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A Law of Large Numbers for Moderately Interacting Diffusion Processes

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Summary. We consider two special models of interacting diffusion processes, and derive in the limit, as the number of different processes tends to infinity and the interaction is rescaled in a suitable ("moderate") way, a law of large numbers for the empirical processes. As limit dynamics we obtain certain nonlinear diffusion equations.

I. Introduction

In the present paper we study the asymptotic behaviour of the time evolution of the empirical distribution of moderately interacting diffusion processes. To explain the term "moderate interaction" and its connections to other forms of interaction used in the modelling of diffusion processes, it seems to be best to begin at once with the introduction of the first of two special models considered in this paper.

For each $N \in \mathbb{N}$ we have $N\ \mathbb{R}^d$-valued diffusion processes $\bar{X}_1^N(t), \ldots, \bar{X}_N^N(t)$, $0 \leqq t < \infty$, which satisfy

$$\begin{aligned} d\bar{X}_l^N(t) &= F\left(\bar{X}_l^N(t), \frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t) - \bar{X}_k^N(t))\right) dt + dW_l(t), \\ \bar{X}_l^N(0) &= \xi_l^N, \\ &\quad l = 1, \ldots, N;\ 0 \leqq t < \infty, \end{aligned} \tag{1.1}$$

where $W_l = (W_l(t))_{t \geqq 0}$, $l \in \mathbb{N}$, are independent $\mathbb{R}^d$-valued Brownian motions, ξ_l^N ($N \in \mathbb{N}$, $l = 1, \ldots, N$) are $\mathbb{R}^d$-valued random variables, and the functions V^N: $\mathbb{R}^d \to \mathbb{R}_+$ and F: $\mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ are continuous and satisfy

i) $V^N(x) = \chi_N^d V^1(\chi_N x)$ for some continuous probability density V^1 on $\mathbb{R}^d$, (1.2)

ii) $\chi_N = N^{\beta/d}$, $\beta \in (0, 1)$, (1.3)

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iii) $|F(x,p)| \leqq c_1 \; (x \in \mathbb{R}^d, p \in \mathbb{R}_+)$ (1.4)

$$|pF(x,p) - p'F(x',p')| + |F(x,p) - F(x',p')| \leqq c_2(|x-x'| + |p-p'|). \tag{1.5}$$

Here and in the sequel we denote by c_1, c_2, ... positive finite constants. If such a constant c_k depends on certain parameters a, b, ... we sometimes write $c_k(a,b,\ldots)$.

The empirical process $X^N = (X^N(t))_{t \geqq 0}$ of the processes $\bar{X}_l^N(t)$, $l=1,\ldots,N$, is defined as the probability measure valued process $X^N(t) = \frac{1}{N} \sum_{l=1}^{N} \delta_{\bar{X}_l^N(t)}$, where δ_a is the Dirac measure concentrated at a.

Our interest lies in the investigation of the behaviour of the dynamics of the processes $t \to X^N(t)$ in the limit $N \to \infty$. For that aim we need to study for any $f \in C_b^2(\mathbb{R}^d)$ (= space of all bounded continuous functions f: $\mathbb{R}^d \to \mathbb{R}$ with bounded continuous partial derivatives of first and second order) the processes $t \to \langle X^N(t), f \rangle = \int_{\mathbb{R}^d} f(x) X^N(t)(dx)$. The dynamics of those processes is obtained from (1.1) and Ito's Formula. Using the abbreviations

$$\begin{aligned} g^N(x,t) &= \frac{1}{N} \sum_{k=1}^{N} V^N(x - \bar{X}_k^N(t)) = \langle X^N(t), V^N(x - .) \rangle \\ &= (X^N(t) * V^N)(x) \end{aligned} \tag{1.6}$$

and

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(x) \, \mu(dx)$$

for any measure μ on $\mathbb{R}^d$ and $f \in C_b(\mathbb{R}^d)$ we have

$$\begin{aligned} &\langle X^N(t), f \rangle - \langle X^N(0), f \rangle - \int_0^t \langle X^N(s), F(., g^N(., s)) \cdot \nabla f + \tfrac{1}{2} \Delta f \rangle \, ds \\ &= \frac{1}{N} \sum_{l=1}^{N} \int_0^t \nabla f(\bar{X}_l^N(s)) \cdot dW_l(s), \\ &f \in C_b^2(\mathbb{R}^d), \; t \in [0, \infty). \end{aligned} \tag{1.7}$$

The right side of (1.7) is a martingale, and so we obtain from Doob's Inequality

$$\begin{aligned} \mathbb{E}&\left[\sup_{t \leqq T} \left(\frac{1}{N} \sum_{k=1}^{N} \int_0^t \nabla f(\bar{X}_k^N(s)) \cdot dW_k(s) \right)^2 \right] \\ &\leqq 4E\left[\left(\frac{1}{N} \sum_{k=1}^{N} \int_0^T \nabla f(\bar{X}_k^N(s)) \cdot dW_k(s) \right)^2 \right] \\ &\leqq \frac{4}{N} T \|\nabla f\|_\infty^2. \end{aligned} \tag{1.8}$$

Therefore in the limit $N \to \infty$, we can neglect this term, and it suffices to study the behaviour of the terms on the left side of (1.7) only.

To get some insight into the structure of the possible limits $X^\infty=(X^\infty(t))_{t\geq 0}$ of the empirical processes $X^N(t)$, and the differences, that appear, when we make another choice for χ_N than that in (1.3), we proceed on a heuristic level.

First we note, that the "strength of the interaction" between two processes $\bar{X}_l^N(t)$ and $\bar{X}_k^N(t)$, $l\neq k$, is measured by $\frac{1}{N}V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$. Let us now consider the case $\beta=0$, i.e. $\chi_N=N^0=1$. Then the strength of the interaction between any two processes is of order $1/N$, while the number of different processes $\bar{X}_k^N(t)$ interacting with one fixed process $\bar{X}_l^N(t)$ is of order N. In analogy with physics [13], we speak in that situation of "weakly" interacting processes $\bar{X}_l^N(t)$, $l=1,\dots,N$. For such processes the asymptotic behaviour in the limit $N\to\infty$ has been studied extensively in the literature (e.g. [1, 9, 10, 14]).

We get the dynamics of the limit process $X^\infty(t)$ by making the formal limit in (1.7).

$$\langle X^\infty(t),f\rangle-\langle X^\infty(0),f\rangle-\int_0^t\langle X^\infty(s),F(.,\int_{\mathbb{R}^d}V^1(.-y)X^\infty(s)(dy))\cdot\nabla f+\tfrac{1}{2}\Delta f\rangle\,ds=0$$
$$f\in C_b^2(\mathbb{R}^d),\quad 0\leq t<\infty. \tag{1.9}$$

Another important case of interest is $\beta=1$, i.e. $\chi_N=N^{1/d}$, where different processes only interact, when their distance is of order $N^{-1/d}$, but then the strength of their interaction is ~ 1. Therefore here we may speak of "strongly" interacting processes. The investigation of the limit $N\to\infty$ presumably will provide similar difficulties as the study of the so-called "hydrodynamic limit" [12].

Our case $\chi_N=N^{\beta/d}$, $\beta\in(0,1)$, obviously describes a situation lying between the "weakly" and "strongly" interacting cases, so that the name "moderate" interaction seems appropriate. Heuristically we can obtain the limit dynamics as follows.

Let us fix some process $t\to\bar{X}_l^N(t)$ and look at the expression $\frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$, which describes the interaction of $\bar{X}_l^N(t)$ with the other processes $\bar{X}_k^N(t)\,(k\neq l)$. Obviously the volume of the "domain of interaction" with $\bar{X}_l^N(t)$, i.e. of that region of space, where the presence of a process $\bar{X}_k^N(t)$ has an influence on the motion of $\bar{X}_l^N(t)$, or where $V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))\not\approx 0$, is of order χ_N^{-d}. If we assume, that all the processes $\bar{X}_k^N(t)$ are distributed "smoothly" over space, then the number of those processes being in the domain of interaction with $\bar{X}_l^N(t)$ is $\sim N\chi_N^{-d}=N^{1-\beta}$. This means, that $\frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$ consists of many ($N^{1-\beta}$) nonvanishing summands, each being small of order $\frac{1}{N}\chi_N^d=N^{\beta-1}$.

Therefore we presume, that in case $\beta\in(0,1)$ a "Law of Large Numbers" holds for $\frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$ (l fixed), i.e. that $\frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$ in the limit $N\to\infty$ behaves more and more deterministically. This fact would imply, that as $N\to\infty$ the processes $\bar{X}_l^N(t)$, $l=1,\dots,N$, evolve independently with deterministic drift vector and diffusion matrix. A look at (1.1), (1.2) and (1.3), which imply

$$\lim_{N\to\infty} V^N(.)=\delta_0 \quad \text{(in the sense of distributions)}, \tag{1.10}$$

shows that the drift vector should be equal to $F(x, g^\infty(x,t))$, where $g^\infty(x,t)$ is the density of the limit $X^\infty(t)=\lim_{N\to\infty} X^N(t)$ of the empirical distributions. More precisely, by (1.7) we expect $X^\infty(t)$ to be absolutely continuous with a density $g^\infty(x,t)$, such that

$$\langle X^\infty(t),f\rangle - \langle X^\infty(0),f\rangle - \int_0^t \langle X^\infty(s), F(.,g^\infty(.,s))\cdot \nabla f + \tfrac{1}{2}\Delta f\rangle\, ds = 0,$$

$$f\in C_b^2(\mathbb{R}^d),\ 0\leqq t<\infty, \tag{1.11}$$

holds.

Such formal arguments of course are possible in the case of weak interaction ($\chi_N=1$) too, and would lead to (1.9) for the limit dynamics. However the situation drastically changes in the strongly interacting case ($\chi_N=N^{1/d}$). Argueing as above, we conclude that the mean number of processes $\bar{X}_k^N(t)$ interacting with a fixed process $\bar{X}_l^N(t)$ is of order 1 and the strength of that interaction is ~ 1 too. In contrast to the weakly or moderately interacting case local fluctuations in the particle density now have a considerable effect on the expression $\frac{1}{N}\sum_{k=1}^N V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))$, and we no longer can expect that a law of large numbers for this expression holds. It seems that the evolution of a fixed process $t\to\bar{X}_l^N(t)$ in the limit $N\to\infty$ looks like that of a process in a rapidly varying random environment.

An essential difference between the weakly and the moderately interacting cases lies in the possible forms of the limit dynamics for the empirical processes. In the weak case the interaction is "nonlocal" uniformly in $N\in\mathbb{N}$, and this nonlocality is preserved in the limit, so that the equation describing the time evolution of the limit $X^\infty(t)$ of the empirical distributions $X^N(t)$ is a nonlinear diffusion equation with an integral term providing the nonlinearity (cf. (1.9)).

In contrast to that, the interaction in the moderately interacting case gets more and more local, and we expect as limit dynamics a nonlinear partial differential equation (cf. (1.11), which is a weak form of such a PDE) and no integro-differential equation. Therefore as far as the limit dynamics is concerned, the moderately interacting case resembles the strongly interacting case, since of course in that situation one also expects a PDE as limit dynamics, provided it exists.

The second model for moderately interacting processes describes a "gradient-system" of diffusions. We assume now, that the processes $\bar{X}_l^N(t)$, $l=1,2,\ldots,N$, satisfy

$$d\bar{X}_l^N(t) = -\frac{1}{2N}\sum_{k=1}^N \nabla V^N(\bar{X}_l^N(t)-\bar{X}_k^N(t))\,dt + dW_l(t),$$

$$\bar{X}_l^N(0)=\xi_l^N, \tag{1.12}$$

where the function V^N is supposed to be of the form

$$V^N(x)=\chi_N^d V^1(\chi_N x) \text{ for some symmetric probability density } V^1\in C_b^1(\mathbb{R}^d) \quad \text{on } \mathbb{R}^d, \tag{1.13}$$

$$\chi_N=N^{\beta/d}, \qquad \beta\in\left(0,\frac{d}{d+2}\right). \tag{1.14}$$

In this situation the empirical process $X^N(t)$ is described by

$$\langle X^N(t),f\rangle-\langle X^N(0),f\rangle-\tfrac{1}{2}\int_0^t\langle X^N(s),(-\nabla g^N(.,s))\cdot\nabla f+\Delta f\rangle\,ds$$
$$=\frac{1}{N}\sum_{k=1}^N\int_0^t \nabla f(\bar{X}_k^N(s))\cdot dW_k(s), \qquad f\in C_b^2(\mathbb{R}^d),\ t\geqq 0. \tag{1.15}$$

where $g^N(.,.)$ is defined by (1.6).

Let us now use some heuristics to derive in this situation too an equation for $X^\infty(t)=\lim\limits_{N\to\infty} X^N(t)$. Again by (1.8) the right side of (1.15) can be neglected in the limit $N\to\infty$. The third expression on the left side of (1.15) represents the interaction between the different processes. Formally we get by (1.10)

$$\langle X^N(t),\nabla g^N(.,t)\cdot\nabla f\rangle=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^N(t)(dx)\,X^N(t)(dy)\,\nabla V^N(x-y)\cdot\nabla f(x)$$
$$\xrightarrow[(N\to\infty)]{}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)\,X^\infty(t)(dy)\,\nabla\delta(x-y)\cdot\nabla f(x)$$
$$=\int_{\mathbb{R}^d} g^\infty(x,t)\,\nabla g^\infty(x,t)\cdot\nabla f(x)\,dx \tag{1.16}$$

(for the density $g^\infty(.,t)$ of $X^\infty(t)$ with respect to Lebesgue measure).

Combining (1.8), (1.15) and (1.16) we formally obtain for $X^\infty(t)$ (resp. its density $g^\infty(x,t)$) the following dynamics.

$$\langle g^\infty(.,t),f\rangle-\langle g^\infty(.,0),f\rangle-\tfrac{1}{2}\int_0^t\langle g^\infty(.,s),(-\nabla g^\infty(.,s))\cdot\nabla f+\Delta f\rangle\,ds=0,$$
$$f\in C_b^2(\mathbb{R}^d),\ t\geqq 0. \tag{1.17}$$

This equation is a weak form of

$$\frac{\partial}{\partial t}g^\infty(x,t)=\tfrac{1}{2}\nabla\cdot[(1+g^\infty(x,t))\,\nabla g^\infty(x,t)]. \tag{1.18}$$

As in the case of (1.11) for the first model, (1.18) indeed describes the limit dynamics of the density $g^\infty(.,t)$ of $X^\infty(t)$.

