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A correction to "Definable principal congruences in varieties of groups and rings"

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In the paper cited above [1], the proofs of Theorems 1.2 and 1.3 are incorrect. We do not know if the results, as stated, are correct. The error lies in assuming that for all $\omega(x, y, u, v, \vec{z})$ in Γ one can claim

$$F_k \models \omega(\bar{x}, \bar{y}, \bar{u}, \bar{v}, \bar{z}_0, \dots, \bar{z}_n)$$

as at the top of page 154. However for a restricted class of ω in Γ this claim holds, and if one replaces the lemma and theorems of § 1 by the following text then one has a result which is sufficiently strong for the study of groups and rings as in § 2,3. (The main results, those of § 2, 3, are correct as presented in [1].)

LEMMA 1. *If K is closed under ultra products, then given formulas $\{\phi_i\}_{i\in I}$ and ϕ, we have* $K \models \bigvee_{i\in J} \phi_i \leftrightarrow \phi$ iff *for some finite* $J \subseteq I$, $K \models \bigvee_{i\in J} \phi_i \leftrightarrow \phi$.

Proof. (Standard.)

DEFINITION 2. Let P be the set of polynomials $p(w, z_0, \dots, z_n)$, $n < \omega$. For $P_0 \subseteq P$, a variety V has P_0*-projective principal congruences* if, for $a, b, c, d \in A \in V$, $(a, b) \in \theta_A(c, d)$ holds iff

$$A \models \exists \vec{z}[a = p(e_1, \vec{z}) \ \& \ b = p(e_2, \vec{z})]$$

for some $p \in P_0$, where $\{e_1, e_2\} = \{c, d\}$.

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Two examples of varieties with P_0-projective principal congruences follow:
(1) For *rings* let $P_0=\{p_n : n \geq 1\}$ where

$$p_n(w, z_0, \ldots, z_{2n+1}) = \sum_{i=0}^{n-1} z_{2i} \cdot (w - z_{2n}) \cdot z_{2i+1} + z_{2n+1}.$$

To see that this works let $a, b, c, d \in R$, R a ring. Then $(a, b) \in \theta_R(c, d)$ iff $a-b$ is in the ideal generated by $c-d$ iff for some $n<\omega$ and some $a_0, \ldots, a_{2n-1}$

$$a-b = \sum_{i=0}^{n-1} a_{2i}(c-d)a_{2i+1}.$$

But then

$$a = p_n(c, a_o, \ldots, a_{2n-1}, d, b)$$
$$b = p_n(d, a_0, \ldots, a_{2n-1}, d, b).$$

(2) For *groups* of finite exponent e let $P_0=\{p_n : n \geq 1\}$ where

$$p_n(w, z_0, \ldots, z_{n+1}) = \left[\prod_{i=0}^{n-1} z_i^{-1} \cdot (w \cdot z_n^{-1}) \cdot z_i)\right] \cdot z_{n+1}.$$

If G is a group of exponent e and $a, b, c, d \in G$ then $(a, b) \in \theta_G(c, d)$ iff ab^{-1} is a product of conjugates of cd^{-1}, hence iff for some n and $a_i \in G$,

$$ab^{-1} = \prod_{i=0}^{n-1} a_i^{-1}(cd^{-1})a_i.$$

But then

$$a = p_n(c, a_0, \ldots, a_{n-1}, d, b)$$
$$b = p_n(d, a_0, \ldots, a_{n-1}, d, b).$$

THEOREM 3. *Let V be a variety with P_0-projective principal congruences, for a given P_0. Then V has DPC* iff *for some finite subset P_0' of P_0 there is, for each $p(w, z_0, \ldots, z_n) \in P_0$, a $q(w, z_0, \ldots, z_k) \in P_0'$ and polynomials $q_i(u, v, z_0, \ldots, z_n)$, $0 \leq i \leq k$, such that V satisfies, for suitable $\{w_1, w_2\}=\{u, v\}$,*

$$p(u, z_0, \ldots, z_n) = q(w_1, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n))$$
$$p(v, z_0, \ldots, z_n) = q(w_2, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)).$$

