to $H(2^r)$.

To study the algebra $\mathcal{C}(r, r+2)$ we use the isomorphism $\mathcal{C}(r, r+2) \cong \mathcal{C}(0, 2) \otimes \mathcal{C}(r, r) \cong H \otimes \mathbb{R}(2^r)$.

Clearly

$$\sigma_1 = \tau_0 \otimes \tau_2, \quad \sigma_2 = \tau_2 \otimes \tau_1$$

satisfy the fundamental relations of $\mathcal{C}(0, 2)$. Let

$$\chi = \sigma_1 \sigma_2 = -\tau_2 \otimes \tau_3,$$

then

$$\chi^2 = -I_4,$$

where I_4 is the 4×4 identity matrix.

We put

$$\begin{aligned}\gamma_i &= \chi \otimes \eta_{i+r} & \text{if } i = 1, \dots, r, \\ \gamma_i &= \chi \otimes \eta_{i-r} & \text{if } i = r+1, \dots, 2r, \\ \gamma_{2r+i} &= \sigma_i \otimes I_\eta & \text{if } i = 1, 2.\end{aligned}$$

Then $(\gamma_1, \dots, \gamma_{2r+2}) \subset \mathbb{R}(2^{r+2})$ satisfy the fundamental relations of the Clifford algebra $\mathcal{C}(r, r+2)$ and (2.1) with respect to the canonical inner product of $\mathbb{R}^{2^{2r+2}}$. Let $(Z_1, \dots, Z_{2r+2})$ be a set of Clifford generators of $\mathcal{C}(r, r+2)$, then

$$J: Z_i \mapsto \gamma_i \quad \text{for } i = 1, \dots, 2r+2$$

is a representation.

Let

$$\mathfrak{J} = \tau_2 \otimes \tau_0 \otimes I_\eta \quad \text{and} \quad \mathfrak{K} = \tau_1 \otimes \tau_2 \otimes I_\eta,$$

then

$$\mathfrak{J}^2 = \mathfrak{K}^2 = -I \quad \text{and} \quad \mathfrak{J}\mathfrak{K} = -\mathfrak{K}\mathfrak{J},$$

thus $(\mathfrak{J}, \mathfrak{K})$ defines a quaternionic structure on $\mathfrak{v} = \mathbb{R}^{2^{r+2}}$ which is the real irreducible module of $\mathcal{C}(r, r+2)$. $\mathfrak{J}$ and $\mathfrak{K}$ commute with γ_i for all $i = 1, \dots, 2r+2$.

If $A \in \mathbb{R}(2^{r+2})$, we already know (see (0.7)) that $A = A_0 + A_{\mathfrak{J}} + A_{\mathfrak{K}} + A_{\mathfrak{J}\mathfrak{K}}$. Moreover, like in the complex case, A can be decomposed into

$$A = \sum_{\alpha, \beta=0}^3 \tau_\alpha \otimes \tau_\beta \otimes \Xi_{\alpha\beta},$$

with $\Xi_{\alpha\beta} \in \mathbb{R}(2^r)$.

We note that ${}^T\gamma_i A = -A\gamma_i$ if and only if ${}^T\gamma_i A_\alpha = -A_\alpha \gamma_i$ with $\alpha = 0, \mathfrak{J}, \mathfrak{K}, \mathfrak{J}\mathfrak{K}$.

We observe that $\mathfrak{J}A_0 = A_0\mathfrak{J}$, if and only if

$$A_0 = \sum_{\beta=0}^3 (\tau_0 \otimes \tau_\beta \otimes \Xi_{0\beta} + \tau_2 \otimes \tau_\beta \otimes \Xi_{2\beta}).$$

If in addition $\mathfrak{K}A_0 = A_0\mathfrak{K}$, then

$$A_0 = \tau_0 \otimes \tau_0 \otimes \Xi_{00} + \tau_0 \otimes \tau_2 \otimes \Xi_{02} + \tau_2 \otimes \tau_1 \otimes \Xi_{21} + \tau_2 \otimes \tau_3 \otimes \Xi_{23}.$$

