Z. Wahrscheinlichkeitstheorie verw. Gebiete
70, 289-306 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Lower Bounds on the Approximation of the Multivariate Empirical Process

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Summary. It is well-known that if $\mathscr{C}$ is the class of rectangles $0 \leqq x_1 \leqq a_1$, $0 \leqq x_2 \leqq a_2$ or the class of circular discs then the normalized empirical measure on $\mathscr{C}$ behaves like a Brownian bridge. Our main result shows that for these two classes the distances between the normalized empirical measure and the nearest Brownian measure have entirely different order of magnitudes.

1. Introduction

Let $\lambda_0(\cdot)$ denote the restriction of the d-dimensional Lebesgue measure to the unit cube I^d, i.e.,

$$\lambda_0(A) = \text{volume}\,(A \cap I^d).$$

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables (i.i.d.r.v.'s) with distribution λ_0 (that is, uniformly distributed over I^d). The random empirical measure $P_n(\cdot)$ is defined as follows: for any $A \subset \mathbb{R}^d$ let

$$P_n(A) = \frac{1}{n} \sum_{X_j \in A} 1.$$

Let $v_n = n^{1/2}(P_n - \lambda_0)$ be the normalized empirical measure.

It is well-known that if $\mathscr{C}$ is some class of "reasonable" subsets of $\mathbb{R}^d$ then the stochastic process v_n indexed by $\mathscr{C}$ (i.e., the stochastic set function $v_n(A), A \in \mathscr{C}$) behaves like a Brownian bridge indexed by $\mathscr{C}$, e.g., if $\mathscr{C}$ is the class of products of intervals or the class of sets having sufficiently smooth boundaries (see Dudley [4]).

The aim of this paper is to give *lower bounds* to the distance between the normalized empirical measure $v_n(A)$ and the Brownian measure $B_n(A)$ (the precise meaning of the latter measure will be formulated in Sect. 2) where A runs over two fundamental classes, namely the classes

$$BOX(d) = \{[0, x_1] \times [0, x_2] \times \ldots \times [0, x_d] : 0 \leqq x_1, x_2, \ldots, x_d \leqq 1\}$$

(i.e., Cartesian product of intervals parallel to the coordinate axes) and

$$BALL(d)=\{G\cap I^d\colon G \text{ is an arbitrary } d\text{-dimensional closed ball of radius } r,\ r\leqq 1\}, \text{ respectively.}$$

From now on ball means closed ball.

If $d=1$ then $BOX(1)$ and $BALL(1)$ represent essentially the same class (i.e., intervals), and a remarkable results of Komlós et al. [5] completely answers the question.

Theorem A ([5]). *Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d.r.v.'s uniformly distributed on $[0,1]$. Assume further that the probability space is "rich enough". Then one can construct a squence $\{B_n\}$, $n\in\mathbb{N}$ of Brownian bridges (B.b.'s) such that*

$$\sup_{0\leqq x\leqq 1} n^{1/2}|v_n([0,x])-B_n([0,x])|=O(\log n)$$

with probability one (w.p.1).

The precise meaning of "rich enough" will not be formulated each time, one can find it e.g. in [7], p. 729.

Note that the right hand side $O(\log n)$ is the best possible apart from constant factor (see [5]).

In contrast to the case $d=1$ for $d=2$ the sequences

$$\inf_{B_n}\sup_{A\in BOX(2)}|v_n(A)-B_n(A)| \quad\text{and}\quad \inf_{B_n}\sup_{A\in BALL(2)}|v_n(A)-B_n(A)|$$

have entirely different order of magnitudes as $n\to+\infty$. First we recall the following theorem of Tusnády [11] (which is based on [5]).

Theorem B ([11]). *Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d.r.v.'s uniformly distributed over the unit square I^2. Then one can construct a sequence $\{B_n\}$, $n\in\mathbb{N}$ of two-dimensional B.b.'s such that*

$$\sup_{A\in BOX(2)} n^{1/2}|v_n(A)-B_n(A)|=O(\log^2 n)\quad \text{w.p.1.}$$

Our main result below yields that for the class $BALL(2)$ (i.e., circular discs) the analogous error term is much greater, namely greater than a constant multiple of $n^{1/4}$.

For any real $\omega, 0<\omega\leqq 1$ we introduce

$$BALL(d,\omega)=\{G\cap I^d\colon G \text{ is an arbitrary } d\text{-dimensional ball of radius } r,\ \omega/2\leqq r\leqq\omega\}.$$

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v.'s uniformly distributed over the unit cube I^d. Then for any version B of the d-dimensional B.b.*

$$\Pr\Big(\text{for any real } \omega, n^{-1/d}\leqq\omega\leqq 1\colon \sup_{A\in BALL(d,\omega)} n^{1/2}|v_n(A)-B(A)| > c_1\cdot(n\omega^d)^{1/2-1/(2d)}\Big)>1-e^{-n}$$

where the positive constant $c_1=c_1(d)$ depends only on the dimension.

Note that using a modification of our method it is not hard to prove the existence of a ball *contained in the unit cube* I^d with slightly weaker "error" term $n^{1/2-1/(2d)-\varepsilon}$.

Let $SM(d)$ be the class of sets $H \cap I^d$ where the smooth set $H \subset \mathbb{R}^d$ has d-times continuously differentiable boundary. Obviously $BALL(d) \subset SM(d)$. For the latter class Révész [7, 8] has proved a strong approximation theorem.

Theorem C ([8]). *Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d.r.v.'s uniformly distributed over the unit cube I^d. Then one can find a sequence $\{B_n\}$, $n \in \mathbb{N}$ of d-dimensional B.b.'s such that*

$$\sup_{A \in SM(d)} n^{1/2} |v_n(A) - B_n(A)| = O(n^{1/2-1/12(d+1)+\varepsilon}) \quad \text{w.p.1.}$$

If $d > 2$ and we are interested in the class $BOX(d)$, then there is a tremendous gap in our knowledge. The best known strong approximation result is due to Csörgő and Révész [3].

Theorem D ([3]). *Let $X_1, X_2, \ldots$ be the same as in Theorem C. Then one can define a sequence $\{B_n\}$, $n \in \mathbb{N}$ of d-dimensional B.b.'s such that*

$$\sup_{A \in BOX(d)} n^{1/2} |v_n(A) - B_n(A)| = O(n^{1/2-1/2(d+1)} \cdot (\log n)^{3/2}) \quad \text{w.p.1.}$$

In the opposite direction, Komlós et al. [5] observed that a variant of Bártfai's proof [1] might show the lower estimate $c_2 \cdot \log n$ for $d \geqq 1$. Tusnády [11] raised the question of improving this lower bound as $d \to +\infty$.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v.'s uniformly distributed over the unit cube I^d. Then for any version B of the d-dimensional B.b.*

$$\Pr\Big(\sup_{A \in BOX(d)} n^{1/2} |v_n(A) - B(A)| > c_3 \cdot (\log n)^{(d-1)/2}\Big) > 1 - e^{-n}$$

where $c_3 = c_3(d)$ depends only on $d (c_3 > 0)$.

Clearly this estimate is an improvement of $c_2 \cdot \log n$ as $d > 3$ (since $(d-1)/2 > 1$ if $d > 3$). But the great open problem is to decide whether for an appropriate sequence $\{B_n\}$, $n \in \mathbb{N}$ of d-dimensional B.b.'s

$$\sup_{A \in BOX(d)} n^{1/2} |v_n(A) - B_n(A)| = O((\log n)^{c(d)})$$

with some constant $c(d)$ or not.

Finally, let $SEG(2)$ denote the class of sets which can be represented as the intersection of a halfplane and the unit square I^2 (i.e., segments). Combining the ideas of this paper and [2] it is not hard to prove the following result: Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v.'s uniformly distributed over the unit square. Then for *any* version B of the two-dimensional B.b.

$$\Pr\left(\sup_{A \in SEG(2)} n^{1/2} |v_n(A) - B(A)| > c_4 \cdot \frac{n^{1/4}}{(\log n)^{7/2}}\right) > 1 - e^{-n}.$$

2. Notations and the Probabilistic Part of the Proofs

The d-dimensional Brownian bridge $B(\underline{x})$ is defined by

$$B(\underline{x}) = W(\underline{x}) - \lambda(\underline{x}) \cdot W(\underline{1})$$

where $\underline{1} = (1, 1, \ldots, 1)$, $\underline{x} = (x_1, x_2, \ldots, x_d)$, $0 \leqq x_1, x_2, \ldots, x_d \leqq 1$, $\lambda(\underline{x}) = \prod_{j=1}^{d} x_j$ is the volume of the box $\prod_{j=1}^{d} [0, x_j]$ and $W(\underline{x})$ is a d-dimensional Wiener-process (i.e., a Gaussian process with independent increments, variance equal to the d-dimensional volume).