Proof. ($\Rightarrow$) Suppose V has *DPC*. Then from Lemma 1 there must be a $P_0' \subseteq P_0$ such that V satisfies

$$(*) \quad \bigvee_{p \in P_0} \exists \vec{z}[x = p(u, \vec{z}) \ \& \ y = p(v, \vec{z}))]$$

$$\leftrightarrow \bigvee_{q \in P_0'} \exists \vec{z}[x = q(w_1, \vec{z}) \ \& \ y = q(w_2, \vec{z})]$$

$$\{w_1, w_2\} = \{u, v\}$$

Given $p(w, z_0, \ldots, z_n) \in P_0$ let F be the free algebra in V freely generated by $u, v, z_0, \ldots, z_n$. In F let $x = p(u, z_0, \ldots, z_n)$, $y = p(v, z_0, \ldots, z_n)$. As

$$F \models \exists \vec{z}[x = p(u, \vec{z}) \ \& \ y = p(v, \vec{z})]$$

it follows by (*) that for some $q(w, z_0, \ldots, z_k) \in P_0'$,

$$F \models \exists \vec{z}[x = q(w_1, \vec{z}) \ \& \ y = q(w_2, \vec{z})]$$

with $\{w_1, w_2\} = \{u, v\}$. Thus we can choose polynomials $q_i(u, v, z_0, \ldots, z_n) \in F$ such that

$$F \models x = q(w_1, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)).$$
$$F \models y = q(w_2, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)).$$

Of course if two polynomials are equal in F then the corresponding identity holds in V.

($\Leftarrow$) Let $a, b, c, d \in A \in V$ with $(a, b) \in \theta_A(c, d)$. Then, for some $p(x, z_0, \ldots, z_n) \in P_0$,

$$A \models \exists \vec{z}[a = p(e_1, \vec{z}) \ \& \ b = p(e_2, \vec{z})]$$

with $\{e_1, e_2\} = \{c, d\}$. Choose $q, q_0, \ldots, q_k$ as in the statement of the theorem. Then, for suitable $\{\bar{e}_1, \bar{e}_2\} = \{c, d\}$,

$$A \models \exists \vec{z}[a = q(\bar{e}_1, q_0(\bar{e}_1, \bar{e}_2, \vec{z}), \ldots, q_k(\bar{e}_1, \bar{e}_1, \vec{z}))$$
$$\& \ b = q(\bar{e}_2, q_0(\bar{e}_1, \bar{e}_2, \vec{z}), \ldots, q_k(\bar{e}_1, \bar{e}_2, \vec{z}))]$$

so

$$A \models \exists \bar{z}[a = q(\bar{e}_1, \bar{z}) \;\&\; b = q(\bar{e}_2, \bar{z})].$$

Thus the formula $\phi(x, y, u, v)$ given by

$$\bigvee_{\substack{q \in P_0' \\ \{w_1, w_2\} = \{u, v\}}} \exists \bar{z}[x = q(w_1, \bar{z}) \;\&\; y = q(w_2, \bar{z})].$$

defines principal congruences in V.

COROLLARY 4. *Let $V(K)$ be a variety with P_0-projective principal congruences. Then $V(K)$ has DPC* iff *$Q(K)$ has DPC.*

Proof. The direction ($\Rightarrow$) is clear. For ($\Leftarrow$) just repeat the first part of the proof of Theorem 3 as $F \in Q(K)$.

PROBLEM: For arbitrary K is it true that $Q(K)$ has *DPC* implies $\mathbf{V}(K)$ has *DPC*?

REFERENCES

[1] S. BURRIS and J. LAWRENCE, *Definable principal congruences in varieties of groups and rings.* Alg. Univ. *9* (1979), 152–164.

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