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Convergence Rates of the Strong Law for Stationary Mixing Sequences

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Summary. In this note we estimate the rate of convergence in Marcinkiewicz-Zygmung strong law, for partial sums S_n of strong stationary mixing sequences of random variables. The results improve the corresponding ones obtained by Tze Leung Lai (1977) and Christian Hipp (1979).

1. Introduction

Let $\{X_n\}_n$ be a sequence of random variables and let $S_n=\sum_{k=1}^{n} X_k$. Baum and Katz (1965) estimated the rate of convergence of the strong law for partial sums S_n of i.i.d., showing that if $\alpha>1/2$, $p\alpha>1$ and assuming $EX_1=0$ if $\alpha\leqq 1$ then

$$E|X_1|^p<\infty \tag{1.1}$$

is equivalent to

$$\sum_n n^{p\alpha-2} P(\max_{j\leqq n}|S_j|>\varepsilon n^\alpha)<\infty. \tag{1.2}$$

Motivated by applications to sequential analysis of time series, Lai (1977) extended this theorem from i.i.d. case to other dependent cases namely for some classes of ϕ and strong mixing sequences of random variables satisfying the following additional assumption: There exists $\beta>1$ and a positive integer m such that as $x\to\infty$

$$\text{(C)} \qquad \sup_{i\geqq m} P(|X_1|>x, |X_i|>x)=O(P^\beta(|X_1|>x)).$$

The purpose of this note is to prove that the equivalence of (1.1) and (1.2) holds for ϕ and ρ-mixing sequences without the additional assumption (C) and under an improved mixing rate (logarithmic).

We shall denote the L_p norm by $\|\cdot\|_p$, the greatest integer contained in x by $[x]$, the indicator of A by I_A and we shall use the Vinogradov symbol $\ll$ instead of O. Denote $\mathfrak{F}_n^m=\sigma(X_i; n\leqq i<m)$.

We shall use the following mixing coefficients:

$$\rho(n)=\sup_m \sup_{\{X\in L_2(\mathfrak{F}_1^m),\, Y\in L_2(\mathfrak{F}_{n+m}^\infty)\}} \frac{|E(X-EX)(Y-EY)|}{\|X-EX\|_2\cdot\|Y-EY\|_2}$$

and

$$\phi(n)=\sup_m \sup_{\{A\in\mathfrak{F}_1^m,\, B\in\mathfrak{F}_{n+m}^\infty,\, P(A)\neq 0\}} |P(B|A)-P(B)|.$$

The sequence $(X_n; n\geqq 1)$ is said to be ρ-mixing or ϕ-mixing, according to $\rho(n)\to 0$ or $\phi(n)\to 0$. Bradley (1983) has shown that the ρ-mixing sequences are equivalent with λ-mixing where

$$\lambda(n)=\sup_m \sup_{\{A\in\mathfrak{F}_1^m,\, B\in\mathfrak{F}_{n+m}^\infty,\, P(A)\neq 0,\, P(B)\neq 0\}} \frac{|P(A\cap B)-P(A)P(B)|}{[P(A)P(B)]^{1/2}}.$$

By Lemma (1.17) of [4], we have

$$\rho(n)\leqq 2\phi^{1/2}(n) \tag{1.3}$$

and by [2]

$$\lambda(n)\leqq\rho(n). \tag{1.4}$$

2. The Results

We shall establish the following result.

Theorem 1. *Let $\{X_i\}_i$ be a strong stationary ρ-mixing sequence of random variables, $\alpha p>1$, $\alpha>1/2$ and assume that $EX_1=0$ for $\alpha\leqq 1$,*
if

$$\sum_i \rho^{2/k}(2^i)<\infty \quad \textit{where } k=\begin{cases} 0 & \text{for } 0<p<1\\ 2 & \text{for } 1\leqq p<2\\ [(\alpha p-1)/(\alpha-1/2)]+1 & \text{for } p\geqq 2\end{cases} \tag{2.1}$$

then (1.1)⇒(1.2)
if

$$\sum_i \rho^m(i)<\infty \quad \textit{where } m=\begin{cases} 2(p\alpha-1)/(2-p\alpha) & 1<p\alpha<4/3\\ 1 & p\alpha\geqq 4/3\end{cases} \tag{2.2}$$

then (1.2)⇒(1.1).