For any integer $k \geqq 0$ and vector $\underline{i} = (i_1, \ldots, i_d)$, $0 \leqq i_1, \ldots, i_d < 2^k$ let

$$I(k; \underline{i}) = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \colon \frac{i_j}{2^k} \leqq x_j < \frac{i_j + 1}{2^k},\ j = 1, \ldots, d \right\}$$

and

$$\underline{p}(k; \underline{i}) = \left(\frac{i_1}{2^k}, \ldots, \frac{i_d}{2^k} \right).$$

For any set $A \subset I^d$ and integer $k \geqq 0$, let

$$A_k = \sum_{\underline{i} \colon I(k; \underline{i}) \subset A} I(k; \underline{i}) \quad \text{and} \quad \tilde{A}_k = \sum_{\underline{i} \colon I(k; \underline{i}) \cap A \neq 0} I(k; \underline{i}).$$

Given a positive real M, let $S^d(M)$ be the class of sets $A \subset I^d$ for which $\lambda(\tilde{A}_k \setminus A_k) < M \cdot 2^{-k}$ for any integer $k \geqq 0$ where $\lambda(\cdot)$ is the d-dimensional Lebesgue measure.

For any $k \geqq 0$ the *Wiener "measure"* $W(I(k; \underline{i}))$ is defined the usual inclusion-exclusion way, and $W(A_k)$ can be also defined by additivity. For any set $A \in S^d(M)$, we define

$$W(A) = \lim_{k \to +\infty} W(A_k).$$

It is quite easy to prove that this limit exists w.p.1. Observe that $BALL(d) \subset S^d(2^d)$, hence the Wiener measure is defined on the class $BALL(d)$.

Now we are in the position to define the *Brownian "measure"*. For any $A \in S^d(M)$, let

$$B(A) = W(A) - \lambda(A) \cdot W(\underline{1}).$$

In order to avoid the technical difficulties arising from the fact $S^d(M)$ is not a σ-algebra, we introduce the following auxiliary random *measure*. Let m be the smallest integer such that $2^m/m \geqq c_0(d) \cdot n^2$ where the positive constant factor $c_0(d)$ will be specified later. For any $A \subset I^d$ let

$$B^*(A) = \sum_{\underline{i} \colon \underline{p}(m; \underline{i}) \in A} B(I(m; \underline{i})).$$

Clearly $B^*(\cdot)$ is a random signed measure (i.e., σ-additive) on all subsets of I^d.

Let $\beta(\cdot)$ be a deterministic signed measure defined on all subsets of I^d. Let $C \geqq 4$ be a real number such that the binary logarithm of $C \cdot n$ is an integer and is divisible by d. We say that $\beta(\cdot)$ satisfies *property* $(d, n, C, *)$ if for any integer $t \geqq 1$,

$$\operatorname{card}\left\{\underline{i}: \sup_G |\beta(G \cap I(l; i))| \geqq \frac{t}{c^*(d) \cdot n^{1/2}}\right\} \leqq \frac{n}{t^{3/2}}$$

where the integer l is defined by $2^{l \cdot d} = C \cdot n$, G is extended over all balls in $\mathbb{R}^d$, and the positive constant $c^*(d)$ (depending only on d) will be specified in Sect. 3.

The proof of Theorem 1 is based on the following purely deterministic lemma.

Lemma 1. *Let $\beta(\cdot)$ be a signed measure defined on all the subsets of I^d with finite total variation. Assume that β satisfies property $(d, n, C, *)$ with some $C \geqq 4$. Furthermore, let there be given n points $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_n \in I^d$. Then for any $\omega, n^{-1/d} \leqq \omega \leqq 1$ there must exist a ball $G \subset \mathbb{R}^d$ of radius r, $\omega/2 \leqq r \leqq \omega$ such that*

$$\left| \sum_{j: \underline{z}_j \in G} 1 - n \cdot \lambda(G \cap I^d) - n^{1/2} \cdot \beta(G \cap I^d) \right| > c_5 \cdot (n \cdot \omega^d)^{1/2 - 1/(2d)}$$

where the positive constant $c_5 = c_5(d, C)$ depends only on d and the value of C.

We postpone the proof to Sect. 3.

For any integral vectors $\underline{k} = (k_1, \ldots, k_d)$, $0 \leqq k_1, \ldots, k_d < +\infty$ and $\underline{i} = (i_1, \ldots, i_d)$, $0 \leqq i_j < 2^{k_j}$ $(j = 1, \ldots, d)$ let

$$I(\underline{k}; \underline{i}) = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d: \frac{i_j}{2^{k_j}} x_j < \frac{i_j + 1}{2^{k_j}}, j = 1, \ldots, d\right\}.$$

Let $ELL(d, \underline{k})$ denote the class of ellipsoids

$$\left\{\underline{x} \in \mathbb{R}^d: \sum_{j=1}^{d} 4^{k_j}(x_j - y_j)^2 \leqq K\right\}$$

where $\underline{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$ and K is a positive real.

Let $C \geqq 8 \cdot \pi^{d/2}$ be a real number such that the binary logarithm of $C \cdot n$ is an integer. We say that the signed measure $\beta(\cdot)$ satisfies *property* $(d, n, C, **)$ if for any integer $t \geqq 1$ and integral vector $\underline{l} = (l_1, \ldots, l_d)$ such that $\prod_{j=1}^{d} 2^{l_j} = C \cdot n$ and $l_j \geqq 0$,

$$\operatorname{card}\left\{\underline{i}: \sup_{A \in ELL(d, \underline{l})} |\beta(A \cap I(\underline{l}; \underline{i}))| \geqq \frac{t}{c^{**}(d) \cdot n^{1/2}}\right\} \leqq \frac{n}{t^{3/2}}$$

where the positive constant $c^{**}(d)$ (depending only on d) will be specified in Sect. 4.

The proof of Theorem 2 is based on

Lemma 2. *Let $\beta(\cdot)$ be a signed measure defined on all the subsets of I^d with finite total variation. Assume that β satisfies property $(d, n, C, **)$ with some $C \geqq 8 \cdot \pi^{d/2}$. Furthermore, let there be given n points $\underline{z}_1, \ldots, \underline{z}_n \in I^d$. Then*

$$\sup_{A \in BOX(d)} \left| \sum_{j: \underline{z}_j \in A} 1 - n \cdot \lambda(A) - n^{1/2} \cdot \beta(A) \right| > c_6 \cdot (\log n)^{(d-1)/2}$$

where the positive constant $c_6 = c_6(d, C)$ depends only on d and the value of C.

We postpone the proof to Sect. 4.

We remark that both Lemma 1–2 belong to the theory of irregularities of point distributions, a theory which was started by van der Corput and Aardenne-Ehrenfest and which was brilliantly continued by K.F. Roth and W.M. Schmidt. Actually, Lemma 1 and Lemma 2 were motivated by the papers of Schmidt [10] and Roth [9], respectively. Our method is, however, different from theirs. The proofs are based on the fact that v_n is a discrete process and B_n is a continuous process.

Lemma 1⇒Theorem 1. *Let B be an arbitrary version of the d-dimensional Brownian bridge (B.b.). We need three further lemmas.*

Lemma 3. *For any $k \geqq 0$ the number of different sets $\{\underline{p}(k;\underline{i}):\underline{p}(k;\underline{i})\in G\}$ where G is extended over all balls in $\mathbb{R}^d$ is less than $4d \cdot 2^{kd(d+1)}$.*

Proof. Let S be an N-element subset of $\mathbb{R}^d$. Let

$$g(S)=\operatorname{card}\{G\cap S\colon G \text{ is an arbitrary ball in } \mathbb{R}^d\}$$

and

$$h(S)=\operatorname{card}\{H\cap S\colon H \text{ is an arbitrary half-space in } \mathbb{R}^d\}.$$

Let

$$g(N,d)=\max_S g(S) \quad \text{and} \quad h(N,d)=\max_S h(S)$$

where the maximum is taken over all $S\subset\mathbb{R}^d$, $\operatorname{card} S=N$.