Let us discuss now the choice (1.14) for β.

Similarly as in the first model $\beta=0$ defines a situation of weak interaction. To understand the cases $\beta>0$ let us assume for the moment, that at some time $t\geqq 0$ the processes $\bar{X}_k^N(t)$, $k=1,2,\ldots,N$, do not "conglomerate" at any point in $\mathbb{R}^d$. For simplicity let us even assume, that the $\bar{X}_k^N(t)$ are i.i.d. with some smooth density $p(.,t)$. Then obviously the "force" $\nabla g^N(x,t)=\frac{1}{N}\sum_{k=1}^{N}\nabla V^N(x-\bar{X}_k^N(t))$ at some point $x\in\mathbb{R}^d$ has mean

$$\int_{\mathbb{R}^d}\nabla V^N(x-y)\,p(y,t)\,dy=\nabla(V^N(.)*p(.,t))(x)$$

and variance

$$\begin{aligned}
&\frac{1}{N^2}\sum_{k=1}^{N}\Big(\int_{\mathbb{R}^d}|\nabla V^N(x-y)|^2 p(y,t)\,dy-\Big(\int_{\mathbb{R}^d}\nabla V^N(x-y)\,p(y,t)\,dy\Big)^2\Big)\\
&=\frac{1}{N}\Big(\int_{\mathbb{R}^d}\chi_N^{2d+2}|\nabla V^1(\chi_N(x-y))|^2 p(y,t)\,dy-(\nabla(V^N(.)*p(.,t))(x))^2\Big)\\
&=\frac{1}{N}\Big(\int_{\mathbb{R}^d}\chi_N^{2d+2}|\nabla V^1(z)|^2 p(x+\chi_N^{-1}z)\,\chi_N^{-d}\,dz-(\nabla(V^N(.)*p(.,t))(x))^2\Big)\\
&=O(N^{-1}\chi_N^{d+2})=O\big(N^{-1+\beta\frac{d+2}{d}}\big),
\end{aligned}$$

which vanishes as $N\to\infty$, if and only if (1.14) is satisfied. So it seems that for $\beta<\frac{d+2}{d}$ the processes $\bar{X}_k^N(t)$ asymptotically will move in a deterministic "force field", whereas for $\beta\geqq\frac{d}{d+2}$ the force field asymptotically will have nonvanishing random fluctuations, whose effect is not clear at present. Thus the value $\beta=\frac{d}{d+2}$ in this gradient system corresponds to $\beta=1$ in the first model.

There is a close relationship between the model (1.12) and the situation studied in [12]. N diffusion processes interacting via (1) (in [12]) and rescaled according to (4) (in [12]), where $\varepsilon=N^{-1/d}$, obey our Eq. (1.12) with $\beta=1$. Indeed it was that relationship, which motivated the present paper. By eliminating some features providing mathematical difficulties (e.g. asymptotic importance of local fluctuations of particle density), we are able to construct models for interacting diffusions, which inherit certain properties of (1.12) with $\beta=1$ (e.g. asymptotic locality of the interaction) and additionally allow the computation of a limit dynamics for the empirical processes.

Let us finally point at an essential difference between the models (1.1) and (1.12). In both cases the interaction between the different components of $X^N(t)$ comes in through a nonlinear drift in the dynamics of $X^N(t)$, but whereas in the first model this nonlinearity remains a part of the drift in the limit, in the other case we obtain asymptotically a nonlinearity in the diffusion coefficient, namely $\frac{1}{2}(1+g^\infty(x,t))$, which is quite different from the diffusion constant $\frac{1}{2}$ of the Brownian motions governing the N-particle-dynamics.

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II. The Main Results

Let us first explain some additional notation, which is needed in the sequel.

For some topological space S we denote by $C_b(S)$ the space of continuous bounded real valued functions on S and by $C_b(S;\mathbb{R}^d)$ the space of all bounded continuous $\mathbb{R}^d$-valued functions on S. Both spaces are equipped with the supremum norm.

$\mathscr{P}(S)$ is the space of probability measures on S. This space is equipped with the usual weak topology, i.e. $\lim_{k\to\infty} \mu_k = \mu$ if and only if

$$\lim_{k\to\infty} \int_S f(x)\,\mu_k(dx) = \int_S f(x)\,\mu(dx) \qquad \forall f \in C_b(S).$$

On the space $\mathscr{P}(\mathbb{R}^d)$ the weak topology is generated by the complete metric

$$|||\mu - \nu|||_1 = \sup_{f\in\mathscr{H}_1} (\langle \mu, f\rangle - \langle \nu, f\rangle),$$

where $\mathscr{H}$ is the set of all $f \in C_b(\mathbb{R}^d)$, which are bounded by 1 and Lipschitz continuous with constant 1 (cf. Theorem 18, [4]).

For any S-valued random variable Y we denote by $\mathscr{L}(Y) \in \mathscr{P}(S)$ its distribution.

For some $T \in (0, \infty)$, which is held fixed throughout the rest of the paper, $\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d))$ is the space of all continuous functions $X = X(t)$, $0 \leqq t \leqq T$ from $[0, T]$ to $\mathscr{P}(\mathbb{R}^d)$, equipped with the metric

$$\rho(X, X') = \sup_{0\leqq t\leqq T} |||X(t) - X'(t)|||_1.$$

Note that the empirical processes $t \to X^N(t)$, $0 \leqq t \leqq T$, are random elements of $\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d))$, so that the distributions $\mathscr{L}(X^N)$ of those processes can be considered as elements of $\mathscr{P}(\mathscr{C}([0, T], \mathscr{P}(\mathbb{R}^d)))$.

If some $\mu \in \mathscr{P}(\mathbb{R}^d)$ has a density $g(x)$ with respect to Lebesgue measure we use

$$\langle \mu, f\rangle = \langle g(.), f\rangle = \int_{\mathbb{R}^d} f(x)\,\mu(dx) \qquad \text{for } f \in C_b(\mathbb{R}^d).$$

Moreover we use the brackets $\langle .,. \rangle$ to denote by $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx$ the scalar product in $L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ we denote by

$$\tilde{f}(\lambda) = \underset{a\to\infty}{\text{l.i.m.}} \left(\frac{1}{2\pi}\right)^{d/2} \int_{\{|x|\leqq a\}} e^{i\lambda\cdot x} f(x)\,dx$$

its Fourier transform.

In connection with Fourier transforms we shall use the relations

$$\int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}^d} \tilde{f}(\lambda)\,\overline{\tilde{g}(\lambda)}\,d\lambda \qquad (f, g \in L^2(\mathbb{R}^d)), \tag{2.1}$$

$$\widetilde{f * g}(\lambda) = (2\pi)^{d/2} \tilde{f}(\lambda)\,\tilde{g}(\lambda) \qquad (f, g \in L^2(\mathbb{R}^d)), \tag{2.2}$$

$$\widetilde{\nabla f}(\lambda) = -i\lambda \tilde{f}(\lambda) \qquad (f \in W_2^1(\mathbb{R}^d)) \tag{2.3}$$

$$\left(W_2^1(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\lambda|^2)\,|\tilde{f}(\lambda)|^2\,d\lambda = \|f\|_2^2 + \|\nabla f\|_2^2 < \infty\}\right).$$

Occasionally we will use $\sigma_p(x) = (2\pi p)^{-d/2} \exp(-x^2/2p)$.

Similar to the definition of $V^N(.)$ (resp. $g^N(.,.)$) let

$$W^N(x) = \chi_N^d\, W^1(\chi_N x)$$

and

$$g_1^N(x,t) = \int_{\mathbb{R}^d} W^N(x-y) X^N(t)(dy) = \langle X^N(t), W^N(x-.)\rangle$$

when W^1 satisfies (2.4) below.

For simplicity we assume, that all our processes and random variables are defined on some common probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leqq t \leqq T}, \mathbb{P})$.

Let us formulate now our results.

Theorem 1. *Let* $\bar{X}_l^N(t)$ *be defined by* (1.1) *and* $X^N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\bar{X}_k^N(t)}$ *be the empirical process of* $\bar{X}_1^N(t), \ldots, \bar{X}_N^N(t)$.

Assume that (1.2)–(1.5) *hold, and moreover that*

i) $V^1(x) = (W^1 * W^1)(x) = \int_{\mathbb{R}^d} W^1(x-y) W^1(y)\, dy$
for some probability density W^1 *on* $\mathbb{R}^d$, (2.4)

ii) $\int_{\mathbb{R}^d} |\widetilde{W^1}(\lambda)|^2 (1 + |\lambda|^\alpha)\, d\lambda < \infty$ *for some* $\alpha > 0$, (2.5)

iii) $\lim_{N \to \infty} \mathscr{L}(X^N(0)) = \delta_{\mu_0}$ *in* $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$, (2.6)

iv) $\sup_{N \in \mathbb{N}} \mathbb{E}[\int_{\mathbb{R}^d} |x| X^N(0)(dx)] < \infty$. (2.7)

Then the sequence $\mathscr{L}(X^N)$, $N \in \mathbb{N}$, *converges as* $N \to \infty$ *to the Dirac measure* $\delta_\mu \in \mathscr{P}(\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d)))$ *concentrated at the uniquely determined* $\mu = \mu(t)$, $0 \leqq t \leqq T$, $\in \mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d))$ *satisfying*

a) $\mu(t)$, $0 < t \leqq T$, *is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^d$ *with density* $p(x,t)$,

b) $\langle p(.,t), f\rangle - \langle \mu_0, f\rangle - \int_0^t \langle p(.,s), F(., p(.,s)) \cdot \nabla f + \frac{1}{2}\Delta f\rangle\, ds = 0$,
$f \in C_b^2(\mathbb{R}^d)$, $0 < t \leqq T$. (2.8)

For the "gradient-system" we have

Theorem 2. *Let* $\bar{X}_k^N(t)$, $k = 1, 2, \ldots, N$, *now be defined by* (1.12) *and let, as in Theorem* 1, $X^N(t)$ *be the empirical process.*

Assume the validity of (1.13), (1.14), *and* (2.4), *where*

i) $W^1(.)$ *is symmetric (i.e.* $W^1(x) = W^1(-x)$)
and has compact support, and (2.9)

ii) $\int_{\mathbb{R}^d} (1 + |\lambda|^2) |\widetilde{W^1}(\lambda)|^2\, d\lambda < \infty$, (2.10)
i.e. W^1 *is differentiable in the* L^2*-sense.*

Next suppose, that in addition to (2.6) *and* (2.7) *we have*

iii) μ_0 *has a density* $p_0(.)$ *with respect to Lebesgue measure which has bounded Hölder continuous (with exponent* h*) partial derivatives of first and second order,* (2.11)

iv) $\sup_{N \in \mathbb{N}} \mathbb{E}[\int_{\mathbb{R}^d} |g_1^N(x,0)|^2\, dx] = \sup_{N \in \mathbb{N}} \mathbb{E}[\|g_1^N(.,0)\|_2^2] < \infty$. (2.12)

Then the sequence $\mathscr{L}(X^N)$, $N \in \mathbb{N}$, *converges as* $N \to \infty$ *to the Dirac measure* $\delta_\mu \in \mathscr{P}(\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d)))$ *concentrated at the uniquely determined* $\mu = \mu(t)$,

$0 \leqq t \leqq T$, which has for any $t \in [0,T]$ a density $p(.,t) \in H^{2+h,1+h/2}(\mathbb{R}^d \times [0,T])$ with respect to Lebesgue measure satisfying

$$\frac{\partial}{\partial t} p(x,t) = \tfrac{1}{2} \nabla \cdot [(1+p(x,t)) \nabla p(x,t)] \tag{2.13}$$

$$p(x,0) = p_0(x). \tag{2.14}$$

(For the definition of $H^{2+h,1+h/2}(\mathbb{R}^d \times [0,T])$ see Remark b below.)

Remark a. By (2.4) and (2.2) we have

$$\widetilde{V^1}(\lambda) = (2\pi)^{d/2}\, \widetilde{W^1}(\lambda)^2. \tag{2.15}$$

(2.15) and (2.5) yield for $|x-y|<1$

$$\begin{aligned} |V^1(x) - V^1(y)| &= \Big| \int_{\mathbb{R}^d} (\exp(-i\lambda x) - \exp(-i\lambda y))\, \widetilde{W^1}(\lambda)^2\, d\lambda \Big| \\ &\leqq \int_{\{|\lambda| \leqq |x-y|^{-1}\}} |\lambda|\, |x-y|\, |\widetilde{W^1}(\lambda)|^2\, d\lambda + 2 \int_{\{|\lambda| > |x-y|^{-1}\}} |\widetilde{W^1}(\lambda)|^2\, d\lambda \\ &\leqq c_3 |x-y|^\alpha, \end{aligned} \tag{2.16}$$

i.e. V^1 is Hölder continuous with exponent α.

If additionally (2.9) is satisfied, then

$$\widetilde{V^1}(\lambda) = (2\pi)^{d/2}\, \widetilde{W^1}(\lambda)^2 \geqq 0, \tag{2.17}$$

since by symmetry $\widetilde{W^1}$ is real valued. Next (2.10) implies

$$V^1(.) \in C_b^2(\mathbb{R}^d), \tag{2.18}$$

because by (2.3)

$$\frac{\partial^2}{\partial x_k \partial x_j} V^1(x) = -\int_{\mathbb{R}^d} \exp(-i\lambda x)\, \lambda_k \lambda_j\, \widetilde{W^1}(\lambda)^2\, d\lambda$$

is absolutely convergent. This representation and (2.17) imply that the matrix $\left(\frac{\partial^2}{\partial x_k \partial x_j} V^1(0)\right)_{1 \leqq k,j \leqq d}$ is negative definite. This means, that the "force" between two different processes is "repelling", at least if the processes approach sufficiently close.