Therefore, if A_0 is $(\mathfrak{J}, \mathfrak{K})$ -quaternionic, imposing that $\gamma_{2r+1}A = A\gamma_{2r+1}$ we obtain

$$A = \tau_0 \otimes \tau_0 \otimes \Xi_{00} + \tau_0 \otimes \tau_2 \otimes \Xi_{02},$$

and imposing also that $\gamma_{2r+2}A = A\gamma_{2r+2}$ we obtain

$$A = \tau_0 \otimes \tau_0 \otimes \Xi_{00}.$$

Then requiring that ${}^T\gamma_i A = -A\gamma_i$ for $i = 1, \dots, 2r$, we get

$${}^T\eta_i \Xi_{00} = \Xi_{00} \eta_i$$

for $i = 1, \dots, 2r$. In fact,

$$\begin{aligned} {}^T\gamma_i A &= ((-\chi) \otimes {}^T\eta_i)(I_4 \otimes \Xi_{00}) = -\chi \otimes {}^T\eta_i \Xi_{00} = \\ &= -A\gamma_i = -(I_4 \otimes \Xi_{00})(\chi \otimes \eta_i) = -\chi \otimes \eta_i \Xi_{00}. \end{aligned}$$

It follows that

$$\Xi_{00} = \begin{cases} \xi Q_\eta, & \text{for } r \text{ even,} \\ \xi P_\eta, & \text{for } r \text{ odd,} \end{cases}$$

with $\xi \in \mathbb{R}$. But ${}^TQ_\eta = Q_\eta$ if and only if $r \equiv 0, 3 \pmod{4}$ and ${}^TP_\eta = P_\eta$ if and only if $r \equiv 0, 1 \pmod{4}$. Thus we find that

$$\begin{aligned} A_0 &= \xi I_4 \otimes Q_\eta & \text{if } r \equiv 0 \pmod{4}, \\ A_0 &= \xi I_4 \otimes P_\eta & \text{if } r \equiv 1 \pmod{4}, \end{aligned}$$

with $\xi \in \mathbb{R}$, while when $r \equiv 2, 3 \pmod{4}$ A_0 does not give an H -algebra.

We consider now $A_\beta \in \mathbb{R}(2^{r+2})$, an argument similar to the above shows that imposing $\mathfrak{A}A_\beta = A_\beta \mathfrak{A}$ and $\mathcal{K}A_\beta = -A_\beta \mathcal{K}$ we obtain

$$A_\beta = \tau_0 \otimes \tau_1 \otimes \Xi_{01} + \tau_0 \otimes \tau_3 \otimes \Xi_{03} + \tau_2 \otimes \tau_0 \otimes \Xi_{20} + \tau_2 \otimes \tau_2 \otimes \Xi_{22}.$$

Therefore requiring that $\gamma_{2r+1} A_\beta = A_\beta \gamma_{2r+1}$ we find

$$A_\beta = \tau_2 \otimes \tau_0 \otimes \Xi_{20} + \tau_2 \otimes \tau_2 \otimes \Xi_{22},$$

and requiring also that $\gamma_{2r+2} A_\beta = A_\beta \gamma_{2r+2}$, we get

$$A_\beta = \tau_2 \otimes \tau_0 \otimes \Xi_{20}.$$

Moreover if ${}^T\gamma_i A_\beta = -A_\beta \gamma_i$ for $i = 1, \dots, 2r$, then

$${}^T\eta_i \Xi_{02} = \Xi_{02} \eta_i$$

for $i = 1, \dots, 2r$. Hence, when r is odd Ξ_{02} is proportional to P_η and when r is even Ξ_{02} is proportional to Q_η . But when $r \equiv 0, 1 \pmod{4}$ ${}^TA_\beta = -A_\beta$. So finally

$$\begin{aligned} A_\beta &= \xi \mathfrak{I}(I_4 \otimes P_\eta) & \text{when } r \equiv 3 \pmod{4}, \\ A_\beta &= \xi \mathfrak{I}(I_4 \otimes Q_\eta) & \text{when } r \equiv 2 \pmod{4}, \end{aligned}$$

while when $r \equiv 0, 1 \pmod{4}$ A_β does not give an H -algebra.

In the same way we could prove that imposing that a symmetric $A_\mathcal{K}$ satisfies $\mathfrak{A}A_\mathcal{K} = -A_\mathcal{K} \mathfrak{A}$, $\mathcal{K}A_\mathcal{K} = A_\mathcal{K} \mathcal{K}$ and ${}^T\gamma_i A_\mathcal{K} = -A_\mathcal{K} \gamma_i$ for $i = 1, \dots, 2r+2$, we obtain

$$\begin{aligned} A_\mathcal{K} &= \xi \mathfrak{I}(I_4 \otimes P_\eta) & \text{when } r \equiv 2 \pmod{4}, \\ A_\mathcal{K} &= \xi \mathfrak{I}(I_4 \otimes Q_\eta) & \text{when } r \equiv 3 \pmod{4}, \end{aligned}$$

with $\zeta \in \mathbb{R}$, and similarly for an operator $A_{\mathfrak{Y}\mathfrak{X}}$ which commutes with $\mathfrak{Y}\mathfrak{X}$ and anti-commutes with $\mathfrak{Y}$ and $\mathfrak{X}$.