We claim

$$g(N,d)\leqq N\cdot h(N-1,d). \tag{1}$$

In order to prove (1) observe that in the definition of $g(S)$ we may assume the surface of the ball G contains at least one point of S. Applying *inversions* with center at each $\underline{x}\in S$ we obtain (1).

Next we use

$$h(N-1,d)<2\sum_{i=0}^{d}\binom{N-2}{i}. \tag{2}$$

(For a proof of (2), see p. 24 in Gänssler: Empirical processes, IMS – Lecture Notes – Monograph Series 1984.)

Combining (1) and (2) we get

$$g(N,d)<N\cdot h(N-1,d)<2(1+d)N\cdot N^d=2(1+d)N^{d+1}. \tag{3}$$

By substituting $N=2^{k\cdot d}$ in (3), we have

$$g(N,d)<2(1+d)\cdot 2^{k\cdot d(d+1)}<4d\cdot 2^{k\cdot d(d+1)}, \tag{4}$$

and Lemma 3 follows.

We recall that $2^{l\cdot d}=C\cdot n$ and m is the least integer with $2^m/m\geqq c_0(d)\cdot n^2$.

Lemma 4. *The probability of the event that for $k=0$ and l, for any vector $\underline{i}=(i_1,\dots,i_d)$, $0\leqq i_1,\dots,i_d<2^k$ and for any ball $G\subset\mathbb{R}^d$,*

$$|B(G\cap I(k;\underline{i}))-B^*(G\cap I(k;\underline{i}))|\leqq c_7(d)\cdot n^{1/2}\cdot m^{1/2}\cdot 2^{-m/2}$$

is greater than $1-e^{-n-2}$.

Proof. We follow the argument of the proof of Lemma 2 in Révész [7]. Let $G(k;\underline{i})=G\cap I(k;\underline{i})$. By definition, we have

$$B(G(k;\underline{i}))-B^*(G(k;\underline{i}))=\sum_{\underline{j}:\,I(m;\underline{j})\subset G(k;\underline{i})} B(I(m;\underline{j}))$$
$$+\sum_{t=m}^{\infty} B(G_{t+1}(k;\underline{i})\setminus G_t(k;\underline{i}))-\sum_{\underline{j}:\,p(m;\underline{j})\in G(k;\underline{i})} B(I(m;\underline{j}))$$
$$=\sum_{t=m}^{\infty} B(G_{t+1}(k;\underline{i})\setminus G_t(k;\underline{i}))-\sum_{\underline{j}\in J(G,m,k,\underline{i})} B(I(m;\underline{j}))$$

where $J(G,m,k,\underline{i})=\{\underline{j}: p(m;\underline{j})\in G(k;\underline{i})$ but $I(m;\underline{j})\not\subset G(k;\underline{i})\}$ (we recall that for any $A\subset I^d$, $A_t=\sum_{\underline{j}:\,I(t;\underline{j})\subset A} I(t;\underline{j})$).

Therefore (the parameters $q>0$, $q_t>0$, $t=m,m+1,\dots$ will be fixed later)

$$\Pr\left(\sup_G |B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq q+\sum_{t=m}^{\infty} q_t\right)$$
$$\leqq \Pr(\sup_G |\sum_{\underline{j}\in J(G,m,k,\underline{i})} B(I(m;\underline{j}))|\geqq q)+\sum_{t=m}^{\infty}\Pr(\sup_G |B(G_{t+1}(k;\underline{i})\setminus G_t(k;\underline{i}))|\geqq q_t). \quad (5)$$

Simple geometric consideration shows that

$$\operatorname{card} J(G,m,k,\underline{i})<2^d\cdot 2^{(m-k)(d-1)} \quad (6)$$

and

$$\operatorname{card}\{\underline{j}: I(t+1;\underline{j})\subset G_{t+1}(k;\underline{i})\setminus G_t(k;\underline{i})\}<2^d\cdot 2^{(t+1-k)(d-1)}. \quad (7)$$

Since for any $A\in S^d(M)$, $B(A)$ has normal distribution with variance $\lambda(A)\cdot(1-\lambda(A))$, by (5)–(7) and Lemma 3 we obtain

$$\Pr\left(\sup_G |B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq q+\sum_{t=m}^{\infty} q_t\right)$$
$$\leqq 4d\cdot 2^{(m-k)d(d+1)}\cdot 2(1-\Phi(q\cdot 2^{(m+k(d-1)-d)/2}))$$
$$+\sum_{t=m}^{\infty} 4d\cdot 2^{(t+1-k)d(d+1)}\cdot 2(1-\Phi(q_t\cdot 2^{(t+1+k(d-1)-d)/2})) \quad (8)$$

where Φ is the unit normal distribution function.

Let $q=c_8(d)\cdot n^{1/2}\cdot m^{1/2}\cdot 2^{-(m+k(d-1)-d)/2}$,

$$q_t=c_8(d)\cdot n^{1/2}\cdot(t+1)^{1/2}\cdot 2^{-(t+1+k(d-1)-d)/2}, \qquad t\geqq m \text{ and } n>n_1(\varepsilon).$$

Then from (8) we see by some elementary calculation

$$\Pr(\sup_G |B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq c_7(d)\cdot n^{1/2}\cdot m^{1/2}\cdot 2^{-m/2})$$
$$\leqq \Pr\left(\sup_G |B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq q+\sum_{t=m}^{\infty} q_t\right)$$
$$\leqq\frac{e^{-n-2}}{4\cdot 2^{k\cdot d}}+\frac{1}{4}\sum_{t=m}^{\infty}\frac{e^{-n-2}}{2^{(t+1)d}}<\frac{1}{2}\,\frac{e^{-n-2}}{2^{k\cdot d}}.$$

Consequently,

$$\begin{aligned}&\Pr(\max_{k=0,l}\max_{\underline{i}}\sup_{G}|B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq c_7(d)\cdot n^{1/2}\cdot m^{1/2}\cdot 2^{-m/2})\\ &\leqq \sum_{k=0,l} 2^{k\cdot d}\cdot \Pr(\sup_{G}|B(G(k;\underline{i}))-B^*(G(k;\underline{i}))|\geqq c_7(d)\cdot n^{1/2}\cdot m^{1/2}\cdot 2^{-m/2})\\ &< \sum_{k=0,l} 2^{k\cdot d}\cdot\frac{1}{2}\frac{e^{-n-2}}{2^{k\cdot d}}=e^{-n-2},\end{aligned}$$

which completes Lemma 4.

Now we show that for some sufficiently large constant $c_9=c_9(d)$,

$$\Pr(\text{the random measure } B^*(\cdot) \text{ satisfies property } (d,n,C=c_9,*))>1-e^{-n-1}. \tag{9}$$

In order to verify it we need

Lemma 5. *For any $q>0$ we have*

$$\Pr(\sup_{G}|W(G\cap I^d)|\geqq q)<c_{10}(d)\cdot\exp(-q^2/3)$$

where $W(\cdot)$ is a Wiener measure and G is extended over all balls in $\mathbb{R}^d$.

For $d=2$ this lemma is a particular case of Theorem 1 in Révész [7]. Since Révész's argument works in higher dimensions without any modification, we omit the proof.