Remark b. By Theorem 8.1, page 494 [8], (2.13), (2.14) has a unique classical solution in the space $H^{2+h,1+h/2}(\mathbb{R}^d \times [0,T])$, where we denote by $H^{2+h,1+h/2}(\mathbb{R}^d \times [0,T])$ the set of all bounded real valued functions on $\mathbb{R}^d \times [0,T]$ having bounded Hölder continuous partial derivatives of first and second order with respect to the spatial variable x and of first order with respect to the time variable t, such that the Hölder exponents are h (with respect to x) and $h/2$ (with respect to t).

Remark c. The result of Theorem 2 would also be true, if W^1 is a positive function with $\int_{\mathbb{R}^d} W^1(x)\,dx = K < \infty$. In that case in (2.13) we would obtain the expression "$1 + K^2 p(x,t)$" instead of "$1 + p(x,t)$".

Remark d. The proofs of both theorems essentially can be divided into three parts.

(i) Relative compactness of the sequence $\mathscr{L}(X^N)$, $N \in \mathbb{N}$.

(ii) Identification of the dynamics of the limit process. All possible limits $X^\infty = X^\infty(t)$, $0 \leqq t \leqq T$, are shown to be almost surely a solution of a certain deterministic equation.

(iii) Uniqueness of the solution of the deterministic equation.

But since the models are so different, there are only in (i) and (ii) some parallel steps in both proofs.

Quite different methods are used for (iii). Both proofs have in common the use of L^2-techniques, especially in (ii), where Fourier transforms are used to study the properties of the functions $(x,t) \to g^N(x,t)$ as random elements of $L^2(\mathbb{R}^d \times [0,T])$. In both cases part (iii) provides many difficulties, since (ii) does not yield enough information (e.g. regularity properties) on the limit processes. Consequently we could not use existing results on nonlinear diffusion equations very effectively.

Remark e. It should be possible to show, that the limit processes (2.8), resp. (2.13), (2.14) can be obtained in the same way as in [2], where Calderoni and Pulvirenti derived "Burgers Equation". Namely we first could replace the N-dependent potential $V^N(.)$ in (1.2) or (1.13) by $V^K(.)$, where $K \in (0,\infty)$ is some fixed number, which is independent of N. By making the limit $N \to \infty$ for each fixed K, which is a limit of a system of weakly interacting processes, we obtain a family $X^{\infty,K} = X^{\infty,K}(t)$, $0 \leqq t \leqq T$, $K \in (0,\infty)$, of deterministic measure valued processes satisfying

$$\langle X^{\infty,K}(t), f\rangle - \langle \mu_0, f\rangle - \int_0^t \langle X^{\infty,K}(s), F(., \int_{\mathbb{R}^d} V^K(.-y)\, X^{\infty,K}(s)(dy)) \cdot \nabla f + \tfrac{1}{2}\Delta f\rangle\, ds = 0$$

resp.

$$\langle X^{\infty,K}(t), f\rangle - \langle \mu_0, f\rangle - \int_0^t \langle X^{\infty,K}(s), (-\tfrac{1}{2}\int_{\mathbb{R}^d} \nabla V^K(.-y)\, X^{\infty,K}(s)(dy)) \cdot \nabla f + \tfrac{1}{2}\Delta f\rangle\, ds = 0,$$

$$f \in C_b^2(\mathbb{R}^d),\quad 0 \leqq t \leqq T.$$

Then we may conclude, that $\lim_{K\to\infty} \sup_{t \leqq T} \| X^{\infty,K}(t) - p(.,t)\|_1 = 0$, where $p(.,.)$ is the solution of (2.8), resp. (2.13), (2.14). Thus the results in this paper show, how one can let K and N simultaneously tend to infinity.

Remark f. (2.8) is a weak form of the following PDE

$$\frac{\partial}{\partial t}\bar{p}(x,t) = -\nabla \bar{p}(x,t) \cdot F(x,\bar{p}(x,t)) - \bar{p}(x,t)\, \nabla \cdot F(x,\bar{p}(x,t)) + \tfrac{1}{2}\Delta \bar{p}(x,t). \quad (2.19)$$

The solution $p(.,.)$ of (2.8) is an element of $H^{2+h^*, 1+h^*/2}(\mathbb{R}^d \times [0,T])$ for some $h^* > 0$ ($H^{2+h^*,1+h^*/2}(\mathbb{R}^d \times [0,T])$ is defined in Remark b), and therefore a classical solution of (2.19), if μ_0 has a density $p(.,0)$ with respect to Lebesgue measure having bounded and Hölder continuous partial derivatives of first and second order, and if the function $(x,p)\to F(x,p)$ in addition to (1.2), (1.3) has Hölder continuous partial derivatives of first order.

To prove this we can first use the technique of Propositions 3.4 and 3.5 to show, that $p(.,.)$ is bounded and Hölder continuous uniformly in $\mathbb{R}^d \times [0,T]$, and then apply Theorem 5.1, page 320 [8].

An example for (2.19) is Burgers Equation, which is studied as a special application at the end of Chap. III.

III. Proof of Theorem 1

A. We begin with the main line of the proof, and present the proofs of several propositions needed there in part B of this chapter.

As the first point we need

Proposition 3.1. *The sequence $\mathscr{L}(X^N)$, $N\in\mathbb{N}$, is relatively compact in $\mathscr{P}(\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d)))$.*

This proposition can be proved by now well-known methods, where the most elegent technique seems to be that of A.S. Sznitman [14]. Nevertheless for completeness we later present another proof. Moreover some partial results of that proof are needed in the sequel.

Proposition 3.1 implies the existence of a subsequence $\mathscr{N}=\{N_k: N_1<N_2<\dots\}\subseteq\mathbb{N}$, such that the sequence $\mathscr{L}(X^{N_k})$, $k\in\mathbb{N}$, converges in $\mathscr{P}(\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d)))$ to some limit $\mathscr{L}^*=\mathscr{L}(X^\infty)$, which is the distribution of some process $X^\infty=X^\infty(t)$, $0\leqq t\leqq T$, with trajectories in $\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d))$. Since we shall be concerned with the properties of the possible limits X^∞, especially such properties, which characterize X^∞ uniquely, we may assume for simplicity that $\mathscr{N}=\mathbb{N}$. Furthermore by a theorem of Skorokhod (Theorem 2.7, p. 9 [7]), we are allowed, eventually after choosing a suitable probability space $(\Omega,\mathscr{F},\mathbb{P})$, where all our random variables can be defined, to assume

$$\lim_{N\to\infty}\sup_{t\leqq T}|||X^N(t)-X^\infty(t)|||_1=0 \quad \mathbb{P}\text{-a.s.} \tag{3.1}$$

In the case of weak interaction ($\chi_N=1$) the convergence of Proposition 3.1, or more precisely of (3.1), is sufficient for characterizing the limit process $X^\infty(t)$.

Indeed by (1.8), (3.1) and since $V^1\in C_b(\mathbb{R}^d)$ in that situation Eq. (1.7) converges to (1.9) in probability as $N\to\infty$. But in the case of moderate interaction (3.1) is too weak, since we get no control over the asymptotic behaviour of

$$\int_0^t \langle X^N(s), F(.,g^N(.,s))\cdot\nabla f\rangle\,ds.$$

To obtain some information about such expressions we need to study the "smoothness-properties" of the functions $(x,t)\to g_1^N(x,t)=\langle X^N(t), W^N(x-.)\rangle$,

$N \in \mathbb{N}$, resp. the moments of its Fourier transforms

$$\widetilde{g_1^N}(\lambda, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g_1^N(x,t)\, e^{i\lambda\cdot x}\, dx = \left(\frac{1}{N}\sum_{k=1}^{N} e^{i\lambda\cdot \bar{X}_k^N(t)}\right) \widetilde{W^N}(\lambda)$$
$$= \langle X^N(t), e^{i\lambda\cdot}\rangle\, \widetilde{W^N}(\lambda).$$

The following proposition provides a first useful result in this direction.

Proposition 3.2. *For any* $\alpha_1 \in [0, (1-\beta)\wedge\alpha)$ *and* $t>0$ *we have*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\int_{\mathbb{R}^d} (1+|\lambda|^{\alpha_1})\, |\widetilde{g_1^N}(\lambda,t)|^2\, d\lambda\Big] \leqq c_4 (t^{-(d+1)/2}+1). \tag{3.2}$$

Now from (3.2) and

$$\lim_{N,N'\to\infty} \sup_{t\leqq T} \sup_{|\lambda|\leqq K} |\langle X^N(t), e^{i\lambda\cdot}\rangle - \langle X^{N'}(t), e^{i\lambda\cdot}\rangle| = 0 \qquad \forall K>0 \quad \mathbb{P}\text{-a.s.}, \tag{3.3}$$

which follows from (3.1), we obtain for any $\delta>0$

$$\lim_{N,N'\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\mathbb{R}^d} |g_1^N(x,t) - g_1^{N'}(x,t)|^2\, dx\, dt\right]$$
$$= \lim_{N,N'\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\mathbb{R}^d} |\widetilde{g_1^N}(\lambda,t) - \widetilde{g_1^{N'}}(\lambda,t)|^2\, d\lambda\, dt\right] \quad \text{(by (2.1))}$$
$$\leqq \lim_{N,N'\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\{|\lambda|\leqq K\}} |\widetilde{g_1^N}(\lambda,t) - \widetilde{g_1^{N'}}(\lambda,t)|^2\, d\lambda\, dt\right]$$
$$+2 \lim_{N,N'\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\{|\lambda|>K\}} (|\widetilde{g_1^N}(\lambda,t)|^2 + |\widetilde{g_1^{N'}}(\lambda,t)|^2)\, d\lambda\, dt\right]$$
$$\leqq \lim_{N,N'\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\{|\lambda|\leqq K\}} |\langle X^N(t), e^{i\lambda\cdot}\rangle - \langle X^{N'}(t), e^{i\lambda\cdot}\rangle|^2\, d\lambda\, dt\right]$$
$$+4(1+K^{\alpha_1})^{-1} \sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_\delta^T \int_{\mathbb{R}^d} |\widetilde{g_1^N}(\lambda,t)|^2 (1+|\lambda|^{\alpha_1})\, d\lambda\, dt\right] \quad (\text{since } |\widetilde{W^N}(\lambda)|\leqq 1)$$
$$\leqq 4c_4(1+K^{\alpha_1})^{-1}(\delta^{-(d+1)/2}+1)T. \tag{3.4}$$

Since for fixed $\delta>0$ this can be made smaller than any given $\varepsilon>0$, we have proved the existence of a positive (random) function $g_1^\infty(x,t)$ with

$$\lim_{N\to\infty} \mathbb{E}\left[\int_\delta^T \int_{\mathbb{R}^d} |g_1^N(x,t) - g_1^\infty(x,t)|^2\, dx\, dt\right] = 0 \qquad \forall \delta>0. \tag{3.5}$$

By (3.1) and (1.10) we have

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} f(x)\, g_1^N(x,t)\, dx = \int_{\mathbb{R}^d} f(x)\, X^\infty(t)(dx), \qquad f\in C_b^2(\mathbb{R}^d),\ 0\leqq t\leqq T \quad \mathbb{P}\text{-a.s.}, \tag{3.6}$$

and so by (3.5) we conclude, that $X^\infty(t)$ is absolutely continuous with respect to Lebesgue measure with density $g_1^\infty(.,t)$ for almost all $t\in(0,T]$ $\mathbb{P}$-a.s.

Let $\mathcal{M}_T^d$ be the set of $\mathbb{R}^d$-valued measures (set functions) on $\mathbb{R}^d\times[0,T]$ of finite total variation, equipped with the weak topology. Then we can prove the following corollary of the proof of Proposition 3.1.

Proposition 3.3. *The distributions $\mathscr{L}(v^N)$, $N\in\mathbb{N}$, of the $\mathscr{M}_T^d$-valued random variables v^N, which are defined by*

$$\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^N(dx,dt)=\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot F(x,g^N(x,t))\,X^N(t)(dx)\,dt,$$
$$f\in C_b(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$$

are relatively compact in $\mathscr{P}(\mathscr{M}_T^d)$.

Similarly as in (3.1), we can assume, that the sequence v^N, $N\in\mathbb{N}$, satisfies

$$\lim_{N\to\infty}\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^N(dx,dt)=\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^\infty(dx,dt),$$
$$f\in C_b(\mathbb{R}^d\times[0,T];\mathbb{R}^d)\quad \mathbb{P}\text{-a.s.} \tag{3.7}$$

for some $\mathscr{M}_T^d$-valued random variable v^∞.

Since for any $f\in C_b(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$

$$\left|\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^\infty(dx,dt)\right|=\lim_{N\to\infty}\left|\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^N(dx,dt)\right|$$
$$\leqq \lim_{N\to\infty}\int_{\mathbb{R}^d}\int_0^T |f(x,t)|\,c_1\,X^N(t)(dx)\,dt \quad \text{(by (1.4))}$$
$$=c_1\int_{\mathbb{R}^d}\int_0^T |f(x,t)|\,X^\infty(t)(dx)\,dt \quad \mathbb{P}\text{-a.s.}\quad \text{(by (3.1))},$$

we may conclude, that $v^\infty(dx,dt)\ll X^\infty(t)(dx)\,dt$ $\mathbb{P}$-a.s., and that there exists a random function F^∞: $\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ with

$$\sup_{t\leqq T,x\in\mathbb{R}^d}|F^\infty(x,t)|\leqq c_1 \tag{3.8}$$

and

$$\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot v^\infty(dx,dt)=\int_{\mathbb{R}^d}\int_0^T f(x,t)\cdot F^\infty(x,t)\,X^\infty(t)(dx)\,dt,$$
$$f\in C_b(\mathbb{R}^d\times[0,T];\mathbb{R}^d)\quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

Let $|v^\infty|$ be the variation of the $\mathbb{R}^d$-valued measure v^∞. Then by (3.9) $|v^\infty|(\mathbb{R}^d\times\{t\})=0$ $\forall t\in[0,T]$, and therefore by (3.7)

$$\lim_{N\to\infty}\int_0^t \langle X^N(s), F(.,g^N(.,s))\cdot\nabla f\rangle\,ds$$
$$=\lim_{N\to\infty}\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot v^N(dx,ds)$$
$$=\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot v^\infty(dx,ds)$$
$$=\int_0^t\langle X^\infty(s),F^\infty(.,s)\cdot\nabla f\rangle\,ds,\qquad f\in C_b^2(\mathbb{R}^d),\ 0\leqq t\leqq T\quad \mathbb{P}\text{-a.s.} \tag{3.10}$$

(3.10), (2.6), (3.1), (1.7) and (1.8) yield

$$\langle X^\infty(t),f\rangle-\langle\mu_0,f\rangle-\int_0^t\langle X^\infty(s),F^\infty(.,s)\cdot\nabla f+\tfrac{1}{2}\Delta f\rangle\,ds=0,$$
$$f\in C_b^2(\mathbb{R}^d),\ 0\leqq t\leqq T\ \ \mathbb{P}\text{-a.s.,} \tag{3.11}$$

i.e. the limit process $t\to X^\infty(t)$ of the sequence $X^N(t)$, $N\in\mathbb{N}$, is $\mathbb{P}$-a.s. a solution of the system (3.11) of integral equations. To obtain more information about $X^\infty(t)$, $0\leqq t\leqq T$, we now investigate systems like (3.11), which obviously is a weak form of a diffusion equation with drift vector $F^\infty(x,t)$ and the identity matrix as diffusion matrix.