This theorem, (1.3) and Lemma (5), (ii) of [5] imply:

Theorem 2. *Let $\{X_i\}_i$ be a strict stationary ϕ-mixing sequence of random variables, $p\alpha>1$, $\alpha>1/2$.*

Assume that $EX_1=0$ *for* $\alpha\leqq 1$, *and* $\Sigma\phi^{1/k}(2^i)<\infty$,

$$\textit{where } k=\begin{cases} 0 & \text{for } 0<p<1\\ 2 & \text{for } 1\leqq p<2\\ [(\alpha p-1)/(\alpha-1/2)]+1 & \text{for } p\geqq 2\end{cases}$$

then (1.1)⇔(1.2).

In order to prove Theorems 1 and 2 we need the following:

Lemma 1. *Suppose* $\{X_i\}_i$ *is a stationary* ρ*-mixing sequence of random variables and for some integer* $k\geqq 2$, $E|X_1|^k<\infty$ *and* $\Sigma\rho^{2/k}(2^i)<\infty$. *Then, there exists a positive constant* K_k, *depending only on* $\{\rho(n)\}_n$ *and on* k *such that for every* $n\geqq 1$,

$$\|S_n\|_k\leqq K_k(n^{1/k}\|X_1\|_k+n^{1/(k-1)}\|X_1\|_{k-1}+\ldots+n^{1/2}\|X_1\|_2+n|EX_1|). \tag{2.3}$$

Proof. We shall prove this Lemma by induction on k. For $k=2$ cf. [7] Lemma (3.4) there exists a constant K_2 depending only on $\{\rho(n)\}_n$ such that

$$\|S_n\|_2\leqq K_2(n^{1/2}\|X_1\|_2+n|EX_1|).$$

We assume (2.3) holds for any integer $l, l<k$. We shall show first that we can find two positive constants C_1 and C_2 depending only on k, such that for every $n\geqq 1$ and $m\geqq 1$,

$$\|S_{2n}\|_k\leqq 2^{1/k}(1+C_1\rho^{2/k}(m))^{1/k}\|S_n\|_k+C_2\|S_n\|_{k-1}+2m\|X_1\|_k. \tag{2.4}$$

Denote by $\bar{S}_n=\sum_{j=n+m+1}^{2n+m}X_j$. From the equation

$$S_{2n}=S_n+\sum_{j=n+1}^{n+m}X_j+\bar{S}_n-\sum_{j=2n+1}^{2n+m}X_j$$

we find that

$$\|S_{2n}\|_k\leqq\|S_n+\bar{S}_n\|_k+2m\|X_1\|_k. \tag{2.5}$$

Obviously there exists a positive constant C, depending on k such that, by stationarity,

$$E|S_n+\bar{S}_n|^k\leqq 2E|S_n|^k+C\sum_{i=1}^{k-1}E|S_n|^i|\bar{S}_n|^{k-i}.$$

Using Hölder inequality and then the definition of ρ-mixing we obtain for $i\leqq k/2$

$$\begin{aligned}E|S_n|^i|\bar{S}_n|^{k-i}&\leqq(E|S_n|^{k/2}|\bar{S}_n|^{k/2})^{2i/k}(E|\bar{S}_n|^k)^{1-2i/k}\\&\leqq[(E|S_n|^{k/2})^2+\rho(m)E|S_n|^k]^{2i/k}(E|S_n|^k)^{1-2i/k}\\&\leqq\rho^{2i/k}(m)E|S_n|^k+\|S_n\|_k^{k-2i}\|S_n\|_{k-1}^{2i}.\end{aligned}$$