To study the algebra $\mathcal{C}(r-1, r+3)$ we use isomorphism $\mathcal{C}(r-1, r+3) \simeq \mathcal{C}(0, 2) \otimes \mathcal{C}(r+1, r-1) \simeq \mathbb{H} \otimes \mathbb{R}(2^r)$. We put

$$\begin{aligned} \bar{\gamma}_i &= \chi \otimes \bar{\eta}_{i+r+1} = \chi \otimes \eta_{i+r} = \gamma_i & \text{if } i = 1, \dots, r-1, \\ \bar{\gamma}_r &= \chi \otimes \bar{\eta}_{1+r} = \chi \otimes \varepsilon_\eta = \varepsilon_\gamma \\ \bar{\gamma}_i &= \chi \otimes \bar{\eta}_{i-r} = \gamma_i & \text{if } i = r+1, \dots, 2r, \\ \bar{\gamma}_{2r+i} &= \sigma_i \otimes I_\eta = \gamma_{2r+i} & \text{if } i = 1, 2, \end{aligned}$$

where the $\bar{\eta}_i$'s have been defined by (3.8). Then $(\bar{\gamma}_1, \dots, \bar{\gamma}_{2r+2})$ is a set of matrixes which are orthogonal and satisfy the fundamental relations of $\mathcal{C}(r-1, r+3)$. Hence, if $(\bar{Z}_1, \dots, \bar{Z}_{2r+2})$ is a set of Clifford generators of $\mathcal{C}(r-1, r+3)$, the linear map

$$\bar{J}: \bar{Z}_i \mapsto \bar{\gamma}_i \quad \text{for } i = 1, \dots, 2r+2$$

defines a representation.

Now requiring that $\bar{A}_0 \in \mathbb{R}(2^{r+2})$ is $(\mathfrak{Y}, \mathfrak{X})$ -quaternionic and satisfies ${}^T \bar{\gamma}_i \bar{A}_0 = -\bar{A}_0 \gamma_i$ for $i = 1, \dots, 2r+2$, we obtain

$$\bar{A}_0 = I_4 \otimes \Xi$$

where $\Xi \in \mathbb{R}(2^r)$ satisfies

$${}^T \eta_i \Xi = \Xi \eta_i$$

for $i = 1, \dots, r-1, r+1, \dots, 2r$, and

$${}^T \varepsilon_\eta \Xi = \Xi \varepsilon_\eta \Leftrightarrow \eta_r \Xi = \Xi \eta_r.$$

Hence,

$$\bar{A}_0 = \zeta I_4 \otimes \bar{Q}_\eta \quad \text{when } r \equiv 0, 2 \pmod{4},$$

$$\bar{A}_0 = \zeta I_4 \otimes \bar{P}_\eta \quad \text{when } r \equiv 1, 3 \pmod{4},$$

with $\zeta \in \mathbb{R}$, but ${}^T \bar{A}_0 = \bar{A}_0$ if and only if $r \equiv 0, 1 \pmod{4}$. Thus if $r \equiv 2, 3 \pmod{4}$ $\bar{A}_0$ does not give an H -algebra.

In the same way we could prove that $\bar{A}_3 \in \mathbb{R}(2^{r+2})$ such that $\mathfrak{Y}\bar{A}_3 = \bar{A}_3 \mathfrak{Y}$, $\mathfrak{X}\bar{A}_3 = -\bar{A}_3 \mathfrak{X}$ gives rise to an H -algebra if and only if

$$\bar{A}_3 = \zeta \mathfrak{Y}(I_4 \otimes P_\eta) \quad \text{when } r \equiv 3 \pmod{4},$$

$$\bar{A}_3 = \zeta \mathfrak{Y}(I_4 \otimes Q_\eta) \quad \text{when } r \equiv 2 \pmod{4},$$

with $\zeta \in \mathbb{R}$, and similarly for an operator which commutes with $\mathfrak{X}$ and anti-commutes with $\mathfrak{Y}$ and $\mathfrak{Y}\mathfrak{X}$, or which commutes with $\mathfrak{Y}\mathfrak{X}$ and anti-commutes with $\mathfrak{Y}$ and $\mathfrak{X}$. Now the statement follows from Lemma 3.2. ■