Proposition 3.4. *Let F_1: $\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ be bounded and measurable, $\mu_0\in\mathscr{P}(\mathbb{R}^d)$. Then any solution $t\to X(t)\in\mathscr{P}(\mathbb{R}^d)$, $0\leqq t\leqq T$, of the system*

$$\langle X(t),f\rangle-\langle\mu_0,f\rangle-\int_0^t\langle X(s),F_1(.,s)\cdot\nabla f+\tfrac{1}{2}\Delta f\rangle\,ds=0,$$
$$f\in C_b^2(\mathbb{R}^d),\ 0\leqq t\leqq T, \tag{3.12}$$

has for any $t>0$ a density $g(x,t)$ with respect to Lebesgue measure on $\mathbb{R}^d$, which satisfies

$$\|g(.,t)\|_\infty\leqq c_5(t^{-d/2}+1) \tag{3.13}$$
$$|g(y,t)-g(y',t)|\leqq c_6(t^{-(d+1)/2}+1)\,|y-y'|^{1/2} \tag{3.14}$$
$$|g(y,t)-g(y,s)|\leqq c_7((t\wedge s)^{-(d+1)/2}+1)\,|t-s|^{1/5}. \tag{3.15}$$

By (3.11) and this proposition $X^\infty(t)$ has a density $g^\infty(x,t)$ with respect to Lebesgue measure for all $t>0$, which in any set

$\mathbb{R}^d\times[\delta,T]$, $\delta>0$, is uniformly bounded and Hölder continuous with exponent 1/5 $\mathbb{P}$-a.s. (3.16)

By (3.6) and (3.16) we can assume for the sequel

$$g^\infty(x,t)=g_1^\infty(x,t)\qquad\forall\,x\in\mathbb{R}^d,\ \forall\,t\in(0,T]\quad\mathbb{P}\text{-a.s.} \tag{3.17}$$

Let us fix now $0<s\leqq t\leqq T$ and $f\in C_b^2(\mathbb{R}^d)$. Then

$$\mathbb{E}\left[\left|\int_s^t\langle X^N(u),F(.,g^N(.,u))\cdot\nabla f\rangle\,du-\int_s^t\langle X^N(u),F(.,(g^\infty(.,u)*W^N)(.))\cdot\nabla f\rangle\,du\right|\right]$$
$$\leqq\mathbb{E}\left[\int_s^t\langle X^N(u),c_2\,|(g_1^N(.,u)*W^N)(.)-(g^\infty(.,u)*W^N)(.)|\,|\nabla f|\rangle\,du\right]$$

(by (1.5) and since $g^N(.,u)=(g_1^N(.,u)*W^N)(.)$)

$$\leqq c_2\,\|\nabla f\|_\infty\,\mathbb{E}\left[\int_s^t\left\langle\int_{\mathbb{R}^d}W^N(y-.)\,X^N(u)(dy),\,|g_1^N(.,u)-g^\infty(.,u)|\right\rangle du\right]$$
$$=c_2\,\|\nabla f\|_\infty\,\mathbb{E}\left[\int_{\mathbb{R}^d}\int_s^t g_2^N(x,u)\,|g_1^N(x,u)-g^\infty(x,u)|\,dx\,du\right]$$
$$\leqq c_2\,\|\nabla f\|_\infty\,\mathbb{E}\left[\left(\int_s^t\int_{\mathbb{R}^d}|g_2^N(x,u)|^2\,dx\,du\right)^{1/2}\left(\int_s^t\int_{\mathbb{R}^d}|g_1^N(x,u)-g^\infty(x,u)|^2\,dx\,du\right)^{1/2}\right]$$

with $g_2^N(x,t)=\langle X^N(t), W^N(.-x)\rangle$. Since $\widetilde{g_2^N}(\lambda,t)=\langle X^N(t), e^{i\lambda\cdot}\rangle\,\overline{\widetilde{W^N}(\lambda)}$ and $\widetilde{g_1^N}(\lambda,t)=\langle X^N(t), e^{i\lambda\cdot}\rangle\,\widetilde{W^N}(\lambda)$ we have $\|g_1^N(.,t)\|_2=\|g_2^N(.,t)\|_2$, $t>0$. Therefore and by (3.2), (3.5) and (3.17) we obtain

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}\Bigg[\Bigg|&\int_s^t\langle X^N(u), F(.,g^N(.,u))\cdot\nabla f\rangle\,du\\&-\int_s^t\langle X^N(u), F(.,(g^\infty(.,u)*W^N)(.))\cdot\nabla f\rangle\,du\Bigg|\Bigg]=0,\\&0<s\leqq t\leqq T,\ f\in C_b^2(\mathbb{R}^d).\end{aligned}\tag{3.18}$$

(1.5), (3.1) and (3.16) imply

$$\begin{aligned}\lim_{N\to\infty}&\int_s^t\langle X^N(u), F(.,(g^\infty(.,u)*W^N)(.))\cdot\nabla f\rangle\,du\\&=\int_s^t\langle X^\infty(u), F(.,g^\infty(.,u))\cdot\nabla f\rangle\,du,\\&\quad 0<s\leqq t\leqq T,\ f\in C_b^2(\mathbb{R}^d)\quad \mathbb{P}\text{-a.s.}\end{aligned}\tag{3.19}$$

Furthermore we have

$$\begin{aligned}&\mathbb{E}\left[\left|\langle X^\infty(t),f\rangle-\langle X^\infty(s),f\rangle-\int_s^t\langle X^\infty(u), F(.,g^\infty(.,u))\cdot\nabla f+\tfrac12\Delta f\rangle\,du\right|\right]\\
&\leqq\mathbb{E}\Bigg[|\langle X^\infty(t),f\rangle-\langle X^N(t),f\rangle|+|\langle X^\infty(s),f\rangle-\langle X^N(s),f\rangle|\\
&\quad+\tfrac12\int_s^t|\langle X^\infty(u),\Delta f\rangle-\langle X^N(u),\Delta f\rangle|\,du+\Bigg|\int_s^t\langle X^\infty(u), F(.,g^\infty(.,u))\cdot\nabla f\rangle\,du\\
&\quad-\int_s^t\langle X^N(u), F(.,(g^\infty(.,u)*W^N)(.))\cdot\nabla f\rangle\,du\Bigg|\\
&\quad+\Bigg|\int_s^t\langle X^N(u), F(.,(g^\infty(.,u)*W^N)(.))\cdot\nabla f\rangle\,du\\
&\quad-\int_s^t\langle X^N(u), F(.,g^N(.,u))\cdot\nabla f\rangle\,du\Bigg|\\
&\quad+\left|\langle X^N(t),f\rangle-\langle X^N(s),f\rangle-\int_s^t\langle X^N(u), F(.,g^N(.,u))\cdot\nabla f+\tfrac12\Delta f\rangle\,du\right|\Bigg].\end{aligned}$$

By (1.7), (1.8), (3.1), (3.18), (3.19) and Lebesgue's Bounded Convergence Theorem the left side of this inequality tends to 0 as $N\to\infty$ for all $0<s\leqq t\leqq T$ and $f\in C_b^2(\mathbb{R}^d)$. Therefore

$$\begin{aligned}\langle X^\infty(t),f\rangle-\langle X^\infty(s),f\rangle-\int_s^t\langle X^\infty(u), F(.,g^\infty(.,u))\cdot\nabla f+\tfrac12\Delta f\rangle\,du=0\quad\mathbb{P}\text{-a.s.}\\0<s\leqq t\leqq T,\ f\in C_b^2(\mathbb{R}^d)\end{aligned}\tag{3.20}$$

(3.20), (2.6), (1.4), and the fact, that

$$t\to X^\infty(t)\in\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d)),$$

imply

$$\langle X^\infty(t), f\rangle - \langle \mu_0, f\rangle - \int_0^t \langle X^\infty(s), F(.,g^\infty(.,s))\cdot \nabla f + \tfrac{1}{2}\Delta f\rangle\, ds = 0,$$

$$f\in C_b^2(\mathbb{R}^d),\ 0\le t\le T \quad \mathbb{P}\text{-a.s.}, \tag{3.21}$$

i.e. $\mathbb{P}$-almost all trajectories $t\to X^\infty(t)$ are solutions of (2.8). So to finish the proof of the theorem, we only need

Proposition 3.5. *The system* (2.8) *of integral equations has a unique solution.*

B. We finish the proof of Theorem 1 by proving Propositions 3.1–3.5.

a) Proof of Proposition 3.1. We essentially follow the lines of the proof of the compactness result in [10].

First we note, that by (1.7) for $0\le s\le t\le T$ and any positive function $\varphi(.)$ in $C^2(\mathbb{R}^d)$ with

$$\varphi(x)=|x| \quad \text{for } |x|\ge 1 \text{ and } \|\nabla\varphi\|_\infty + \|\Delta\varphi\|_\infty < \infty \tag{3.22}$$

we have

$$\mathbb{E}[\langle X^N(t),\varphi\rangle|\mathscr{F}_s] = \langle X^N(s),\varphi\rangle + \mathbb{E}\left[\int_s^t \langle X^N(v), F(.,g^N(.,v))\cdot\nabla\varphi + \tfrac{1}{2}\Delta\varphi\rangle\, dv|\mathscr{F}_s\right]$$

$$\le \langle X^N(s),\varphi\rangle + \mathbb{E}\left[\int_s^t \langle X^N(v), c_8\rangle\, dv|\mathscr{F}_s\right] \quad \text{(by (1.4) and (3.22))}$$

$$= \langle X^N(s),\varphi\rangle + c_8(t-s). \tag{3.23}$$

This means, that the process

$$t\to\langle X^N(t),\varphi\rangle - c_8 t \quad \text{is a supermartingale.} \tag{3.24}$$

Similar to (3.23) we obtain

$$\mathbb{E}[\langle X^N(t),\varphi\rangle|\mathscr{F}_s] \ge \langle X^N(s),\varphi\rangle - c_8(t-s),$$

so that we obtain

$$\text{the submartingale property of} \quad t\to\langle X^N(t),\varphi\rangle + c_8 t. \tag{3.25}$$

Using the semimartingales of (3.24) and (3.25) we get for any set $B_\lambda^c = \{x\in\mathbb{R}^d: |x|>\lambda\}$, $\lambda>1$ and $\delta>0$

$$\mathbb{P}[\sup_{t\le T}\langle X^N(t), \mathbb{1}_{B_\lambda^c}\rangle > \delta] \le \mathbb{P}[\sup_{t\le T}(\langle X^N(t),\varphi\rangle + c_8 t) > \lambda\delta]$$

$$\le \frac{1}{\lambda\delta}\mathbb{E}[\langle X^N(T),\varphi\rangle + c_8 T]$$

(by Doob's Inequality and (3.25))

$$\le \frac{1}{\lambda\delta}\mathbb{E}[\langle X^N(0),\varphi\rangle + 2c_8 T] \quad \text{(by (3.23))}$$

$$\le c_9/\lambda\delta \quad \text{(by (2.7)).} \tag{3.26}$$

Let us take now $\varepsilon>0$ and two sequences μ_k and δ_k of positive numbers such that $\sum_{k=1}^{\infty} \mu_k=\varepsilon$ and $\delta_k \searrow 0$. Let $\lambda_k=\frac{c_9}{\mu_k \delta_k} \to \infty$. Then (3.26) yields

$$\begin{aligned} \mathbb{P}[\sup_{t\leqq T} \langle X^N(t), \mathbb{1}_{B^c_{\lambda_k}}\rangle \geqq \delta_k \text{ for any } k\in\mathbb{N}] &\leqq \sum_{k=1}^{\infty} \mathbb{P}[\sup_{t\leqq T}\langle X^N(t), \mathbb{1}_{B^c_{\lambda_k}}\rangle \geqq \delta_k] \\ &\leqq \sum_{k=1}^{\infty} \frac{c_9}{\lambda_k \delta_k} = \sum_{k=1}^{\infty} \mu_k = \varepsilon. \end{aligned} \tag{3.27}$$

By Prohorov's Theorem we know, that the set

$$Q=\{\mu\in\mathscr{P}(\mathbb{R}^d)\colon \langle \mu, \mathbb{1}_{B^c_{\lambda_k}}\rangle \leqq \delta_k\ \forall k\in\mathbb{N}\}$$

is compact in $\mathscr{P}(\mathbb{R}^d)$. So (3.27) implies:

For all $\varepsilon>0$ there exists a compact set $K_\varepsilon \subseteq \mathscr{P}(\mathbb{R}^d)$ with

$$\inf_{N\in\mathbb{N}} \mathbb{P}[X^N(t)\in K_\varepsilon\ \forall t\in[0,T]] \geqq 1-\varepsilon. \tag{3.28}$$

Furthermore we have for $0\leqq s\leqq t\leqq T$

$$\begin{aligned} |||X^N(t)-X^N(s)|||_1^4 &= \sup_{f\in\mathscr{H}_1} (\langle X^N(t), f\rangle - \langle X^N(s), f\rangle)^4 \\ &= \sup_{f\in\mathscr{H}_1} \left(\frac{1}{N}\sum_{k=1}^N (f(\bar{X}_k^N(t)) - f(\bar{X}_k^N(s)))\right)^4 \\ &\leqq \left(\frac{1}{N}\sum_{k=1}^N |\bar{X}_k^N(t)-\bar{X}_k^N(s)|\right)^4 \\ &\leqq \frac{1}{N}\sum_{k=1}^N |\bar{X}_k^N(t)-\bar{X}_k^N(s)|^4 \\ &= \frac{1}{N}\sum_{k=1}^N \left|\int_s^t F(\bar{X}_k^N(u), g^N(\bar{X}_k^N(u),u))\,du + \int_s^t dW_k(u)\right|^4 \quad \text{(by (1.1))} \\ &\leqq c_{10}\left(\frac{1}{N}\sum_{k=1}^N \left|\int_s^t F(\bar{X}_k^N(u), g^N(\bar{X}_k^N(u),u))\,du\right|^4 + \frac{1}{N}\sum_{k=1}^N |W_k(t)-W_k(s)|^4\right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}[|||X^N(t)-X^N(s)|||_1^4] &\leqq c_{11}(c_1^4|t-s|^4 + 3|t-s|^2) \quad \text{(by (1.4))} \\ &\leqq c_{12}|t-s|^2. \end{aligned} \tag{3.29}$$

From (3.28), (3.29) and [5] (Chap. VI, §4) we conclude the validity of Proposition 3.1.

b) Proof of Proposition 3.2. We begin with an estimate for $\mathbb{E}[\int_{\mathbb{R}^d} |g_1^N(x,t)|^2\,dx]$, which by (2.1) equals $\mathbb{E}[\int_{\mathbb{R}^d} |\widetilde{g_1^N}(\lambda,t)|^2\,d\lambda]$.