From Lemma 5 immediately follows that for any vector $\underline{i}=(i_1,\ldots,i_d)$, $0\leqq i_1,\ldots,i_d<2^l$,

$$\Pr(\sup_{G}|W(G\cap I(l;\underline{i}))|\geqq q\cdot 2^{-l\cdot d/2})<c_{10}(d)\cdot\exp(-q^2/3). \tag{10}$$

Let $E(*)$ denote the event that for some integer $t\geqq 1$,

$$\operatorname{card}\left\{\underline{i}\colon \sup_{G}|W(G\cap I(l;\underline{i}))|\geqq\frac{t}{2c^*(d)\cdot n^{1/2}}\right\}>\frac{n}{t^{3/2}}.$$

Then using (10) and the *independence* of the events we obtain via some elementary calculation that

$$\begin{aligned}\Pr(E(*))&\leqq\sum_{t=1}^{\infty}\sum_{R\geqq j>r}\binom{R}{j}\cdot(c_{10}\cdot\exp(-q^2/3))^j\cdot(1-(c_{10}\cdot\exp(-q^2/3))^{R-j}\\ &\leqq e^{-n-3}\end{aligned} \tag{11}$$

where $q=t\cdot C^{1/2}/(2c^*(d))$, $R=2^{l\cdot d}=C\cdot n$, $r=n\cdot t^{-3/2}$, $C=c_9(d)$ and $c_9(d)$ is sufficiently large.

Since $|W(\underline{1})|\leqq C\cdot n^{1/2}/(4c^*(d))$ with probability $>1-e^{-n-2}$ if $C=c_9(d)$ is sufficiently large, by Lemma 4 we conclude that if $2^m/m\geqq c_0(d)\cdot n^2 =(4c_7(d)\cdot c^*(d))^2\cdot n^2$ (it defines $c_0(d)$) then

$$\max_{\underline{i}} \sup_{G} |B^*(G \cap I(l;\underline{i})) - W(G \cap I(l;\underline{i}))|$$
$$\leqq \max_{\underline{i}} \sup_{G} |B^*(G \cap I(l;\underline{i})) - B(G \cap I(l;\underline{i}))|$$
$$+ \max_{\underline{i}} \sup_{G} |B(G \cap I(l;\underline{i})) - W(G \cap I(l;\underline{i}))|$$
$$\leqq c_7(d) \cdot n^{1/2} \cdot m^{1/2} \cdot 2^{-m/2} + |W(\underline{1})| \cdot \lambda(I(l;\underline{i}))$$
$$\leqq c_7(d) \cdot n^{1/2} \cdot m^{1/2} \cdot 2^{-m/2} + \frac{C \cdot n^{1/2}}{4c^*(d)} \cdot \lambda(I(l;\underline{i}))$$
$$\leqq \frac{1}{4c^*(d) \cdot n^{1/2}} + \frac{1}{4c^*(d) \cdot n^{1/2}} \leqq \frac{1}{2c^*(d) \cdot n^{1/2}} \tag{12}$$

with probability $> 1 - e^{-n-2} - e^{-n-2}$.

Combining (11) and (12) we conclude (9).

By means of (9) and Lemma 1 we obtain (note that $\underline{z}_1, \ldots, \underline{z}_n$ are the actual values of the r.v.'s $X_1, \ldots, X_n$)

$$\Pr(\text{for any real } \omega, n^{-1/d} \leqq \omega \leqq 1: \sup_{A \in BALL(d,\omega)} n^{1/2} |v_n(A) - B^*(A)|$$
$$> c_5(d) \cdot (n \cdot \omega^d)^{1/2 - 1/(2d)}) > 1 - e^{-n-1}. \tag{13}$$

Again by Lemma 4,

$$1 - e^{-n-2} < \Pr(\sup_{A \in BALL(d)} |B^*(A) - B(A)| \leqq c_7(d) \cdot n^{1/2} \cdot m^{1/2} \cdot 2^{-m/2})$$
$$\leqq \Pr\left(\sup_{A \in BALL(d)} |B^*(A) - B(A)| = O\left(\frac{1}{n^{1/2}}\right)\right). \tag{14}$$

Finally, (13) and (14) complete the deduction of Theorem 1 from Lemma 1.

Lemma 2 $\Rightarrow$ Theorem 2. This deduction is quite similar to the previous one. Let B be an arbitrary version of the d-dimensional Brownian bridge (B.b.).

Since $2^m \geqq C \cdot n = \prod_{j=1}^{d} 2^{l_j}$ if $n > n_2(d, C)$, we see that any cell $I(\underline{l};\underline{i})$ can be perfectly filled by some cubes $I(m;\underline{j})$ without gap whenever $n > n_2(d, C)$. From now on we assume $n > n_2(d, C)$.

We need

Lemma 6. *The probability of the event that for* $\underline{k} = \underline{0} = (0, \ldots, 0)$ *and for any* $\underline{k} = \underline{l} = (l_1, \ldots, l_d)$ *with* $\prod_{j=1}^{d} 2^{l_j} = C \cdot n$, $l_j \geqq 0$ *integers, for any* $\underline{i} = (i_1, \ldots, i_d)$ *with* $0 \leqq i_j < 2^{k_j}$, $j = 1, \ldots, d$,

$$\sup_{A \in ELL(d,\underline{k}) \cup BOX(d)} |B(A \cap I(\underline{k};\underline{i})) - B^*(A \cap I(\underline{k};\underline{i}))| \leqq c_{11}(d) \cdot n^{1/2} \cdot m^{1/2} \cdot 2^{-m/2}$$

is greater than $1 - e^{-n-2}$ $(n > n_2(d, C))$.

We omit the proof since it proceeds along the same lines as that of Lemma 4 (note that applying a suitable linear transformation an ellipsoid becomes a ball).

Exactly the same proof as that of Lemma 5 yields that for any $\underline{l}=(l_1,\dots,l_d)$ with $\prod_{j=1}^{d} 2^{l_j}=C\cdot n, l_j\geqq 0$ integers, and any $\underline{i}=(i_1,\dots,i_d)$ with $0\leqq i_j<2^{l_j}, j=1,\dots,d$,

$$\Pr\left(\sup_{A\in ELL(d,\underline{l})} |W(A\cap I(\underline{l};\underline{i}))|\geqq q\cdot\left(\prod_{j=1}^{d} 2^{l_j}\right)^{-1/2}\right)<c_{12}(d)\cdot\exp\{-q^2/3\}. \tag{15}$$

Now we show that for some sufficiently large constant $c_{13}=c_{13}(d)$,

$$\Pr(\text{the random measure } B^*(\cdot) \text{ satisfies property } (d,n,C=c_{13},**))>1-e^{-n-1}.$$

Let $E(**)$ denote the event that for some integer $t\geqq 1$ and for some vector $\underline{l}=(l_1,\dots,l_d)$ with $\prod_{j=1}^{d} 2^{l_j}=C\cdot n, l_j\geqq 0$,

$$\operatorname{card}\left\{\underline{i}:\ \sup_{A\in ELL(d,\underline{l})} |W(A\cap I(\underline{l};\underline{i}))|\geqq\frac{t}{2c^{**}(d)\cdot n^{1/2}}\right\}>\frac{n}{t^{3/2}}.$$

Then using (15) and the *independence* of the events we obtain via some elementary calculation that

$$\Pr(E(**))\leqq\sum_{t=1}^{\infty}\sum_{\underline{l}}\left(\sum_{R\geqq j\geqq r}\binom{R}{j}(c_{12}\cdot\exp(-q^2/3))^j(1-c_{12}\cdot\exp(-q^2/3))^{R-j}\right)\leqq e^{-n-3}$$

where

$$q=\frac{t\cdot C^{1/2}}{2c^{**}(d)},\qquad R=\prod_{j=1}^{d} 2^{l_j},\qquad r=\frac{n}{t^{3/2}},\qquad C=c_{13}(d)$$

and $c_{13}(d)$ is sufficiently large.

From now on we can straightly follow the argument of the previous deduction but we have to apply Lemma 6 instead of Lemma 4.

3. Proof of Lemma 1

In the proof we shall employ the Fourier transform technique. First we introduce two measures. For any $A\subset\mathbb{R}^d$ denote by $Z(A)=\sum_{\underline{z}_j\in A} 1$ how many of the given points $\underline{z}_1,\underline{z}_2,\dots,\underline{z}_n$ lie in A. For any measurable $A\subset\mathbb{R}^d$ let

$$\mu(A)=n\cdot\lambda(A\cap I^d)+n^{1/2}\cdot\beta(A\cap I^d) \tag{16}$$

where $\lambda(\cdot)$ is the Lebesgue measure (i.e., the d-dimensional volume).