From the proof of Lemma 3.5 it follows also that in the case of the family of algebras

$\mathcal{C}(r, r+2)$ A is given by

$$\begin{aligned} A &= \zeta(I_4 \otimes Q_\eta) && \text{when } r \equiv 0 \pmod{4}, \\ A &= \zeta(I_4 \otimes P_\eta) && \text{when } r \equiv 1 \pmod{4}, \\ A_\alpha &= \zeta \mathfrak{J}_\alpha(I_4 \otimes Q_\eta) && \text{when } r \equiv 2 \pmod{4}, \\ A_\alpha &= \zeta \mathfrak{J}_\alpha(I_4 \otimes P_\eta) && \text{when } r \equiv 3 \pmod{4}, \end{aligned}$$

with $\zeta \in \mathbb{R}$ and $\alpha = 1, 2, 3$, $\mathfrak{J}_1 = \mathfrak{J}$, $\mathfrak{J}_2 = \mathfrak{K}$, $\mathfrak{J}_3 = \mathfrak{J}\mathfrak{K}$.

LEMMA 3.6. – *The Clifford algebras which are isomorphic to $\mathbb{H}(2^r) \otimes \mathbb{H}(2^r)$ with $r = 0, 1, \dots$ have real irreducible $\mathcal{C}$ -modules which are admissible if and only if r is even.*

PROOF. – By Lemma 3.2 it is enough to prove the statement for the algebra $\mathcal{C}(r, r+3) \cong \mathbb{H}(2^r) \oplus \mathbb{H}(2^r)$, since if $\mathcal{C}(p, q) \cong \mathbb{H}(2^r) \oplus \mathbb{H}(2^r)$, then there exists an integer k such that $p = r + 4k$, $q = r - 4k$. The algebras $\mathcal{C}(r, r+3)$, as the algebras $\mathcal{C}(r+1, r) \cong \mathbb{R}(2^r) \oplus \mathbb{R}(2^r)$ (see the proof of Lemma 3.4), have two irreducible modules $\mathfrak{v}_1$ and $\mathfrak{v}_2$ which are isomorphic as vector spaces to $\mathfrak{v} = \mathbb{R}^{2^{r+2}}$ which is the space carrying the irreducible representation of $\mathcal{C}(r, r+2)$. The proof that if r is odd $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are non-admissible is, except for minor modifications, identical to the first half of the proof of Lemma 3.2. Therefore we will only prove that if r is even the scalar product which makes $\mathfrak{v}$ admissible as $\mathcal{C}(r, r+2)$ -module makes it also admissible as $\mathcal{C}(r, r+3)$ -module. Let $(\gamma_1, \dots, \gamma_{2r+2}) \subset \mathbb{R}(2^{r+2})$ satisfy the fundamental relations of $\mathcal{C}(r, r+2)$ and (2.1). Set

$$\widehat{\varepsilon} = \widehat{\gamma}_1 \dots \widehat{\gamma}_{2r+2},$$

then

$$\widehat{\gamma}_i \widehat{\varepsilon} = -\widehat{\varepsilon} \widehat{\gamma}_i, \quad \text{for } i = 1, \dots, 2r+2,$$

and

$$\widehat{\varepsilon}^2 = (-1)^{r+2} (-1)^{(2r+2)(2r+1)/2} I_{\widehat{\gamma}} = -I_{\widehat{\gamma}}.$$

We put

$$\widetilde{\gamma}_i = \widehat{\gamma}_i \quad \text{for } i = 1, \dots, 2r+2,$$

$$\widetilde{\gamma}_{2r+3} = \widehat{\varepsilon},$$

then $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_{2r+3})$ is a set of orthogonal matrixes satisfying the fundamental relations of $\mathcal{C}(r, r+3)$.

Let $A \in \mathbb{R}(2^{r+2})$. Suppose that ${}^T \widetilde{\gamma}_i A = -A \widetilde{\gamma}_i$ for $i = 1, \dots, 2r+2$. Then we obtain

$${}^T \widetilde{\gamma}_{2r+3} A = -A \widetilde{\gamma}_{2r+3}.$$

this concludes the proof of the lemma. ■

Finally we summarize the last five lemmas in a unique statement, which concerns explicitly only the algebras $\mathcal{C}(p, q)$ with $p, q = 1, \dots, 8$ by the modulo 8 periodicity of Clifford algebras.