By Ito's Formula for all $\lambda \in \mathbb{R}^d$ and $T' \leqq T$ the process

$$\begin{aligned}
t \to & |\langle X^N(t), e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-T')} \\
& -\int_0^t \Big(\Big(\langle X^N(s), e^{-i\lambda\cdot}\rangle \Big\langle X^N(s), F(.,g^N(.,s))\cdot(i\lambda)e^{i\lambda\cdot} - \frac{\lambda^2}{2} e^{i\lambda\cdot}\Big\rangle \\
& + \langle X^N(s), e^{i\lambda\cdot}\rangle \Big\langle X^N(s), F(.,g^N(.,s))\cdot(-i\lambda)e^{-i\lambda\cdot} - \frac{\lambda^2}{2} e^{-i\lambda\cdot}\Big\rangle \Big) e^{\lambda^2(s-T')} \\
& + |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2 \lambda^2 e^{\lambda^2(s-T')} + \frac{1}{N}\lambda^2 e^{\lambda^2(s-T')} \Big) ds \\
= & |\langle X^N(t), e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-T')} \\
& -\int_0^t \Big((\langle X^N(s), e^{-i\lambda\cdot}\rangle \langle X^N(s), F(.,g^N(.,s))\cdot(i\lambda)e^{i\lambda\cdot}\rangle \\
& + \langle X^N(s), e^{i\lambda\cdot}\rangle \langle X^N(s), F(.,g^N(.,s))\cdot(-i\lambda)e^{-i\lambda\cdot}\rangle) e^{\lambda^2(s-T')} \\
& + \frac{1}{N}\lambda^2 e^{\lambda^2(s-T')} \Big) ds
\end{aligned} \tag{3.30}$$

is a martingale.

Setting $T' = t+h$ and $\kappa_h^N(\lambda, t) = |\langle X^N(t), e^{i\lambda\cdot}\rangle|^2 |\widetilde{W^N}(\lambda)|^2 e^{-\lambda^2 h}$ we obtain from that martingale property

$$\begin{aligned}
\mathbb{E}[\kappa_h^N(\lambda,t)] = & \mathbb{E}[\kappa_{t+h}^N(\lambda,0)] \\
& + \int_0^t \mathbb{E}\Big[\langle X^N(s), e^{-i\lambda\cdot}\rangle \langle X^N(s), F(.,g^N(.,s))\cdot(i\lambda)e^{i\lambda\cdot}\rangle \\
& + \langle X^N(s), e^{i\lambda\cdot}\rangle \langle X^N(s), F(.,g^N(.,s))\cdot(-i\lambda)e^{-i\lambda\cdot}\rangle + \frac{\lambda^2}{N}\Big] \\
& \cdot e^{-\lambda^2(t+h-s)} |\widetilde{W^N}(\lambda)|^2 ds \\
\leqq & \mathbb{E}[\kappa_{t+h}^N(\lambda,0)] \\
& + \int_0^t (\mathbb{E}[2(|\langle X^N(s), e^{i\lambda\cdot}\rangle| e^{-\lambda^2(t+h-s)/3} |\widetilde{W^N}(\lambda)|) \\
& \cdot (|\langle X^N(s), F(.,g^N(.,s)) e^{i\lambda\cdot}\rangle| e^{-\lambda^2(t+h-s)/3} |\widetilde{W^N}(\lambda)|)] \\
& \cdot |\lambda|(t+h-s)^{1/2} e^{-\lambda^2(t+h-s)/3}(t+h-s)^{-1/2} \\
& + \frac{1}{N}\lambda^2 e^{-\lambda^2(t+h-s)} |\widetilde{W^N}(\lambda)|^2) ds.
\end{aligned} \tag{3.31}$$

Noting that for $b, d > 0$

$$(|\lambda| t^{1/2})^d e^{-\lambda^2 t/b} \leqq \sup_{y\in\mathbb{R}^d} |y|^d e^{-y^2/b} = c_{13}(b,d) < \infty \tag{3.32}$$

we obtain after integrating over λ

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda,t)]\,d\lambda \\
&\quad\leqq \int_{\mathbb{R}^d} \mathbb{E}[\kappa_{t+h}^N(\lambda,0)]\,d\lambda \\
&\qquad + c_{14}\int_0^t (t+h-s)^{-1/2}\,\mathbb{E}\Big[\int_{\mathbb{R}^d} (|\langle X^N(s), e^{i\lambda\cdot}\rangle|\, e^{-\lambda^2(t+h-s)/3}\,|\widetilde{W^N}(\lambda)|) \\
&\qquad \cdot (|\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|\, e^{-\lambda^2(t+h-s)/3}\,|\widetilde{W^N}(\lambda)|)\,d\lambda\Big]\,ds \\
&\qquad + \frac{1}{N}\int_{\mathbb{R}^d} \lambda^2\,|\widetilde{W^N}(\lambda)|^2 \left(\int_0^t e^{-\lambda^2(t+h-s)}\,ds\right) d\lambda
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
&\quad\leqq \int_{\mathbb{R}^d} \mathbb{E}[\kappa_{t+h}^N(\lambda,0)]\,d\lambda \\
&\qquad + c_{14}\int_0^t (t+h-s)^{-1/2}\,\mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, e^{-2\lambda^2(t+h-s)/3}\,d\lambda\Big)^{1/2} \\
&\qquad \cdot \Big(\int_{\mathbb{R}^d} |\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, e^{-2\lambda^2(t+h-s)/3}\,d\lambda\Big)^{1/2}\Big]\,ds \\
&\qquad + \frac{1}{N}\int_{\mathbb{R}^d} \lambda^2\,|\widetilde{W^N}(\lambda)|^2\, e^{-\lambda^2 h}\,\frac{1}{\lambda^2}\,d\lambda.
\end{aligned} \tag{3.34}
$$

The last term in (3.34) is less than

$$
\frac{1}{N}\int_{\mathbb{R}^d} |\widetilde{W^N}(\lambda)|^2\,d\lambda = \frac{1}{N}\int_{\mathbb{R}^d} |\widetilde{W^1}(\lambda/\chi_N)|^2\,d\lambda = \frac{\chi_N^d}{N}\int_{\mathbb{R}^d} |\widetilde{W^1}(\mu)|^2\,d\mu \leqq \frac{\chi_N^d}{N}\,c_{15} \leqq c_{15} \tag{3.35}
$$

(by (2.5) and (1.3)).

Moreover we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, e^{-2\lambda^2(t+h-s)/3}\,d\lambda \\
&\quad = \int_{\mathbb{R}^d} \Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(y,g^N(y,s))\,W^N(x-y)\,\sigma_{2(t+h-s)/3}(z-y)\,e^{i\lambda z}\,X^N(S)(dy)\,dx\,dz\Big|^2\,d\lambda \\
&\qquad \text{(by (2.2))} \\
&\quad = \int_{\mathbb{R}^d} \Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(y,g^N(y,s))\,W^N(x-y)\,\sigma_{2(t+h-s)/3}(z-x)\,X^N(s)(dy)\,dx\Big|^2\,dz \\
&\qquad \text{(by (2.1))} \\
&\quad = \int_{\mathbb{R}^d} |\langle X^N(s), F(.,g^N(.,s))(W^N * \sigma_{2(t+h-s)/3})(z-.)\rangle|^2\,dz,
\end{aligned} \tag{3.36}
$$

since $\sigma_v(x) = (2\pi v)^{-d/2}\,e^{-x^2/2v}$ is the inverse Fouriertransform of $e^{-\lambda^2 v/2}(2\pi)^{-d/2}$.

Since $W^N(x) \geqq 0$ and $\sigma_v(x) \geqq 0$ for all $x \in \mathbb{R}^d$ and by (1.4) the right side of (3.36) is less than

$$
\begin{aligned}
&c_1^2 \int_{\mathbb{R}^d} |\langle X^N(s), (W^N * \sigma_{2(t+h-s)/3})(x-.)\rangle|^2\,dx \\
&\quad = c_1^2 \int_{\mathbb{R}^d} |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, e^{-2\lambda^2(t+h-s)/3}\,d\lambda \\
&\quad = c_1^2 \int_{\mathbb{R}^d} \kappa_{2(t+h-s)/3}^N(\lambda,s)\,d\lambda.
\end{aligned} \tag{3.37}
$$

Therefore we obtain from (3.34), (3.35) and

$$\int_{\mathbb{R}^d} \kappa_{t+h}^N(\lambda, 0)\, d\lambda \leqq \int_{\mathbb{R}^d} e^{-\lambda^2(t+h)}\, d\lambda \leqq c_{16}(t+h)^{-d/2} \tag{3.38}$$

the inequality

$$\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda, t)]\, d\lambda \leqq c_{16}(t+h)^{-d/2} + c_{17}\int_0^t (t+h-s)^{-1/2} \cdot \Big(\int_{\mathbb{R}^d} \mathbb{E}[\kappa_{2(t+h-s)/3}^N(\lambda, s)]\, d\lambda\Big)\, ds + c_{15}. \tag{3.39}$$

Using (3.39) we give by some iteration procedure increasingly better estimates for $\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda, t)]\, d\lambda$. As a first crude estimate we obtain from (3.34), (3.35) and (3.38)

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda, t)]\, d\lambda &\leqq c_{16}(t+h)^{-d/2} + c_{18}\int_0^t (t+h-s)^{-1/2}\Big(\int_{\mathbb{R}^d} e^{-2\lambda^2(t+h-s)/3}\, d\lambda\Big)\, ds + c_{15}\\ &\leqq c_{16}(t+h)^{-d/2} + c_{19}\int_0^t (t+h-s)^{-1/2}(t+h-s)^{-d/2}\, ds + c_{15}\\ &\leqq c_{20}((t+h)^{-d/2} + h^{-(d-1)/2} + \delta_{d,1}\,|\log h| + 1).\end{aligned} \tag{3.40}$$

We insert (3.40) into (3.39), thus obtaining an improvement of (3.40), where "improvement" means, that the exponent of $1/h$ is less.

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda, t)]\, d\lambda &\leqq c_{16}(t+h)^{-d/2} + c_{15}\\ &\quad + c_{21}\int_0^t (t+h-s)^{-1/2}((s+2(t+h-s)/3)^{-d/2}\\ &\quad + (t+h-s)^{-(d-1)/2} + \delta_{d,1}\,|\log(t+h-s)| + 1)\, ds\\ &\leqq c_{22}((t+h)^{-d/2}(1+T^{1/2}) + \int_0^t (t+h-s)^{-d/2}\, ds + T^{1/3} + 1)\\ &\leqq c_{23}((t+h)^{-d/2} + h^{-(d-2)/2} + \delta_{d,2}\,|\log h| + 1).\end{aligned} \tag{3.41}$$

We may insert this improvement of (3.40) into (3.39) to get another improvement of (3.41). Continuing in this way, we reduce step by step the exponent of h^{-1} in (3.41), until finally

$$\int_{\mathbb{R}^d} \mathbb{E}[\kappa_h^N(\lambda, t)]\, d\lambda \leqq c_{24}((t+h)^{-d/2} + 1) \qquad \text{(uniformly in } h>0). \tag{3.42}$$

In the limit $h \to 0$ we obtain from (3.42) the desired estimate for

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}[|g_1^N(\lambda, t)|^2]\, d\lambda &= \int_{\mathbb{R}^d} \mathbb{E}[|\langle X^N(t), e^{i\lambda\cdot}\rangle|^2\, |\widetilde{W^N}(\lambda)|^2]\, d\lambda\\ &= \int_{\mathbb{R}^d} \mathbb{E}[\kappa_0^N(\lambda, t)]\, d\lambda,\end{aligned}$$

namely

$$\int_{\mathbb{R}^d} \mathbb{E}[|\widetilde{g_1^N}(\lambda, t)|^2] \, d\lambda \leqq c_{24}(t^{-d/2}+1). \tag{3.43}$$