Let

$$G(\underline{x},r)=\left\{\underline{y}\in\mathbb{R}^d:\ \sum_{j=1}^{d}(x_j-y_j)^2\leqq r^2\right\},$$

and denote by χ_r the characteriristic function of the ball $G(\underline{0}, r)$ $\left(\text{i.e., } \chi_r(\underline{y})=1 \text{ if } \sum_{j=1}^{d} y_j^2 \leqq r^2 \text{ and 0 otherwise}\right)$.

Now consider the function

$$F_r = \chi_r * (dZ - d\mu) \tag{17}$$

where $*$ denotes the *convolution operation.*

More explicitly (see (16))

$$\begin{aligned} F_r(\underline{x}) &= \int_{\mathbb{R}^d} \chi_r(\underline{x}-\underline{y})(dZ-d\mu)(\underline{y}) = \sum_{\underline{z}_j \in G(\underline{x}, r)} 1 - \mu(G(\underline{x}, r)) \\ &= \sum_{\underline{z}_j \in G(\underline{x}, r)} 1 - n \cdot \lambda(G(\underline{x}, r) \cap I^d) - n^{1/2} \cdot \beta(G(\underline{x}, r) \cap I^d). \end{aligned} \tag{18}$$

By Parseval-Plancherel identity (see (21) below)

$$\int_{\rho/2}^{\rho} \left(\int_{\mathbb{R}^d} F_r^2(\underline{x})\, d\underline{x}\right) dr = \int_{\rho/2}^{\rho} \left(\int_{\mathbb{R}^d} |\hat{F}_r(\underline{t})|^2\, d\underline{t}\right) dr \tag{19}$$

where $\hat{F}_r$ denotes the Fourier transform of F_r.

We recall some well-known facts from Fourier analysis. Given a function $f \in L^2(\mathbb{R}^d)$, we denote by

$$\hat{f}(\underline{t}) = \pi^{-d/2} \int_{\mathbb{R}^d} e^{-i\underline{x}\cdot\underline{t}} f(\underline{x})\, d\underline{x}$$

the Fourier transform of f ($i=\sqrt{-1}$ and $\underline{x}\cdot\underline{t}$ is the Euclidean inner product). We shall use the following identities

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \qquad \text{(where $*$ is the convolution operation)} \tag{20}$$

$$\int_{\mathbb{R}^d} |f(\underline{x})|^2\, d\underline{x} = \int_{\mathbb{R}^d} |\hat{f}(\underline{t})|^2\, d\underline{t}. \tag{21}$$

By (17), (20) and (19) we have $(0<\rho \leqq 1)$ is a parameter)

$$\begin{aligned} \Delta(\rho) &= \frac{2}{\rho} \int_{\rho/2}^{\rho} \left(\int_{\mathbb{R}^d} F_r^2(\underline{x})\, d\underline{x}\right) dr \\ &= \int_{\mathbb{R}^d} \left(\frac{2}{\rho} \int_{\rho/2}^{\rho} |\hat{\chi}_r(\underline{t})|^2\, dr\right) \cdot |\widehat{(dZ - d\mu)}(\underline{t})|^2\, d\underline{t} = \int_{\mathbb{R}^d} h(\rho, |\underline{t}|) \cdot |\phi(\underline{t})|^2\, d\underline{t} \end{aligned} \tag{22}$$

where $h(\rho, |\underline{t}|) = \frac{2}{\rho} \int_{\rho/2}^{\rho} |\hat{\chi}_r(\underline{t})|^2\, dr$ and $\phi = \widehat{(dZ - d\mu)}$, i.e.,

$$\phi(\underline{t}) = \pi^{-d/2} \int_{\mathbb{R}^d} e^{-i\underline{x}\cdot\underline{t}} (dZ - d\mu)(\underline{x}).$$

First we investigate $h(\rho, |\underline{t}|)$. For the sake of brevity, let $t = |\underline{t}|$ (Euclidean length) and $g(r, t) = \hat{\chi}_r(\underline{t})$.

By definition,

$$g(r,t)=\pi^{-d/2}\int_{\mathbb{R}^d} e^{-i\underline{x}\cdot\underline{t}}\cdot\chi_r(\underline{x})\,d\underline{x}=\pi^{-d/2}\int_{G(\underline{0},r)} e^{-i\underline{x}\cdot\underline{t}}\,d\underline{x}$$

$$=c_{14}(d)\cdot\int_{-r}^{r} e^{-ity}\cdot(r^2-y^2)^{(d-1)/2}\cdot dy$$

$$=c_{14}(d)\cdot r^d\cdot\int_{-1}^{+1}\cos(t\cdot r\cdot u)\cdot(1-u^2)^{(d-1)/2}\cdot du. \qquad (23)$$

On the other hand, the classical Bessel function $J_k(x)$ has the following integral representation (Poisson's integral, see e.g. in [6], p. 241)

$$J_k(x)=\frac{1}{\pi^{1/2}\cdot\Gamma(k+1/2)}\cdot\left(\frac{x}{2}\right)^k\cdot\int_{-1}^{+1}\cos(x\cdot u)\cdot(1-u^2)^{k-1/2}\cdot du \qquad (k>-1/2). \quad (24)$$

Hence, by (23) and (24) we get the following explicit form of $g(r,t)$,

$$g(r,t)=c_{15}(d)\cdot\left(\frac{r}{t}\right)^{d/2}\cdot J_{d/2}(r\cdot t). \qquad (25)$$

By Hankel's asymptotic expansion (see e.g. in [6], p. 133),

$$J_{d/2}(x)\sim\left(\frac{2}{\pi x}\right)^{1/2}\cdot\left(\cos(x-d\cdot\pi/4-\pi/4)\cdot\sum_{j=0}^{+\infty}(-1)^j\cdot\frac{A_{2j}(d/2)}{x^{2j}}\right.$$
$$\left.-\sin(x-d\cdot\pi/4-\pi/4)\cdot\sum_{j=0}^{+\infty}(-1)^j\cdot\frac{A_{2j+1}(d/2)}{x^{2j+1}}\right) \quad \text{if } x\to+\infty \qquad (26)$$

where

$$A_j(d/2)=\frac{(d^2-1^2)(d^2-3^2)\dots(d^2-(2j-1)^2)}{j!\,8^j}.$$

The asymptotic expansion (26) says that if x is sufficiently large depending on d then $J_{d/2}(x)$ has essentially the form of $x^{-1/2}\cdot\cos(x-d\cdot\pi/4-\pi/4)$. On the other hand, from formula (24) it is easy to see that if x is sufficiently small depending on d then $J_{d/2}(x)$ almost equals $\pi^{-1/2}\cdot\Gamma^{-1}((d+1)/2)\cdot 2^{1-d/2}\cdot x^{d/2}$. Consequently, with (25) we obtain that for any $\varepsilon>0$,

$$\left|g(r,t)-c_{16}(d)\frac{r^{(d-1)/2}}{t^{(d+1)/2}}\cos(r\cdot t-d\pi/4-\pi/4)\right|<\varepsilon\cdot c_{16}(d)\frac{r^{(d-1)/2}}{t^{(d+1)/2}}$$

for $r\cdot t>c_{17}(d,\varepsilon)$ (where $c_{17}>0$ is a "large" constant) (27)

and

$$|g(r,t)-c_{18}(d)\cdot r^d|<\varepsilon\cdot c_{18}(d)\cdot r^d$$

for $r\cdot t<c_{19}(d,\varepsilon)$ (where $c_{19}>0$ is a "small" constant). (28)

Although $g(r,t)$ has the form of a slightly perturbed cosine-function (and so has infinitely many zeros), the quadratic average $h(\rho,t)=\frac{2}{\rho}\int_{\rho/2}^{\rho} g^2(r,t)\,dr$ is already *uniformly large*.