Therefore:

$$\begin{aligned}E|S_n+\bar{S}_n|^k&\leqq 2(1+kC\rho^{2/k}(m))\|S_n\|_k^k+2C\sum_{i=1}^{[k/2]}\|S_n\|_{k-1}^{2i}\|S_n\|_k^{k-2i}\\&\leqq(2^{1/k}(1+kC\rho^{2/k}(m))^{1/k}\|S_n\|_k+2C\|S_n\|_{k-1})^k\end{aligned} \tag{2.6}$$

and (2.4) follows from (2.5) and (2.6). Taking now into account the induction assumption, from (2.4) we deduce

$$\begin{aligned}\|S_{2n}\|_k\leqq&\, 2^{1/k}(1+C_1\rho^{2/k}(m))^{1/k}\|S_n\|_k\\&+C_2K_{k-1}\left[\sum_{i=1}^{k-2}n^{1/(k-i)}\|X_1\|_{k-i}+n|EX_1|\right]+2m\|X_1\|_k.\end{aligned}$$

Writing this inequality for $n=2^{r-1}$ and $m=[n^{1/(k+1)}]$, and denoting $[1+C_1\rho^{2/k}([2^{i/(k+1)}])]^{1/k}=a_i$ we obtain

$$\|S_{2^r}\|_k \leqq \left(\prod_{i=0}^{r-1} a_i\right)(2^{r/k}\|X_1\|_k + \sum_{j=0}^{r-1} 2^{j/k}\left[C_2 K_{k-1}\left(\sum_{i=1}^{k-2} 2^{(r-j-1)/(k-i)}\|X_1\|_{k-i} + 2^{r-j-1}|EX_1|\right) + 2\times 2^{(r-j-1)/(k+1)}\|X_1\|_k\right].$$

Therefore there exists a positive constant C depending only on k and $\{\rho(n)\}_n$ such that:

$$\|S_{2^r}\|_k \leqq C\left(\prod_{i=0}^{r-1} a_i\right)\left(2^{r/k}\|X_1\|_k + \sum_{i=1}^{k-2} 2^{r/(k-i)}\|X_1\|_{k-i} + 2^r|EX_1|\right).$$

Since $\sum_i \rho^{2/k}(2^i) < \infty$ we have $\prod_{i=1}^{\infty} a_i < \infty$.

Writing n in binary form we obtain from the preceding inequality that for every n, the relation (2.3) holds.

We also need the following variant of Theorem 5 in [6]:

Lemma 2. *Let $r \geqq 1$ be a given real. Suppose that*

$$E|S_m|^r \leqq m\lambda^r(m) \qquad \textit{for all } m \leqq n$$

where $\lambda(n)$ is a nondecreasing sequence of positive numbers. Then

$$E(\max_{i\leqq n}|S_i|^r) \ll n\left(\sum_{j=0}^{[\log_2 n]} \lambda([n/2^{j+1}])\right)^r.$$

Proof of Theorem 1.

I. We prove first that (1.1) implies (1.2). Let us denote by $b_k = P\{k \leqq |X_1| < k+1\}$. We note that

$$E|X_1|^p < \infty \Leftrightarrow \sum_k k^p b_k < \infty.$$

1. We consider first the case $p \geqq 1$. Without loss of generality we may assume that the random variables are centered at expectations for all $\alpha > 1/2$, because for $\alpha > 1$, and n large enough we have

$$P(\max_{i\leqq n}|S_i| > \varepsilon n^\alpha) \leqq P\left(\max_{i\leqq n}\left|\sum_{j=1}^{i}(X_j - EX_j)\right| > \varepsilon n^\alpha - nE|X_1|\right)$$
$$\leqq P\left(\max_{i\leqq n}\left|\sum_{j=1}^{i}(X_j - EX_j)\right| > (\varepsilon/2)n^\alpha\right).$$