Using (3.43) it is easy to finish the proof of Proposition 3.2. From the martingale (3.30) we obtain analogously to (3.31)

$$\begin{aligned}
&\mathbb{E}[\kappa_0^N(\lambda, t)] \\
&\quad \leqq \mathbb{E}[\kappa_{t/2}^N(\lambda, t/2)] \\
&\qquad + \int_{t/2}^{t} 2\,\mathbb{E}[(|\langle X^N(s), e^{i\lambda\cdot}\rangle|\,|\widetilde{W^N}(\lambda)|)(|\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|\,|\widetilde{W^N}(\lambda)|)] \\
&\qquad \cdot |\lambda|\, e^{-\lambda^2(t-s)}\, ds \\
&\qquad + \int_{t/2}^{t} \frac{1}{N}\,\lambda^2 e^{-\lambda^2(t-s)}\,|\widetilde{W^N}(\lambda)|^2\, ds.
\end{aligned}$$

We multiply both sides with $(1+|\lambda|^{\alpha_1})$ and take into account (3.32) to obtain

$$\begin{aligned}
&(1+|\lambda|^{\alpha_1})\,\mathbb{E}[\kappa_0^N(\lambda, t)] \\
&\quad \leqq (1+|\lambda|^{\alpha_1})\,\mathbb{E}[\kappa_{t/2}^N(\lambda, t/2)] \\
&\qquad + \int_{t/2}^{t} 2\,\mathbb{E}[(|\langle X^N(s), e^{i\lambda\cdot}\rangle|\,|\widetilde{W^N}(\lambda)|)(|\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|\,|\widetilde{W^N}(\lambda)|)] \\
&\qquad \cdot c_{25}(t-s)^{-(1+\alpha_1)/2}\, ds \\
&\qquad + \frac{1}{N}\,|\widetilde{W^N}(\lambda)|^2(1+|\lambda|^{\alpha_1}).
\end{aligned}$$

Integrating over λ we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} (1+|\lambda|^{\alpha_1})\,\mathbb{E}[\kappa_0^N(\lambda, t)]\, d\lambda \\
&\quad \leqq \int_{\mathbb{R}^d} (1+|\lambda|^{\alpha_1})\,\mathbb{E}[\kappa_{t/2}^N(\lambda, t/2)]\, d\lambda \\
&\qquad + \int_{t/2}^{t} 2c_{25}(t-s)^{-(1+\alpha_1)/2}\,\mathbb{E}[(\int_{\mathbb{R}^d} |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, d\lambda)^{1/2} \\
&\qquad \cdot (\int_{\mathbb{R}^d} |\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, d\lambda)^{1/2}]\, ds \\
&\qquad + \frac{1}{N}\int_{\mathbb{R}^d} |\widetilde{W^N}(\lambda)|^2(1+|\lambda|^{\alpha_1})\, d\lambda.
\end{aligned} \tag{3.44}$$

By (3.37) we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\langle X^N(s), F(.,g^N(.,s))\,e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, d\lambda \\
&\quad \leqq c_1^2 \int_{\mathbb{R}^d} |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2\,|\widetilde{W^N}(\lambda)|^2\, d\lambda.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\frac{1}{N}\int_{\mathbb{R}^d}|\widetilde{W^N}(\lambda)|^2(1+|\lambda|^{\alpha_1})\,d\lambda\\
&=\frac{1}{N}\int_{\mathbb{R}^d}|\widetilde{W^1}(\lambda/\chi_N)|^2(1+|\lambda|^{\alpha_1})\,d\lambda\\
&=\frac{1}{N}\int_{\mathbb{R}^d}|\widetilde{W^1}(\mu)|^2(1+|\chi_N\mu|^{\alpha_1})\,\chi_N^d\,d\mu\\
&\leqq\frac{\chi_N^{d+\alpha_1}}{N}\int_{\mathbb{R}^d}|\widetilde{W^1}(\mu)|^2(1+|\mu|^{\alpha_1})\,d\mu\leqq c_{26}\qquad\text{(by (1.3) and (2.5)).}
\end{aligned}\tag{3.45}$$

Therefore (3.44) implies

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{R}^d}(1+|\lambda|^{\alpha_1})|\widetilde{g_1^N}(\lambda,t)|^2\,d\lambda\Big]\\
&=\int_{\mathbb{R}^d}(1+|\lambda|^{\alpha_1})\,\mathbb{E}[\kappa_0^N(\lambda,t)]\,d\lambda\\
&\leqq\int_{\mathbb{R}^d}(1+|\lambda|^{\alpha_1})\,\mathbb{E}[|\langle X^N(t/2),e^{i\lambda\cdot}\rangle|^2]\,|\widetilde{W^N}(\lambda)|^2\,e^{-t\lambda^2/2}\,d\lambda\\
&\quad+\int_{t/2}^{t}c_{27}(t-s)^{-(1+\alpha_1)/2}\Big(\int_{\mathbb{R}^d}\mathbb{E}[|\langle X^N(s),e^{i\lambda\cdot}\rangle|^2]\,|\widetilde{W^N}(\lambda)|^2\,d\lambda\Big)\,ds+c_{26}\\
&\leqq\int_{\mathbb{R}^d}(1+|\lambda|^{\alpha_1})\,e^{-t\lambda^2/2}\,d\lambda\\
&\quad+\int_{t/2}^{t}c_{27}(t-s)^{-(1+\alpha_1)/2}\,c_{24}\left(\frac{2^{d/2}}{t^{d/2}}+1\right)ds+c_{26}\qquad\text{(by (3.43))}\\
&\leqq c_{28}(t^{-(d+1)/2}+1).
\end{aligned}$$

This finishes the proof of Proposition 3.2.

Remark. An essential point in the proof of Proposition 3.2 are the estimates (3.35) and (3.45). The term $\frac{1}{N}\int_{\mathbb{R}^d}(1+|\lambda|^{\alpha_1})\,|\widetilde{W^N}(\lambda)|^2\,d\lambda$ would not be bounded for any $\alpha_1>0$ as $N\to\infty$ if $\chi_N\geqq N^{1/d}$. This corresponds to the observation made in the introduction, that in that case local fluctuations of the particle density have a considerable effect on the dynamics of $X^N(t)$ in the limit $N\to\infty$.

c) Proof of Proposition 3.3. We use the same notation as in the proof of Proposition 3.1. Let $|\nu|$ be the variation of $\nu\in\mathcal{M}_T^d$.

By (3.26) and (1.4) we have

$$\begin{aligned}
\mathbb{P}\Big[\int_{B_\lambda^c}\int_0^T|\nu^N|(dx,dt)\geqq\delta\Big]&\leqq\mathbb{P}\Big[\int_{B_\lambda^c}\int_0^T c_1\,X^N(t)(dx)\,dt\geqq\delta\Big]\\
&\leqq\mathbb{P}\Big[\sup_{t\leqq T}\langle X^N(t),\mathbb{1}_{B_\lambda^c}\rangle\geqq\frac{\delta}{c_1T}\Big]\leqq\frac{c_9c_1T}{\lambda\delta}.
\end{aligned}\tag{3.46}$$

From (3.46) we conclude as in the proof of (3.27), that for any $\varepsilon>0$ there exist sequences $\delta_k \searrow 0$ and $\lambda_k \nearrow \infty$, such that

$$\mathbb{P}\left[\int_{B^c_{\lambda_k}} \int_0^T |\nu^N|(dx, dt) \geqq \delta_k \text{ for any } k \in \mathbb{N}\right] \leqq \varepsilon. \tag{3.47}$$

(3.47), the fact, that $|\nu^N|(\mathbb{R}^d \times [0, T]) \leqq c_1 T$, $\forall N \in \mathbb{N}$, and Prohorov's Theorem imply Proposition 3.3.

The next Propositions 3.4 and 3.5 would be fairly standard, if we could consider "classical" solutions of (3.12) resp. (2.8). But since these equations are "weak", we have to do additional work.

d) Proof of Proposition 3.4. Let $t \to X(t)$ be any solution of (3.12) with $X(t) \in \mathscr{P}(\mathbb{R}^d)$, $\forall t \leqq T$. Obviously (3.12) implies, that for any $\varphi \in C_b^2(\mathbb{R}^d \times [0, T])$

$$\langle X(t), \varphi(.,t)\rangle - \langle \mu_0, \varphi(.,0)\rangle - \int_0^t \left\langle X(s), F_1(.,s) \cdot \nabla\varphi(.,s) + \tfrac{1}{2}\Delta\varphi(.,s) + \frac{\partial}{\partial s}\varphi(.,s)\right\rangle ds = 0, \quad 0 \leqq t \leqq T. \tag{3.48}$$

For any $\gamma \in L^1(\mathbb{R}^d)$, $t \in [0, T]$, and $h>0$ the function $(x,s) \to (\gamma * \sigma_{t+h-s})(x)$ is in $C_b^2(\mathbb{R}^d \times [0, t])$, and therefore (3.48), and

$$\frac{\partial}{\partial s}\sigma_s - \tfrac{1}{2}\Delta\sigma_s = 0, \qquad s>0 \tag{3.49}$$

yield the equation

$$\langle X(t), \sigma_h * \gamma\rangle - \langle \mu_0, \sigma_{t+h} * \gamma\rangle - \int_0^t \langle X(s), F_1(.,s) \cdot \nabla(\sigma_{t+h-s} * \gamma)\rangle ds = 0. \tag{3.50}$$

(3.50) implies

$$\begin{aligned}
&|\langle X(t), \sigma_h * \gamma\rangle| \\
&\quad \leqq \langle X(t), \sigma_h * |\gamma|\rangle \\
&\quad \leqq \langle \mu_0, \sigma_{t+h} * |\gamma|\rangle \\
&\qquad + \int_0^t \left\langle X(s), c_{29}\left((2\pi(t+h-s))^{-d/2} \int_{\mathbb{R}^d} \frac{|.-y|}{t+h-s}\right.\right. \\
&\qquad \left.\left. \cdot \exp\left(-\frac{(.-y)^2}{2(t+h-s)}\right) |\gamma(y)|\, dy\right)\right\rangle ds \\
&\quad \leqq \int_{\mathbb{R}^d} \mu_0(dx) \left(\int_{\mathbb{R}^d} |\gamma(y)| (2\pi(t+h))^{-d/2} \exp\left(-\frac{(x-y)^2}{2(t+h)}\right) dy\right) \\
&\qquad + \int_0^t \left\langle X(s), c_{29}\left((2\pi(t+h-s))^{-d/2} \int_{\mathbb{R}^d} \frac{|.-y|}{(t+h-s)^{1/2}} \exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)\right.\right. \\
&\qquad \left.\left. \cdot \exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right) |\gamma(y)|\, dy\right)\right\rangle (t+h-s)^{-1/2} ds \\
&\quad \leqq c_{30}\left((t+h)^{-d/2} \|\gamma\|_1 + \int_0^t \langle X(s), (\sigma_{2(t+h-s)} * |\gamma|)\rangle (t+h-s)^{-1/2} ds\right) \\
&\qquad \text{(by (3.32)).}
\end{aligned} \tag{3.51}$$

Now we continue in a similar way as in the proof of Proposition 3.2 after (3.39).

In a first step we obtain from (3.51)

$$\begin{aligned}\langle X(t), \sigma_h * |\gamma|\rangle &\leqq c_{31}\left((t+h)^{-d/2}\|\gamma\|_1 + \int_0^t \langle X(s), \int_{\mathbb{R}^d} |\gamma(y)|\,dy\rangle (t+h-s)^{-(d+1)/2}\,ds\right)\\ &\leqq c_{32}\|\gamma\|_1((t+h)^{-d/2} + h^{-(d-1)/2} + \delta_{d,1}|\log h| + 1).\end{aligned} \tag{3.52}$$

Inserting (3.52) into (3.51) we obtain

$$\begin{aligned}\langle X(t), \sigma_h * |\gamma|\rangle &= c_{30}\Big((t+h)^{-d/2}\|\gamma\|_1\\ &\quad + \int_0^t c_{32}\|\gamma\|_1((s+2(t+h-s))^{-d/2} + (2(t+h-s))^{-(d-1)/2}\\ &\quad + \delta_{d,1}|\log(t+h-s)| + 1)(t+h-s)^{-1/2}\,ds\Big)\\ &\leqq c_{35}\|\gamma\|_1((t+h)^{-d/2} + h^{-(d-2)/2} + \delta_{d,2}|\log h| + 1).\end{aligned} \tag{3.53}$$

Since the exponent of $1/h$ is less, this is an improvement of (3.52). We may continue by inserting (3.53) into (3.51) to improve (3.53), etc. Finally we get

$$\begin{aligned}|\langle X(t), \sigma_h * \gamma\rangle| &\leqq \langle X(t), \sigma_h * |\gamma|\rangle\\ &\leqq c_{36}\|\gamma\|_1((t+h)^{-d/2} + 1)\\ &\leqq c_{36}\|\gamma\|_1(t^{-d/2} + 1) \quad \text{(uniformly in } h>0).\end{aligned} \tag{3.54}$$

Since $\lim_{h\to 0} X(t) * \sigma_h = X(t)$ in $\mathscr{P}(\mathbb{R}^d)$ we obtain from (3.54) for any open set $G \subseteq \mathbb{R}^d$ with finite Lebesgue measure $|G|$

$$\begin{aligned}\langle X(t), \mathbb{1}_G\rangle &\leqq \liminf_{h\to 0}\langle X(t) * \sigma_h, \mathbb{1}_G\rangle\\ &= \liminf_{h\to 0}\langle X(t), \mathbb{1}_G * \sigma_h\rangle \leqq c_{36}(t^{-d/2} + 1)|G|.\end{aligned} \tag{3.55}$$

This proves the absolute continuity of $X(t)$, $t>0$, with respect to Lebesgue measure. Let $g(.,t)$ be the density of $X(t)$. For $\varepsilon \in (0,1)$ let

$$B_{\varepsilon,t} = \{x \in \mathbb{R}^d\colon g(x,t) > c_{36}(t^{-d/2}+1)(1+\varepsilon)\}.$$

There exists, if $|B_{\varepsilon,t}|>0$, an open set A, such that

$$A \supseteq B_{\varepsilon,t} \quad \text{and} \quad |A\setminus B_{\varepsilon,t}| \leqq |A|\varepsilon/2. \tag{3.56}$$