Using (27) and (28) one can easily obtain the following inequality:

$$\frac{h(\omega,t)}{h(\rho,t)} > c_{20}(d)\cdot\left(\frac{\omega}{\rho}\right)^{d-1} \quad \text{uniformly for every } t \geqq 0 \text{ and } 0<\rho\leqq\omega\leqq 1. \tag{29}$$

Next we investigate $\Delta(\rho)$ for small values of ρ. Let $\rho_0 = 2^{-l-1}$ (we recall that $2^{l\cdot d} = C\cdot n$), and assume $r\in[\rho_0/2, \rho_0]$. Let $f(\underline{x}, r) = \sum_{\underline{z}_j \in G(\underline{x}, r)} 1$. Then clearly

$$\int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x} = n\cdot\lambda(G(\underline{0}, r)) \geqq n\cdot\lambda(G(\underline{0}, \rho_0/2)) = \frac{c_{21}(d)}{C}. \tag{30}$$

Since the ball $G(\underline{x}, r)$ intersects $\leqq 2^d$ cubes $I(l, \underline{i})$, by property $(d, n, C, *)$ we know

$$\int_{\mathbb{R}^d} |\beta(G(\underline{x}, r)\cap I^d)|\, d\underline{x} \leqq 2^d\cdot\sum_{k=0}^{+\infty}\frac{2^{k+1}}{c^*(d)\cdot n^{1/2}}\cdot\frac{n}{2^{3k/2}}\cdot 2^{-l\cdot d}$$
$$= \frac{\sqrt{2}}{\sqrt{2}-1}\cdot 2^{d+1}\cdot\frac{1}{c^*(d)\cdot C\cdot n^{1/2}} = \frac{c_{21}(d)}{4\cdot C\cdot n^{1/2}} \tag{31}$$

if $c^*(d) = \frac{\sqrt{2}}{\sqrt{2}-1}\cdot\frac{2^{d+3}}{c_{21}(d)}$ (see (30)).

Let

$$\mathscr{A}(r) = \{\underline{x}\in\mathbb{R}^d : f(\underline{x}, r) > 2n^{1/2}|\beta(G(\underline{x}, r)\cap I^d)|\}.$$

Obviously

$$\int_{\mathscr{A}(r)} f(\underline{x}, r)\, d\underline{x} = \int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x} - \int_{\mathbb{R}^d\setminus\mathscr{A}(r)} f(\underline{x}, r)\, d\underline{x} \geqq \int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x}$$
$$-2n^{1/2}\int_{\mathbb{R}^d\setminus\mathscr{A}(r)} |\beta(G(\underline{x}, r)\cap I^d)|\, d\underline{x} \geqq \int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x} - 2n^{1/2}\int_{\mathbb{R}^d}|\beta(G(\underline{x}, r)\cap I^d)|\, d\underline{x},$$

and by (30) and (31),

$$\int_{\mathscr{A}(r)} f(\underline{x}, r)\, d\underline{x} \geqq \int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x} - 2n^{1/2}\int_{\mathbb{R}^d}|\beta(G(\underline{x}, r)\cap I^d)|\, d\underline{x} \geqq \tfrac{1}{2}\int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x}. \tag{32}$$

Since $\lambda(G(\underline{x}, r)\cap I^d) \leqq \lambda(G(\underline{0}, \rho_0)) \leqq 2^{-l\cdot d} = \frac{1}{C\cdot n} \leqq \frac{1}{4n}$ (note that $C\geqq 4$), and since $f(\underline{x}, r) = \sum_{\underline{z}_j\in G(\underline{x}, r)} 1$ has only *integral values* we conclude that for any $\underline{x}\in\mathscr{A}(r)$

$$f(\underline{x}, r) - n^{1/2}\beta(G(\underline{x}, r)\cap I^d) - n\cdot\lambda(G(\underline{x}, r)\cap I^d) \geqq \tfrac{1}{2}f(\underline{x}, r) - \tfrac{1}{4}f(\underline{x}, r) = \tfrac{1}{4}f(\underline{x}, r). \tag{33}$$

Thus by (18), (33), (32) and (30) $(\rho_0/2 \leqq r \leqq \rho_0)$

$$\int_{\mathbb{R}^d} F_r^2(\underline{x})\, d\underline{x} = \int_{\mathbb{R}^d} (f(\underline{x}, r) - n\cdot\lambda(G(\underline{x}, r)\cap I^d) - n^{1/2}\cdot\beta(G(\underline{x}, r)\cap I^d))^2\, d\underline{x}$$
$$\geqq \int_{\mathscr{A}(r)} (f(\underline{x}, r) - n\cdot\lambda(G(\underline{x}, r)\cap I^d) - n^{1/2}\cdot\beta(G(\underline{x}, r)\cap I^d))^2\, d\underline{x} \geqq \frac{1}{4^2}\int_{\mathscr{A}(r)} f^2(\underline{x}, r)\, d\underline{x}$$
$$\geqq \frac{1}{4^2}\int_{\mathscr{A}(r)} f(\underline{x}, r)\, d\underline{x} \geqq \frac{1}{32}\int_{\mathbb{R}^d} f(\underline{x}, r)\, d\underline{x} \geqq \frac{c_{21}(d)}{32\, C}. \tag{34}$$

Summarizing, by (34) and (22)

$$\Delta(\rho_0)=\frac{2}{\rho_0}\int_{\rho_0/2}^{\rho_0}\left(\int_{\mathbb{R}^d} F_r^2(\underline{x})\,d\underline{x}\right)dr \geqq \frac{c_{21}(d)}{32C}. \tag{35}$$

Now we are ready to end the proof. Combining (22), (29) and (35) we see

$$\begin{aligned}\Delta(\omega)&=\int_{\mathbb{R}^d} h(\omega,|\underline{t}|)\cdot|\phi(\underline{t})|^2\,d\underline{t}\\ &\geqq c_{20}(d)\cdot\left(\frac{\omega}{\rho_0}\right)^{d-1}\cdot\int_{\mathbb{R}^d} h(\rho_0,|\underline{t}|)\cdot|\phi(\underline{t})|^2\,d\underline{t}\\ &=c_{20}(d)\cdot\left(\frac{\omega}{\rho_0}\right)^{d-1}\cdot\Delta(\rho_0)\geqq c_{20}(d)\cdot\left(\frac{\omega}{\rho_0}\right)^{d-1}\cdot\frac{c_{21}(d)}{32C}\\ &=c_{22}(d,C)\cdot(n\cdot\omega^d)^{1-1/d}\end{aligned} \tag{36}$$

(we recall that $\rho_0=2^{-l-1}=(C\cdot n/2)^{-1/d}$).

Since $F_r(\underline{x})=0$ whenever $0\leqq r\leqq 1$ and $\underline{x}\notin[-5/2,5/2]^d$, from (18), (22) and (36) it follows that

$$\begin{aligned}\sup_{G\in BALL(d,\omega)}&\left|\sum_{\underline{z}_j\in G}1-n\cdot\lambda(G\cap I^d)-n^{1/2}\cdot\beta(G\cap I^d)\right|\\ &\geqq(5^{-d}\cdot\Delta(\omega))^{1/2}\geqq c_{23}(d,C)\cdot(n\cdot\omega^d)^{1/2-1/(2d)},\end{aligned}$$

which completes the proof of Lemma 1.

4. Proof of Lemma 2

We shall again use the Fourier analysis. Similarly as in Sect. 3, let $Z(A)=\sum_{\underline{z}_j\in A}1$, $A\subset\mathbb{R}^d$ and $\mu(A)=n\cdot\lambda_0(A)+n^{1/2}\cdot\beta_0(A)$ where $\lambda_0(A)=\lambda(A\cap I^d)$ and $\beta_0(A)=\beta(A\cap I^d)$ for any Lebesgue measurable set $A\subset\mathbb{R}^d$.