Let k be an integer as in (2.1). Obviously $k > p$. For some $\beta > (1+k)/(k-p)$ let us define:

$$X_i^n(1)=X_i I_{\{|X_i|>n^\alpha\}}-EX_i I_{\{|X_i|>n^\alpha\}}$$

$$X_i^n(2)=X_i I_{\{|X_i|\leqq n^\alpha/(\log_2 n)^\beta\}}-EX_i I_{\{|X_i|\leqq n^\alpha/(\log_2 n)^\beta\}}$$

and

$$X_i^n(3)=X_i I_{\{n^\alpha/(\log_2 n)^\beta<|X_i|\leqq n^\alpha\}}-EX_i I_{\{n^\alpha/(\log_2 n)^\beta<|X_i|\leqq n^\alpha\}}.$$

Let us put $S_m^n(j)=\sum_{i=1}^m X_i^n(j)$ for $j=1,2,3$.

We note that

$$\sum_n n^{p\alpha-2}P(\max_{i\leqq n}|S_i|>\varepsilon n^\alpha)\leqq\sum_{j=1}^3\sum_n n^{p\alpha-2}P(\max_{i\leqq n}|S_i^n(j)|>(\varepsilon/3)n^\alpha).$$

i) We prove first that for every $\varepsilon>0$, $\sum_n n^{p\alpha-2}P(\max_{i\leqq n}|S_i^n(1)|>\varepsilon n^\alpha)=I<\infty$.

Indeed we have successively

$$\begin{aligned}I&\leqq\varepsilon^{-1}\sum_n n^{p\alpha-\alpha-1}E|X_1^n(1)|\leqq\varepsilon^{-1}\sum_n n^{p\alpha-\alpha-1}\sum_{k\geqq n^\alpha-1}(k+1)b_k\\&\leqq\varepsilon^{-1}\sum_k(k+1)b_k\sum_{n\leqq(k+1)^{1/\alpha}}n^{p\alpha-\alpha-1}\ll\sum_k(k+1)^p b_k<\infty.\end{aligned}$$

ii) We prove now that $\sum_n n^{p\alpha-2}P(\max_{i\leqq n}|S_i^n(2)|>\varepsilon n^\alpha)=II<\infty$, for every $\varepsilon>0$.

Taking into account that the random variables $X_i^n(2)$ are centered, by Lemma 1 we have for every $m\leqq n$,

$$E|S_m^n(2)|^k\leqq\left(K_k\sum_{i=0}^{k-2}m^{1/(k-i)}\|X_1^n(2)\|_{k-i}\right)^k.$$

Using Lemma 2 with $\lambda(m)=K_k\sum_{i=0}^{k-2}m^{i/k(k-i)}\|X_1^n(2)\|_{k-i}$ we obtain that:

$$\begin{aligned}E(\max_{i\leqq n}|S_i^n(2)|^k)&\leqq n\left\{\sum_{j=0}^{[\log_2 n]}\left(K_k\sum_{i=0}^{k-2}[n/2^{j+1}]^{i/k(k-i)}\|X_1^n(2)\|_{k-i}\right)\right\}^k\\&\ll n(\log_2 n)^k\|X_1^n(2)\|_k^k+\sum_{i=1}^{k-2}n^{k/(k-i)}\|X_1^n(2)\|_{k-i}^k.\end{aligned}$$

Therefore

$$\begin{aligned}II&\ll\sum_n n^{p\alpha-2-\alpha k}E(\max_{i\leqq n}|S_i^n(2)|)^k\\&\ll\sum_n n^{\alpha(p-k)-2}\left\{n(\log_2 n)^k\|X_1^n(2)\|_k^k+\sum_{i=1}^{k-[p]-1}n^{k/(k-i)}\|X_1^n(2)\|_{k-i}^k\right.\\&\quad\left.+\sum_{i=k-[p]}^{k-2}n^{k/(k-i)}\|X_1\|_p\right\}=A+B+C.\end{aligned}$$

By the definition of $X_i^n(2)$ we have

$$A\ll\sum_n n^{\alpha(p-k)-1}(\log_2 n)^k[n^\alpha/(\log_2 n)^\beta]^{k-p}\ll\sum n^{-1}(\log_2 n)^{k-\beta(k-p)}$$

which converges for the chosen value of β.