The contradiction

$$\begin{aligned}|A|c_{36}(t^{-d/2}+1) &< |A|c_{36}(t^{-d/2}+1)(1+\varepsilon)(1-\varepsilon/2)\\ &= c_{36}(t^{-d/2}+1)(1+\varepsilon)(|A| - |A|\varepsilon/2)\\ &\leqq c_{36}(t^{-d/2}+1)(1+\varepsilon)(|A| - |A\setminus B_{\varepsilon,t}|) \quad \text{(by (3.56))}\\ &= c_{36}(t^{-d/2}+1)(1+\varepsilon)|B_{\varepsilon,t}| < \int_{B_{\varepsilon,t}} g(x,t)\,dx \leqq \int_A g(x,t)\,dx\\ &\leqq c_{36}(t^{-d/2}+1)|A| \quad \text{(by (3.55))}\end{aligned}$$

proves, that $g(.,t)\in L^{\infty}(\mathbb{R}^d)$ for any $t>0$, and

$$\|g(.,t)\|_{\infty}\leqq c_{36}(t^{-d/2}+1). \tag{3.57}$$

Then we obtain from (3.48) and (3.49) for $0<p=t/2$, $y,y'\in\mathbb{R}^d$, with $\delta=|y-y'|<1\wedge t$

$$\begin{aligned}
&|\langle g(.,t),\sigma_h(.-y)\rangle-\langle g(.,t),\sigma_h(.-y')\rangle|^2\\
&=\Bigg|\langle g(.,p),\sigma_{t-p+h}(.-y)\rangle-\langle g(.,p),\sigma_{t-p+h}(.-y')\rangle\\
&\quad+\int_p^t\Bigg\langle g(.,s),F_1(.,s)\cdot\left(-\frac{(.-y)}{t+h-s}\exp\left(-\frac{(.-y)^2}{2(t+h-s)}\right)\right.\\
&\quad\left.+\frac{(.-y')}{t+h-s}\exp\left(-\frac{(.-y')^2}{2(t+h-s)}\right)\right)(2\pi(t+h-s))^{-d/2}\Bigg\rangle ds\Bigg|^2\\
&\leqq\Bigg(|\langle g(.,p),\sigma_{t-p+h}(.-y)\rangle-\langle g(.,p),\sigma_{t-p+h}(.-y')\rangle|^2\\
&\quad+\Bigg(\int_p^{t-\delta}\Bigg\langle g(.,s),|F_1(.,s)|(t+h-s)^{-1/2}\left|\frac{(.-y)}{(t+h-s)^{1/2}}\exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)\right.\\
&\quad\left.-\frac{(.-y')}{(t+h-s)^{1/2}}\exp\left(-\frac{(.-y')^2}{4(t+h-s)}\right)\right|(2\pi(t+h-s))^{-d/2}\exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)\Bigg\rangle ds\Bigg)^2\\
&\quad+\Bigg(\int_p^{t-\delta}\Bigg\langle g(.,s),|F_1(.,s)|(t+h-s)^{-1/2}\frac{|.-y'|}{(t+h-s)^{1/2}}\exp\left(-\frac{(.-y')^2}{8(t+h-s)}\right)\\
&\quad\cdot(2\pi(t+h-s))^{-d/2}\exp\left(-\frac{(.-y')^2}{8(t+h-s)}\right)\left|\exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)\right.\\
&\quad\left.-\exp\left(-\frac{(.-y')^2}{4(t+h-s)}\right)\right|\Bigg\rangle ds\Bigg)^2\\
&\quad+\Bigg(\int_{t-\delta}^{t}\Bigg\langle g(.,s),|F_1(.,s)|(t+h-s)^{-1/2}\left(\frac{|.-y|}{(t+h-s)^{1/2}}\exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)\right.\\
&\quad\cdot\exp\left(-\frac{(.-y)^2}{4(t+h-s)}\right)+\frac{|.-y'|}{(t+h-s)^{1/2}}\\
&\quad\left.\cdot\exp\left(-\frac{(.-y')^2}{4(t+h-s)}\right)\exp\left(-\frac{(.-y')^2}{4(t+h-s)}\right)\right)(2\pi(t+h-s))^{-d/2}\Bigg\rangle ds\Bigg)^2\Bigg)c_{37}.
\end{aligned}$$

Since

$$\begin{aligned}
|\sigma_{t+h-p}(z)-\sigma_{t+h-p}(z')|&\leqq c_{38}|z-z'|(t-p)^{-(d+1)/2},\\
|z\,e^{-z^2/4}-z'\,e^{-z'^2/4}|&\leqq c_{39}|z-z'|,\\
|e^{-z^2/4}-e^{-z'^2/4}|&\leqq c_{40}|z-z'|,
\end{aligned}$$

$$\sup_{r>0}\int_{\mathbb{R}^d}(2\pi r)^{-d/2}\exp(-x^2/\sigma_1 r)\,dx\leqq c_{41}(\sigma_1),$$

and because of (3.57) and (3.32) the last expression is less than

$$\Bigg(|y-y'|^2+|y-y'|^2\left(\int_p^{t-\delta}(t+h-s)^{-1}\,ds\right)^2 + \left(\int_{t-\delta}^{t}(t+h-s)^{-1/2}\,ds\right)^2\Bigg)\, c_{42}(t^{-(d+1)/2}+1)^2$$
$$\leqq (|y-y'|^2(1+(\log\delta)^2)+\delta)\, c_{43}(t^{-(d+1)/2}+1)^2$$
$$\leqq |y-y'|\, c_{44}^2(t^{-(d+1)/2}+1)^2.$$

This estimate holds for any fixed $t>0$ uniformly in $h>0$, and therefore is valid for the weak limit of the functions $y\to(g(.,t)*\sigma_h)(y)$, $h\to 0$, namely $y\to g(.,t)$, too, i.e.

$$|g(y,t)-g(y',t)|\leqq|y-y'|^{1/2}\, c_{44}(t^{-(d+1)/2}+1). \tag{3.58}$$

We still have to study the continuity properties of $g(x,t)$ with respect to t.

For $0<s\leqq t\leqq T$ and $h=|t-s|^{4/5}$ we obtain

$$|g(y,t)-g(y,s)|\leqq|g(y,t)-\langle g(.,t),\sigma_h(.-y)\rangle|$$
$$+|\langle g(.,t),\sigma_h(.-y)\rangle-\langle g(.,s),\sigma_h(.-y)\rangle|$$
$$+|\langle g(.,s),\sigma_h(.-y)\rangle-g(y,s)|=I_1+I_2+I_3.$$

$$I_1\leqq(2\pi h)^{-d/2}\left|\int_{\mathbb{R}^d}(g(y,t)-g(x,t))\exp\left(-\frac{(x-y)^2}{2h}\right)dx\right|$$
$$\leqq(2\pi h)^{-d/2}\int_{\mathbb{R}^d}|x-y|^{1/2}\, c_{44}(t^{-(d+1)/2}+1)\exp\left(-\frac{(x-y)^2}{2h}\right)dx \qquad \text{(by (3.58))}$$
$$\leqq h^{1/4}\, c_{44}(s^{-(d+1)/2}+1).$$

Similarly we obtain

$$I_3\leqq h^{1/4}\, c_{44}(s^{-(d+1)/2}+1).$$

Furthermore

$$I_2=\int_s^t\langle g(.,u), F_1(.,u)\cdot\nabla\sigma_h(.-y)+\tfrac{1}{2}\Delta\sigma_h(.-y)\rangle\,du$$
$$\leqq\int_s^t\left\langle g(.,u),\left(c_{45}\frac{|.-y|}{h}+\frac{1}{2}\left(\frac{(.-y)^2}{h^2}+\frac{d}{h}\right)\right)(2\pi h)^{-d/2}\exp\left(-\frac{(.-y)^2}{2h}\right)\right\rangle du$$
$$\leqq c_{36}(s^{-d/2}+1)\,|t-s|\, c_{46}(h^{-1/2}+h^{-1})(2\pi h)^{-d/2}\int_{\mathbb{R}^d}\exp\left(-\frac{(x-y)^2}{4h}\right)dx$$
(by (3.57) and (3.32))
$$\leqq|t-s|\, c_{47}h^{-1}(s^{-(d+1)/2}+1).$$

From the estimates for I_1, I_2, and I_3 we infer (3.15). This concludes the proof of Proposition 3.4.

e) Proof of Proposition 3.5. The system (2.8) satisfies the assumptions of Proposition 3.4. Therefore for $t>0$ any two solutions $\mu^1(t)$, $\mu^2(t)$ of (2.8) possess densities $g^1(.,t)$, $g^2(.,t)$ with respect to Lebesgue measure which satisfy (3.13), (3.14), and (3.15). Then from (2.8) and (3.50) we obtain for $t\in(0,T]$ any $y\in\mathbb{R}^d$

$$|g^1(y,t)-g^2(y,t)|=\left|\int_0^t\left(\left\langle g^1(.,s),F(.,g^1(.,s))\cdot\left(-\frac{(.-y)}{t-s}\right)\sigma_{t-s}(.-y)\right\rangle\right.\right.$$
$$\left.\left.-\left\langle g^2(.,s),F(.,g^2(.,s))\cdot\left(-\frac{(.-y)}{t-s}\right)\sigma_{t-s}(.-y)\right\rangle\right)ds\right|. \tag{3.59}$$

Now we take the convolution of both sides of this equation with the function $\sigma_h(.)$, $h>0$. This gives the inequality

$$\int_{\mathbb{R}^d}|g^1(y,t)-g^2(y,t)|\,\sigma_h(x-y)\,dy$$
$$\leqq\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|g^1(z,s)F(z,g^1(z,s))-g^2(z,s)F(z,g^2(z,s))|\frac{|z-y|}{t-s}$$
$$\cdot\,\sigma_{t-s}(z-y)\,\sigma_h(x-y)\,dz\,dy\,ds$$
$$\leqq c_2\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|g^1(z,s)-g^2(z,s)|(t-s)^{-1/2}\frac{|z-y|}{(t-s)^{1/2}}\exp\left(-\frac{(z-y)^2}{4(t-s)}\right)$$
$$\cdot\,(2\pi(t-s))^{-d/2}\exp\left(-\frac{(z-y)^2}{4(t-s)}\right)\sigma_h(x-y)\,dz\,dy\,ds \qquad \text{(by (1.5))}$$
$$\leqq c_{48}\int_0^t\int_{\mathbb{R}^d}|g^1(z,s)-g^2(z,s)|(t-s)^{-1/2}\sigma_{2(t-s)+h}(z-x)\,dz\,ds \qquad \text{(by (3.32))}.$$

In other words, with

$$\int_{\mathbb{R}^d}|g^1(z,s)-g^2(z,s)|\,\sigma_p(z-y)\,dz=Q(p,s,y)$$

this means

$$Q(h,t,x)\leqq c_{48}\int_0^t(t-s)^{-1/2}Q(2(t-s)+h,s,x)\,ds. \tag{3.60}$$

Similarly as in the proof of Propositions 3.2 and 3.4 this inequality can be used in an iteration procedure to estimate the quantity $Q(p,s,y)$. As starting point we use

$$Q(h,s,y)\leqq c_{49}h^{-d/2}. \tag{3.61}$$

Inserted into (3.60) this gives

$$Q(h,t,x)\leqq c_{48}c_{49}\int_0^t(t-s)^{-1/2}(2(t-s)+h)^{-d/2}\,ds$$
$$\leqq c_{50}\left(\int_0^{t-h}(t-s)^{-(d+1)/2}\,ds+\int_{t-h}^t(t-s)^{-1/2}h^{-d/2}\,ds\right)$$
$$\leqq c_{51}(h^{-(d-1)/2}+\delta_{d,1}|\log h|+1)$$
$$\text{(uniformly in } t\in(0,T],\ x\in\mathbb{R}^d) \tag{3.62}$$

(3.62) is an improvement of (3.61) and if we insert this improvement into (3.60) we obtain

$$Q(h, t, x) \leqq c_{52}(h^{-(d-2)/2} + \delta_{d,2}|\log h| + 1)$$
$$\text{(uniformly in } t \in (0, T],\ x \in \mathbb{R}^d).$$

Continuing in this way we finally obtain

$$Q(h, t, x) \leqq c_{53} \quad \text{(uniformly in } h > 0,\ t \in (0, T],\ x \in \mathbb{R}^d).$$

Since by (3.14)

$$\lim_{h \to 0} Q(h, t, x) = |g^1(x, t) - g^2(x, t)|,$$

this yields

$$\sup_{x \in \mathbb{R}^d, t \in (0, T]} |g^1(x, t) - g^2(x, t)| < \infty. \tag{3.63}$$

Furthermore we obtain from (3.59) and (1.5) for $t \in (0, T']$

$$|g^1(y, t) - g^2(y, t)| \leqq \int_0^t c_2 \sup_{z \in \mathbb{R}^d, 0 < v \leqq T'} |g^1(z, v) - g^2(z, v)| \Bigg((t-s)^{-1/2}(2\pi(t-s))^{-d/2}$$
$$\cdot \int_{\mathbb{R}^d} \left(\frac{|x-y|}{(t-s)^{1/2}} \exp\left(-\frac{(x-y)^2}{4(t-s)}\right)\right) \exp\left(-\frac{(x-y)^2}{4(t-s)}\right) dx\Bigg) ds$$
$$\leqq c_{54} \sup_{z \in \mathbb{R}^d, 0 < v \leqq T'} |g^1(z, v) - g^2(z, v)| \int_0^t (t-s)^{-1/2} ds \qquad \text{(by (3.32))}$$
$$\leqq c_{55} \sup_{z \in \mathbb{R}^d, 0 < v \leqq T'} |g^1(z, v) - g^2(z, v)| \sqrt{T'}$$

uniformly in $t \in (0, T']$, $y \in \mathbb{R}^d$, and therefore

$$\sup_{\substack{z \in \mathbb{R}^d \\ 0 < t \leqq T'}} |g^1(z, t) - g^2(z, t)| \leqq c_{55} \sqrt{T'} \sup_{z \in \mathbb{R}^d, 0 < v \leqq T'} |g^1(z, v) - g^2(z, v)|.$$

For $T' < c_{55}^{-2}$ this yields by (3.63)

$$\sup_{\substack{z \in \mathbb{R}^d \\ 0 < t \leqq T'}} |g^1(z, t) - g^2(z, t)| = 0.$$

Iteration of the above argument in the interval $[T', 2T']$, etc. provides the desired result, namely

$$\sup_{\substack{z \in \mathbb{R}^d \\ 0 < t \leqq T}} |g^1(z, t) - g^2(z, t)| = 0$$

and Proposition 3.5 is proved.