THEOREM 3.1. – *The $\mathcal{C}(p, q)$ algebras whose irreducible modules are admissible, are those listed below:*

$$p - q \equiv 0, 6 \pmod{8} \quad \text{for } r \equiv 0, 6 \pmod{4}$$

$$\left(\mathcal{C}(p, q) \cong \mathbb{R}(2^r) \text{ with } r = \frac{p+q}{2} \right),$$

$$p - q \equiv 2, 4 \pmod{8} \quad \text{for all } r$$

$$\left(\mathcal{C}(p, q) \cong \mathbb{H}(2^r) \text{ with } r = \frac{p+q}{2} - 1 \right),$$

$$p - q \equiv 1, 5 \pmod{8} \quad \text{for } r \equiv 0, 2, 3 \pmod{4}$$

$$\left(\mathcal{C}(p, q) \cong \mathbb{C}(2^r) \text{ with } r = \frac{p+q}{2} - \frac{1}{2} \right),$$

$$p - q \equiv 3 \pmod{8} \quad \text{for } r \equiv 0, 2 \pmod{4}$$

$$\left(\mathcal{C}(p, q) \cong \mathbb{H}(2^r) \oplus \mathbb{H}(2^r) \text{ with } r = \frac{p+q}{2} - \frac{3}{2} \right),$$

$$p - q \equiv 7 \pmod{8} \quad \text{for } r \equiv 3 \pmod{4}$$

$$\left(\mathcal{C}(p, q) \cong \mathbb{R}(2^r) \oplus \mathbb{R}(2^r) \text{ with } r = \frac{p+q}{2} - \frac{1}{2} \right).$$

Collecting the results of the two last section we obtain the minimal dimension of the H -type-algebra associated to the Clifford algebra $\mathcal{C}(p, q)$. We will show in which way only in the case $p - q \equiv 0, 6 \pmod{8}$. First we see from Theorem 3.1 that $\mathcal{C}(p, q) \cong \mathbb{R}(2^r)$ with

$$(3.17) \quad r = \frac{p+q}{2}.$$

Hence, the irreducible $\mathcal{C}(p, q)$ -module is $v = \mathbb{R}^{2^r}$. We also see from Theorem 3.1 that v is admissible if and only if $r \equiv 0, 3 \pmod{4}$, which is equivalent by (3.17) to $p+q \equiv 0, 6 \pmod{8}$. Therefore, when $p+q \equiv 0, 6 \pmod{8}$, the H -type-algebra of minimal dimension is $n = v \oplus z$, and its dimension is given by

$$\dim n = \dim z + \dim v = p + q + 2^{(p+q)/2}.$$

On the other hand, from the proof of Theorem 2.1 we see that when the irreducible

$\mathcal{C}(p, q)$ -module $\mathfrak{v}$ is non-admissible, then the lowest dimensional admissible module is $\mathfrak{v} \oplus \mathfrak{v}$. Hence, when $r \equiv 1, 2 \pmod{4}$, which is equivalent to $p + q \equiv 2, 4 \pmod{8}$, the H -type-algebra of minimal dimension is $\mathfrak{n} = \mathfrak{g} \oplus (\mathfrak{v} \oplus \mathfrak{v})$, and

$$\dim \mathfrak{n} = \dim \mathfrak{g} + 2 \dim \mathfrak{v} = p + q + 2^{(p+q+2)/2}.$$

The same argument which can be used in all cases, leads to the following theorem.

THEOREM 3.2. – *The dimensions $h_{(p,q)}$ of the lowest dimensional (p, q) - H -algebras are given by the following table:*

$p - q \equiv 0, 6 \pmod{8}$	and $p + q \equiv 0, 6 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q)/2},$
$p - q \equiv 0, 6 \pmod{8}$	and $p + q \equiv 2, 4 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+2)/2},$
$p - q \equiv 2, 4 \pmod{8}$	and $p + q$ even	$h_{(p,q)} = p + q + 2^{(p+q+2)/2},$
$p - q \equiv 1, 5 \pmod{8}$	and $p + q \equiv 1, 5, 7 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+1)/2},$
$p - q \equiv 1, 5 \pmod{8}$	and $p + q \equiv 3 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+3)/2},$
$p - q \equiv 3 \pmod{8}$	and $p + q \equiv 3, 7 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+3)/2},$
$p - q \equiv 3 \pmod{8}$	and $p + q \equiv 1, 5 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+3)/2},$
$p - q \equiv 7 \pmod{8}$	and $p + q \equiv 7 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+1)/2},$
$p - q \equiv 7 \pmod{8}$	and $p + q \equiv 1, 3, 5 \pmod{8}$	$h_{(p,q)} = p + q + 2^{(p+q+1)/2},$

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