The series obtained for $1 \leqq i \leqq k-2$ appear only for $p \geqq 2$.

For $1 \leqq i \leqq k-[p]-1$ we have

$$B \ll \sum_n \sum_{i=1}^{k-[p]-1} n^{\alpha(p-k)-2+k/(k-i)+\alpha k(k-i-p)/(k-i)}$$
$$\ll \sum_n \sum_{i=1}^{k-[p]-1} n^{\alpha p-2+k(1-p\alpha)/(k-i)} \ll \sum_n n^{-1-(p\alpha-1)(k/(k-1)-1)},$$

which converge because $p\alpha > 1$.

For the series obtained for $k-[p] \leqq i \leqq k-2$ we have

$$C \ll \sum_n n^{\alpha(p-k)-2+k/2}$$

which converge by the definition of k.

iii) We prove that $\sum_n n^{p\alpha-2} P(\max_{i \leqq n} |S_i^n(3)| > \varepsilon n^\alpha) = III < \infty$.

By Lemma 1 we have

$$II \ll \sum_n n^{p\alpha-2} P\left(\sum_{i=1}^n |X_i^n(3)| > \varepsilon n^\alpha\right) \ll \sum_n n^{p\alpha-2-k\alpha} \sum_{i=0}^{k-1} n^{k/(k-i)}$$
$$\times (E|X_1^n(3)|^{k-i})^{k/(k-i)}.$$

For $i=0$ we have successively

$$\sum_n n^{p\alpha-1-k\alpha} E|X_1^n(3)|^k \ll \sum_n n^{p\alpha-1-k\alpha} \sum_{j \leqq n^\alpha} j^k b_j$$
$$\ll \sum_j j^k b_j \sum_{n \geqq j^{1/\alpha}} n^{p\alpha-1-k\alpha} \ll \sum_j j^p b_j < \infty.$$

The proof of the fact that the series obtained for $1 \leqq i \leqq k-2$ converge is similar with the proof of the convergence of the series A and B which appear at the point ii) of this proof. For $i=k-1$ we have

$$\sum_n n^{p\alpha-2-k\alpha} n^k (E|X_1^n(3)|)^k \ll \sum_n n^{p\alpha-2-k\alpha+k} ((\log_2 n)^\beta / n^\alpha)^{k(p-1)}$$
$$\ll \sum_n n^{-1-(k-1)(\alpha p-1)} (\log_2 n)^{\beta k(p-1)} < \infty.$$

2. We consider now the case $p<1$. We have

$$S_m = \sum_{i=1}^m X_i I_{\{|X_i| \leqq n^\alpha\}} + \sum_{i=1}^m X_i I_{\{|X_i| > n^\alpha\}} = S_m^n + \bar{S}_m^n.$$

We have

$$\sum_n n^{p\alpha-2} P(\max_{i \leqq n} |S_i^n| > \varepsilon n^\alpha) \leqq \varepsilon^{-1} \sum_n n^{p\alpha-\alpha-1} \sum_{k \leqq n^\alpha} (k+1) b_k \ll \sum_k (k+1)^p b_k < \infty.$$

We also have

$$\sum_n n^{p\alpha-2} P(\max_{i \leqq n} |\bar{S}_i^n| > \varepsilon n^\alpha) \leqq \varepsilon^{-p/2} \sum_n n^{p\alpha/2-1} \sum_{k \geqq n^\alpha - 1} (k+1)^{p/2} b_k \ll \sum_k (k+1)^p b_k < \infty.$$

We note that for $0<p<1$, (1.1)⇒(1.2) was proved without mixing assumptions.