For any positive function $Q(\underline{x})$, $\underline{x}\in\mathbb{R}^d$ and for any real α, $1/2\leqq\alpha\leqq 1$ let $Q_\alpha(\cdot)$ be defined by $Q_\alpha(\underline{x})=Q(\underline{x}/\alpha)$, $\underline{x}\in\mathbb{R}^d$. Consider the function

$$F(Q,\alpha;\cdot)=Q_\alpha(\cdot)*(dZ-d\mu)=Q_\alpha(\cdot)*(dZ-n\cdot d\lambda_0-n^{1/2}\cdot d\beta_0). \tag{37}$$

By Parseval-Plancherel identity (21), (37) and (20)

$$\begin{aligned}\Delta(Q)&=2\int_{1/2}^{1}\left(\int_{\mathbb{R}^d}F^2(Q,\alpha;\underline{x})\,d\underline{x}\right)d\alpha=2\int_{1/2}^{1}\left(\int_{\mathbb{R}^d}|\hat{F}(Q,\alpha;\underline{t})|^2\,d\underline{t}\right)d\alpha\\ &=\int_{\mathbb{R}^d}\left(2\int_{1/2}^{1}|\hat{Q}_\alpha(\underline{t})|^2\,d\alpha\right)\cdot|\widehat{(dZ-d\mu)}(\underline{t})|^2\,d\underline{t}=\int_{\mathbb{R}^d}h(Q;\underline{t})\cdot|\phi(\underline{t})|^2\,d\underline{t}\end{aligned} \tag{38}$$

where $h(Q;\underline{t})=2\int_{1/2}^{1}|\hat{Q}_\alpha(\underline{t})|^2\,d\alpha$ and $\phi=\widehat{(dZ-d\mu)}$.

Let $\mathscr{L}=\left\{\underline{l}=(l_1,\ldots,l_d)\colon \prod_{j=1}^{d} 2^{l_j}=C\cdot n, l_j\geqq 0 \text{ integers}\right\}$. For any $\underline{l}\in\mathscr{L}$ let $Q(\underline{l};\underline{x})=\exp\left\{-\sum_{j=1}^{d} 4^{l_j}\cdot x_j^2\right\}$, $x\in\mathbb{R}^d$. Then, as it is well-known,
$\hat{Q}(\underline{l};\underline{t})=\left(\prod_{j=1}^{d} 2^{-l_j}\right)\cdot\exp\left\{-\sum_{j=1}^{d} 4^{-l_j}\cdot t_j^2\right\}$.

Clearly

$$\int_{\mathbb{R}^d} Q(\underline{l};\underline{x})\,d\underline{x}=\int_0^1 \lambda\{\underline{x}\in\mathbb{R}^d\colon Q(\underline{l};\underline{x})\geqq R\}\,dR=\int_0^{+\infty} \lambda\{\underline{x}\in\mathbb{R}^d\colon Q(\underline{l};\underline{x})\geqq e^{-r^2}\}\cdot 2r\cdot e^{-r^2}\,dr$$
$$=\int_0^{+\infty}\lambda\left\{\underline{x}\in\mathbb{R}^d\colon \sum_{j=1}^{d} 4^{l_j}\cdot x_j^2\leqq r^2\right\}\cdot 2r\cdot e^{-r^2}\,dr.$$

Since the level-set $\{\underline{x}\in\mathbb{R}^d\colon Q(\underline{l};\underline{x})\geqq e^{-r^2}\}$ is an *ellipsoid* and belongs to $ELL(d,\underline{l})$, and furthermore

$$\sup_{\underline{x}\in\mathbb{R}^d}\operatorname{card}\{\underline{i}\colon \max_{y\in I(\underline{l};\underline{i})} Q(\underline{l};\underline{x}-\underline{y})\geqq e^{-r^2}\}<(2r+1)^d,$$

by property $(d,n,C,**)$ we obtain for any α, $1/2\leqq\alpha\leqq 1$,

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})|d\beta_0(\underline{y})|\right)d\underline{x}\leqq\int_0^{+\infty}(2r+1)^d\cdot\left\{\sum_{k=0}^{+\infty}\frac{2^{k+1}}{c^{**}(d)\cdot n^{1/2}}\cdot\frac{n}{2^{3k/2}}\cdot\prod_{j=1}^{d}2^{-l_j}\right\}$$
$$\cdot 2r\cdot e^{-r^2}\,dr=\frac{c_{24}(d)}{c^{**}(d)\cdot C\cdot n^{1/2}}. \tag{39}$$

In the last step we used $\prod_{j=1}^{d} 2^{l_j}=C\cdot n$.

Let $\underline{l}\in\mathscr{L}$ and $\alpha\in[1/2,1]$ be fixed. Let

$$f_\alpha(\underline{x};\underline{l})=\sum_{\underline{z}_j} Q_\alpha(\underline{l};\underline{x}-\underline{z}_j)$$

(we recall that $\underline{z}_1,\underline{z}_2,\ldots,\underline{z}_n$ are the given points in I^d), and let

$$\mathscr{A}(\alpha,\underline{l})=\{\underline{x}\in\mathbb{R}^d\colon f_\alpha(\underline{x};\underline{l})\geqq 1/2\}.$$

Then we have (see the definition of $Q(\underline{l};\underline{x})$)

$$\int_{\mathscr{A}(\alpha,\underline{l})} f_\alpha(\underline{x};\underline{l})\,d\underline{x}\geqq n\cdot 1/2\cdot\lambda\{Q_{1/2}(\underline{l};\underline{y})\geqq 1/2\}=\frac{c_{25}(d)}{C}. \tag{40}$$

Choosing $c^{**}(d)=\dfrac{c_{24}(d)}{4c_{25}(d)}$, from (39) and (40) it follows that

$$\int_{\mathbb{R}^d} n^{1/2}\left(\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})|d\beta_0(\underline{y})|\right)d\underline{x}\leqq\tfrac{1}{4}\int_{\mathscr{A}(\alpha,\underline{l})} f_\alpha(\underline{x};\underline{l})\,d\underline{x}. \tag{41}$$

Let

$$\tilde{\mathscr{A}}(\alpha,\underline{l})=\{\underline{x}\in\mathscr{A}(\alpha,\underline{l})\colon f_\alpha(\underline{x};\underline{l})>2n^{1/2}\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,|d\beta_0(\underline{y})|\}.$$

Then from (41) one can easily deduce the inequality

$$\int_{\tilde{\mathscr{A}}(\alpha,\underline{l})} f_\alpha(\underline{x};\underline{l})\,d\underline{x} \geqq \tfrac{1}{2} \int_{\mathscr{A}(\alpha,\underline{l})} f_\alpha(\underline{x},\underline{l})\,d\underline{x}. \tag{42}$$

Since

$$\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,d\lambda_0(\underline{y}) \leqq \int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{y})\,d\underline{y} \leqq \int_{\mathbb{R}^d} Q(\underline{l};\underline{y})\,d\underline{y} = \pi^{d/2}\cdot\prod_{j=1}^{d} 2^{-l_j} = \pi^{d/2}\cdot(C\cdot n)^{-1} \leqq \frac{1}{8n}$$

(note that $C \geqq 8\cdot\pi^{d/2}$), and since $f_\alpha(\underline{x};\underline{l}) \geqq 1/2$ whenever $\underline{x}\in\mathscr{A}(\alpha,\underline{l})$ we conclude for any $\underline{x}\in\tilde{\mathscr{A}}(\alpha,\underline{l})$

$$\begin{aligned} f_\alpha(\underline{x};\underline{l}) - n^{1/2}\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,d\beta_0(\underline{y}) - n\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,d\lambda_0(\underline{y}) \\ \geqq \tfrac{1}{2} f_\alpha(\underline{x};\underline{l}) - \tfrac{1}{4} f_\alpha(\underline{x};\underline{l}) = \tfrac{1}{4} f_\alpha(\underline{x};\underline{l}). \end{aligned} \tag{43}$$

Let (see (37))

$$F(\alpha,\underline{l};\underline{x}) = F(Q(\underline{l};\cdot),\alpha;\underline{x}).$$

Thus by (37), (43), (42) and (40),

$$\begin{aligned} \int_{\mathbb{R}^d} F^2(\alpha,\underline{l};\underline{x})\,d\underline{x} &= \int_{\mathbb{R}^d} \Big(f_\alpha(\underline{x};\underline{l}) - n^{1/2}\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,d\beta_0(\underline{y}) - n\int_{\mathbb{R}^d} Q_\alpha(\underline{l};\underline{x}-\underline{y})\,d\lambda_0(\underline{y})\Big)^2 d\underline{x} \\ &\geqq \frac{1}{4^2}\int_{\tilde{\mathscr{A}}(\alpha,\underline{l})} f_\alpha^2(\underline{x};l)\,dx \geqq \frac{1}{32}\int_{\mathscr{A}(\alpha,\underline{l})} f_\alpha^2(\underline{x};\underline{l})\,d\underline{x} \\ &\geqq \frac{1}{32}\int_{\mathscr{A}(\alpha,\underline{l})} \frac{1}{2} f_\alpha(\underline{x};\underline{l})\,d\underline{x} \geqq \frac{1}{64}\cdot\frac{c_{25}(d)}{C}. \end{aligned} \tag{44}$$

Summarizing, by (38) and (44)

$$\Delta(\underline{l}) = \Delta(Q(\underline{l};\cdot)) = 2\int_{1/2}^{1}\Big(\int_{\mathbb{R}^d} F^2(\alpha,\underline{l};\underline{x})\,d\underline{x}\Big)d\alpha \geqq \frac{c_{25}(d)}{64\,C} \tag{45}$$

for any $\underline{l}\in\mathscr{L}$.