II. We prove now that (1.2) implies (1.1). This proof is inspired from Lemma (5) of [5]. First we show that

$$nP(|X_1|>\varepsilon n^\alpha)\to 0. \tag{2.7}$$

By (1.2) we have

$$\sum_n n^{p\alpha-2} P(\max_{j\leq n}|X_j|>\varepsilon n^\alpha)<\infty$$

for every $\varepsilon>0$.

Then, as $n\to\infty$, we have

$$n^{p\alpha-1} P(\max_{j\leq n}|X_j|>\varepsilon n^\alpha)=O\left(\sum_{k=n}^{2n} k^{p\alpha-2} P(\max_{j\leq k}|X_j|>\varepsilon(k/2)^\alpha)\right)\to 0. \tag{2.8}$$

If $p\alpha\geqq 2$, (2.8) implies (2.7). If $1<p\alpha<2$ we put $q=n^{p\alpha-1}$. Then

$$\begin{aligned}
P(\max_{j\leq n}|X_j|>\varepsilon n^\alpha)&\geqq P\left(\bigcup_{i=1}^{[n/q]}|X_{iq}|>\varepsilon n^\alpha\right)\\
&=\sum_{i=1}^{[n/q]} P(\max_{j\leq i}|X_{jq}|\leqq\varepsilon n^\alpha, |X_{iq}|>\varepsilon n^\alpha)\\
&\geqq\sum_{i=1}^{[n/q]}\{P(\max_{j\leq i}|X_{jq}|\leqq\varepsilon n^\alpha)P(|X_1|>\varepsilon n^\alpha)-\rho(q)P^{1/2}(\max_{j\leq i}|X_{jq}|>\varepsilon n^\alpha)\\
&\quad\times P^{1/2}(|X_1|>\varepsilon n^\alpha)\}.
\end{aligned}$$

The last relation follows from (1.4), taking into account that

$$P(A\cap B)-P(A)P(B)=P(A\cap CB)-P(A)P(CB).$$

Obviously $P(\max_{j\leq i}|X_{jq}|>\varepsilon n^\alpha)\leqq iP(|X_1|>\varepsilon n^\alpha)$. Therefore

$$P(\max_{j\leq n}|X_j|>\varepsilon n^\alpha)\geqq[n/q]P(|X_1|>\varepsilon n^\alpha)\{P(\max_{1\leq j\leq n}|X_j|\leqq\varepsilon n^\alpha)-[n/q]^{1/2}\rho(q)\}.$$

Because by (2.8) $P(\max_{1\leq j\leq n}|X_j|\leqq\varepsilon n^\alpha)\to 1$ and by (2.2) $[n/q]^{1/2}\rho(q)\to 0$ from this last inequality we deduce (2.7).

By condition (2.2), we also deduce that we can choose an integer r such that

$$\sum_i \rho(ri)<1. \tag{2.9}$$

By (1.4) we have

$$\begin{aligned}
P(\max_{j\leq n}|X_j|>\varepsilon n^\alpha)&\geqq[n/r]P(|X_1|>\varepsilon n^\alpha)-\sum_{1\leq j<i\leq[n/r]}P(|X_{ri}|>\varepsilon n^\alpha,|X_{rj}|>\varepsilon n^\alpha)\\
&\geqq[n/r]P(|X_1|>\varepsilon n^\alpha)-[n/r]^2P^2(|X_1|>\varepsilon n^\alpha)-[n/r]P(|X_1|>\varepsilon n^\alpha)\times\sum_{i=1}^{[n/r]}\rho(ri)
\end{aligned}$$

whence by (2.7) and (2.9) we obtain the existence of a constant K such that for *n sufficiently large*

$$[n/r]\, P(|X_1| > \varepsilon n^\alpha) \leqq K P(\max_{j \leqq n} |X_j| > \varepsilon n^\alpha).$$

Therefore by (1.2) $\sum_n n^{p\alpha-1} P(|X_1| > \varepsilon n^\alpha) < \infty$, which implies (1.1).

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