Next let (see (38))

$$h(\underline{l};\underline{t}) = h(Q(\underline{l};\cdot);\underline{t}) \quad \text{and} \quad h(I^d;\underline{t}) = h(\chi_{I^d};\underline{t})$$

where χ_{I^d} denotes the characteristic function of the unit cube I^d. Clearly

$$h(\underline{l};\underline{t}) \leqq c_{26}(d)\cdot|\hat{Q}(\underline{l};t)|^2 = c_{26}(d)\cdot\prod_{j=1}^{d}(2^{-l_j}\cdot\exp\{-4^{-l_j}\cdot t_j^2\})^2,$$

and so we have

$$\begin{aligned} \sum_{\underline{l}\in\mathscr{L}} h(\underline{l};\underline{t}) &\leqq c_{26}(d)\cdot\sum_{\underline{l}\in\mathscr{L}}\prod_{j=1}^{d}(2^{-l_j}\cdot\exp\{-4^{-l_j}\cdot t_j^2\})^2 \\ &\leqq c_{26}(d)\cdot\sum_{\substack{\underline{l}:l_j\geqq 0\\ 1\leqq j\leqq d}}\prod_{j=1}^{d}(2^{-l_j}\cdot\exp\{-4^{-l_j}\cdot t_j^2\})^2 \\ &= c_{26}(d)\cdot\prod_{j=1}^{d}\left\{\sum_{l=0}^{+\infty}(2^{-l}\cdot\exp\{-4^{-l}\cdot t_j^2\})^2\right\}. \end{aligned} \tag{46}$$

It is easy to see that for some positive absolute constant M_0,

$$\sum_{l=0}^{+\infty} (2^{-l} \cdot \exp\{-4^{-l} \cdot t^2\})^2 \leqq \frac{M_0}{1+t^2} \quad \text{for } every \text{ real } t \in \mathbb{R}.$$

Thus by (46),

$$\sum_{\underline{l} \in \mathscr{L}} h(\underline{l}; \underline{t}) \leqq c_{26}(d) \cdot (M_0)^d \cdot \prod_{j=1}^{d} \left(\frac{1}{1+t_j^2}\right) \quad \text{for all } \underline{t} \in \mathbb{R}^d. \tag{47}$$

On the other hand,

$$h(I^d; \underline{t}) = 2 \int_{1/2}^{1} \left(\prod_{j=1}^{d} \frac{2\sin(\alpha \cdot t_{j/2})}{\pi^{1/2} \cdot t_j}\right)^2 d\alpha \geqq c_{27}(d) \cdot \prod_{j=1}^{d} \left(\frac{1}{1+t_j^2}\right). \tag{48}$$

Therefore, by (47) and (48) we see

$$h(I^d; \underline{t}) \geqq c_{28}(d) \cdot \sum_{l \in \mathscr{L}} h(\underline{l}; \underline{t}) \quad \text{uniformly for all } \underline{t} \in \mathbb{R}^d. \tag{49}$$

Now we are in the position to end the proof. Let $\Delta(I^d) = \Delta(\chi_{I^d})$. Combining (38), (45) and (47) we get

$$\begin{aligned} \Delta(I^d) &= \int_{\mathbb{R}^d} h(I^d; \underline{t}) \cdot |\phi(\underline{t})|^2 \, dt \geqq c_{28}(d) \cdot \sum_{\underline{l} \in \mathscr{L}} \int_{\mathbb{R}^d} h(\underline{l}; \underline{t}) \cdot |\phi(\underline{t})|^2 \, d\underline{t} \\ &= c_{28}(d) \cdot \sum_{\underline{l} \in \mathscr{L}} \Delta(\underline{l}) \geqq c_{28}(d) \cdot \frac{c_{25}(d)}{64\,C} \cdot \operatorname{card} \mathscr{L} = c_{29}(d, C) \cdot (\log n)^{d-1}. \end{aligned} \tag{50}$$

In the last step we used the trivial fact that the cardinality of $\mathscr{L}$ is greater than a positive constant multiple of $(\log n)^{d-1}$.

Since $F(\chi_{I^d}, \alpha; \underline{x}) = 0$ whenever $\underline{x} \notin [-1, 2]^d$ and $1/2 \leqq \alpha \leqq 1$ (see (37)), from (37), (38) and (50) it follows that

$$\begin{aligned} \sup_{A \in INT(d)} \Big| \sum_{z_j \in A} 1 - n \cdot \lambda(A) - n^{1/2} \cdot \beta(A) \Big| &\geqq (3^{-d} \cdot \Delta(I^d))^{1/2} \\ &\geqq c_{30}(d, C) \cdot (\log n)^{(d-1)/2} \end{aligned} \tag{51}$$

where $INT(d)$ denotes the class of products of intervals $\subset [0, 1]$ with sides parallel to the coordinate axes.

Finally, using the inclusion-exclusion formula we obtain

$$\begin{aligned} &\sup_{A \in BOX(d)} \Big| \sum_{z_j \in A} 1 - n \cdot \lambda(A) - n^{1/2} \cdot \beta(A) \Big| \\ &\geqq 2^{-d} \cdot \sup_{A \in INT(d)} \Big| \sum_{z_j \in A} 1 - n \cdot \lambda(A) - n^{1/2} \cdot \beta(A) \Big| \geqq c_{31}(d, C) \cdot (\log n)^{(d-1)/2}, \end{aligned}$$

which completes the proof of Lemma 2.

Acknowledgements. We are indebted to G. Halász, P. Major and P. Révész for their valuable remarks. We are especially grateful to G. Tusnády for many useful conversations during this research. The idea of introducing the auxiliary random measure $B^*(\cdot)$ is due to him. Last but not least the author is indebted to the referee who simplified the proof of Lemma 3.

References

1. Bártfai, P.: Über die Entfernung der Irrfahrtswege. Stud. Sci. Math. Hungar. **5**, 41–49 (1970)
2. Beck, J.: On a problem of K.F. Roth concerning irregularities of point distribution. Invent. Math. **74**, 477–487 (1983)
3. Csörgő, M., Révész, P.: A new method to prove Strassen type laws of invariance principle. II. Z. Wahrscheinlichkeitstheor. Verw. Geb. **31**, 261–269 (1975)
4. Dudley, R.M.: Central limit theorems for empirical measures. Ann. Probab. **6**, 899–929 (1978); Correction ibid. **7**, 909–911 (1979)
5. Komlós, J., Major, P., Tusnády, G.: An approximation of partial sums of independent r.v.'s and the sample d.f. Z. Wahrscheinlichkeitstheor. Verw. Geb. **32**, 111–131 (1975)
6. Olver, F.W.J.: Asymptotics and Special Functions. New York and London: Academic Press 1974
7. Révész, P.: On strong approximation of the multidimensional empirical process. Ann. Probab. **4**, 729–743 (1976)
8. Révész, P.: Three theorem of multivariate empirical process. Empirical Distributions and Processes. Lect. Notes Math. **566**, 106–126. Berlin-Heidelberg-New York: Springer 1976
9. Roth, K.F.: On irregularities of distribution. Mathematika **1**, 73–79 (1954)
10. Schmidt, W.M.: Irregularities of distribution. IV. Invent. Math. **7**, 55–82 (1969)
11. Tusnády, G.: A remark on the approximation of the sample DF in the multidimensional case. Period. Math. Hung. **8**, 53–55 (1977)

Received July 2, 1984; in revised form March 1, 1985