C. Example

To conclude this chapter, we show, how the (one dimensional) "*Burgers Equation*"

$$\frac{\partial}{\partial t} u(x, t) + u(x, t) \frac{\partial}{\partial x} u(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x, t),$$
$$u(x, 0) = u_0(x) \tag{3.64}$$

appears as limit dynamics of the empirical distributions of certain moderately interacting diffusions. Obviously (3.64) can be written in a weak form as

$$\langle u(.,t), f\rangle - \langle u_0(.), f\rangle - \int_0^t \langle u(.,s), \tfrac{1}{2}u(.,s) f' + \tfrac{1}{2} f''\rangle\, ds = 0, \quad f \in C_b^2(\mathbb{R}). \tag{3.65}$$

The solution of (3.64) is explicitely given by

$$u(x,t) = \left(\int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{2t}\right) \exp\left(-\int_0^y u_0(z)\,dz\right) u_0(y)\,dy\right) \cdot \left(\int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{2t}\right) \exp\left(-\int_0^y u_0(z)\,dz\right) dy\right)^{-1} \tag{3.66}$$

(cf. [3]).

This representation shows, that $\|u(.,t)\|_\infty \leqq \|u_0\|_\infty$ for $t \geqq 0$. Therefore we may rewrite (3.65) in the form

$$\langle u(.,t), f\rangle - \langle u_0(.), f\rangle - \int_0^t \langle u(.,s), F_2(u(.,s)) f' + \tfrac{1}{2} f''\rangle\, ds = 0, \quad f \in C_b^2(\mathbb{R}), \tag{3.67}$$

where $F_2(.) \in C_b^1(\mathbb{R}_+)$ satisfies $F_2(p) = \frac{1}{2}p$ for $p \leqq \|u_0\|_\infty$. (3.67) is of the form (2.8), so that we have proved:

Suppose $d=1$, $F(x,p) = F_2(p)$, and $\lim_{N\to\infty} \mathscr{L}(X^N(0)) = \delta_{u_0^*}$, with $u_0^*(dx) = u_0(x)\,dx$ for some $u_0 \in C_b(\mathbb{R})$.

Then the density $p(.,t)$ of the limit $\mu(t)$, appearing in Theorem 1, is the solution (3.66) of Burgers Equation (3.64).

For other derivations of Burgers Equation as limit of the empirical distributions of interacting diffusion processes see [2, 6] and [11].

IV. Proof of Theorem 2

As in the proof of Theorem 1 we begin by showing the relative compactness of the sequence $\mathscr{L}(X^N)$, $N \in \mathbb{N}$, of the distributions $\mathscr{L}(X^N)$ of the processes $X^N(t)$, $0 \leqq t \leqq T$, in the space $\mathscr{P}(\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d)))$. Similar to (3.30) we obtain from (1.12) and Ito's Formula for any $\lambda \in \mathbb{R}^d$ the martingale property of the process

$$\begin{aligned} t \to &\langle X^N(t), e^{i\lambda\cdot}\rangle \langle X^N(t), e^{-i\lambda\cdot}\rangle \\ &- \int_0^t \Big\{ \tfrac{1}{2}\langle X^N(s), e^{i\lambda\cdot}\rangle \langle X^N(s), (-\nabla g^N(.,s)) \cdot \lambda(-i) e^{-i\lambda\cdot} - \lambda^2 e^{-i\lambda\cdot}\rangle \\ &+ \tfrac{1}{2}\langle X^N(s), e^{-i\lambda\cdot}\rangle \langle X^N(s), (-\nabla g^N(.,s)) \cdot \lambda i e^{i\lambda\cdot} - \lambda^2 e^{i\lambda\cdot}\rangle \\ &+ \frac{1}{N}\langle X^N(s), (i\lambda)\cdot(-i\lambda)\, e^{i\lambda\cdot}\, e^{-i\lambda\cdot}\rangle \Big\}\, ds. \end{aligned}$$

Therefore for $0 \leqq s \leqq t \leqq T$

$$\begin{aligned}
\mathbb{E}[|\langle X^N(t), e^{i\lambda\cdot}\rangle|^2 | \mathscr{F}_s] &= |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2 \\
&+ \int_s^t \mathbb{E}\Big[\tfrac{1}{2}\langle X^N(u), e^{i\lambda\cdot}\rangle \langle X^N(u), (-\nabla g^N(.,u))\cdot\lambda(-i)\,e^{-i\lambda\cdot}\rangle \\
&+ \tfrac{1}{2}\langle X^N(u), e^{-i\lambda\cdot}\rangle \langle X^N(u), (-\nabla g^N(.,u))\cdot\lambda\, i\, e^{i\lambda\cdot}\rangle \\
&- \lambda^2 |\langle X^N(u), e^{i\lambda\cdot}\rangle|^2 + \frac{\lambda^2}{N}\Big|\mathscr{F}_s\Big]\,du. \qquad (4.1)
\end{aligned}$$

Now we multiply both sides of (4.1) with $\widetilde{V^N}(\lambda)/(2\pi)^{d/2} = |\widetilde{W^N}(\lambda)|^2$ and integrate over $\lambda \in \mathbb{R}^d$.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}[|\langle X^N(t), e^{i\lambda\cdot}\rangle|^2 | \mathscr{F}_s]\, |\widetilde{W^N}(\lambda)|^2\, d\lambda \\
&= \int_{\mathbb{R}^d} |\langle X^N(s), e^{i\lambda\cdot}\rangle|^2 \,|\widetilde{W^N}(\lambda)|^2\, d\lambda \\
&+ \int_s^t \Big(\int_{\mathbb{R}^d} \mathbb{E}\Big[\tfrac{1}{2}\langle X^N(u), e^{i\lambda\cdot}\rangle\, \widetilde{V^N}(\lambda)(-i\lambda)\langle X^N(u), (-\nabla g^N(.,u))\,e^{-i\lambda\cdot}\rangle/(2\pi)^{d/2} \\
&+ \tfrac{1}{2}\langle X^N(u), e^{-i\lambda\cdot}\rangle\, \widetilde{V^N}(\lambda)(i\lambda)\langle X^N(u), (-\nabla g^N(.,u))\,e^{i\lambda\cdot}\rangle/(2\pi)^{d/2} \\
&- \lambda^2 |\langle X^N(u), e^{i\lambda\cdot}\rangle|^2 |\widetilde{W^N}(\lambda)|^2 + \frac{1}{N}\lambda^2\, \widetilde{V^N}(\lambda)(2\pi)^{-d/2} | \mathscr{F}_s\Big]\, d\lambda\Big)\, du \qquad (4.2)
\end{aligned}$$

(4.2), (2.1), (2.2) and (2.3) yield

$$\begin{aligned}
\mathbb{E}\Big[\int_{\mathbb{R}^d} |\widetilde{g_1^N}(\lambda, t)|^2\, d\lambda | \mathscr{F}_s\Big] &= \int_{\mathbb{R}^d} |\widetilde{g_1^N}(\lambda, s)|^2\, d\lambda \\
&+ \int_s^t \mathbb{E}\Big[-\tfrac{1}{2}\int_{\mathbb{R}^d} \widetilde{\nabla g^N(.,u)}(\lambda)\,(\widetilde{\nabla g^N(.,u)X^N(u)})(\lambda)\, d\lambda \\
&- \tfrac{1}{2}\int_{\mathbb{R}^d} \overline{\widetilde{\nabla g^N(.,u)}}(\lambda)\,(\widetilde{\nabla g^N(.,u)X^N(u)})(\lambda)\, d\lambda \\
&- \int_{\mathbb{R}^d} |\widetilde{\nabla g_1^N(.,u)}(\lambda)|^2\, d\lambda + \frac{1}{N}\int_{\mathbb{R}^d} \lambda^2\, \widetilde{V^N}(\lambda)(2\pi)^{-d/2}\, d\lambda | \mathscr{F}_s\Big]\, du
\end{aligned}$$

or

$$\begin{aligned}
\mathbb{E}[\|g_1^N(.,t)\|_2^2 | \mathscr{F}_s] &= \|g_1^N(.,s)\|_2^2 \\
&- \int_s^t \mathbb{E}[\langle X^N(u), |\nabla g^N(.,u)|^2\rangle + \|\nabla g_1^N(.,u)\|_2^2 | \mathscr{F}_s]\, du \\
&+ \frac{t-s}{N}\int_{\mathbb{R}^d} \lambda^2\, \widetilde{V^N}(\lambda)(2\pi)^{-d/2}\, d\lambda. \qquad (4.3)
\end{aligned}$$

The application of relations (2.1) or (2.2) to a measure like $X^N(t)$, which is not in $L^2(\mathbb{R}^d)$, can be justified by replacing $X^N(t)$ by $(X^N(t) * \sigma_h)(.)$, then applying the above relations, and finally letting h tend to 0. Note, that by (2.10) and (2.18) for any fixed $N \in \mathbb{N}$ all terms in (4.3) are well defined.

By (1.13), (1.14), (2.4) and (2.10) the last term in (4.3) equals

$$\frac{t-s}{N}\int_{\mathbb{R}^d}\lambda^2\,\widetilde{V^1}(\lambda/\chi_N)(2\pi)^{-d/2}\,d\lambda=\frac{t-s}{N}\int_{\mathbb{R}^d}\chi_N^2\mu^2\,\widetilde{V^1}(\mu)(2\pi)^{-d/2}\chi_N^d\,d\mu$$

$$=\frac{t-s}{N}\chi_N^{d+2}c_{56}=(t-s)N^{\beta(d+2/d)-1}c_{56}. \quad (4.4)$$

In the case $\beta\in\left(0,\frac{d}{d+2}\right)$ the expression (4.4) asymptotically vanishes. This fact essentially corresponds to

$$\lim_{N\to\infty}\operatorname{Var}(\nabla g^N(x,t))=0,$$

which has been derived in the introduction on a heuristic level.

With the definition

$$A^N(t)=\int_0^t(\langle X^N(s),|\nabla g^N(.,s)|^2\rangle+\|\nabla g_1^N(.,s)\|_2^2)\,ds$$

(4.3) and (4.4) mean, that the process

$$t\to\|g_1^N(.,t)\|_2^2+A^N(t)-c_{56}tN^{\beta(d+2/d)-1}$$

is a martingale. Consequently

$$t\to\|g_1^N(.,t)\|_2^2+A^N(t)$$

is a submartingale. Therefore we obtain from Doob's Inequality

$$\begin{aligned}&\mathbb{P}[\sup_{t\le T}(\|g_1^N(.,t)\|_2^2+A^N(t))>K]\\&\le\frac{1}{K}\mathbb{E}[\|g_1^N(.,T)\|_2^2+A^N(T)]\\&=\frac{1}{K}\mathbb{E}[\|g_1^N(.,0)\|_2^2+c_{56}TN^{\beta(d+2/d)-1}]\\&\le c_{57}/K\qquad(\text{uniformly in } N\in\mathbb{N})\end{aligned}\quad (4.5)$$

(by (2.12) and (1.14)).

Let $\tau_K^N=\inf\{t\ge0:\ \|g_1^N(.,t)\|_2^2+A^N(t)>K\}$. (4.5) implies, that

$$\lim_{K\to\infty}\inf_{N\in\mathbb{N}}\mathbb{P}[\tau_K^N>T]=1. \quad (4.6)$$

(4.5) shows, that the functions $g^N(.\,.\,.)$ and $g_1^N(.,.)$ in a certain L^2-sense behave "regular" uniformly in N. Of course (4.5) is a consequence of the very special form of our model (1.12).

Because of (4.6) it suffices, to prove for any fixed $K>0$ the relative compactness of the distributions $\mathscr{L}(X^{N,K})$ of the processes $t\to X^{N,K}(t)=X^N(t\wedge\tau_K^N)$, $0\le t\le T$, $N\in\mathbb{N}$.

By (1.12) we have for $0 \leqq s \leqq t \leqq T$

$$
\begin{aligned}
&|||X^{N,K}(t)-X^{N,K}(s)|||_1 \\
&= \sup_{f\in\mathscr{H}_1} \frac{1}{N}\sum_{l=1}^{N} (f(\bar{X}_l^{N,K}(t))-f(\bar{X}_l^{N,K}(s))) \qquad (\bar{X}_l^{N,K}(t)=\bar{X}_l^N(t\wedge\tau_K^N)) \\
&\leqq \frac{1}{N}\sum_{l=1}^{N} |\bar{X}_l^{N,K}(t)-\bar{X}_l^{N,K}(s)| \\
&= \frac{1}{N}\sum_{l=1}^{N} \left| \int_{s\wedge\tau_K^N}^{t\wedge\tau_K^N} (-\tfrac{1}{2}\nabla g^N(\bar{X}_l^{N,K}(u),u))\,du + W_l(t\wedge\tau_K^N)-W_l(s\wedge\tau_K^N)\right| \\
&\leqq \int_{s\wedge\tau_K^N}^{t\wedge\tau_K^N} \tfrac{1}{2}\langle X^N(u), |\nabla g^N(.,u)|\rangle\,du + \frac{1}{N}\sum_{l=1}^{N} |W_l(t\wedge\tau_K^N)-W_l(s\wedge\tau_K^N)| \\
&\leqq (t-s)^{1/2}\left(\int_0^{\tau_K^N\wedge T} \tfrac{1}{4}\langle X^N(u), |\nabla g^N(.,u)|^2\rangle\,du\right)^{1/2} \\
&\quad + \frac{1}{N}\sum_{l=1}^{N} |W_l(t\wedge\tau_K^N)-W_l(s\wedge\tau_K^N)| \\
&\leqq ((t-s)K/4)^{1/2} + \frac{1}{N}\sum_{l=1}^{N} |W_l(t\wedge\tau_K^N)-W_l(s\wedge\tau_K^N)|.
\end{aligned}
$$

This implies

$$\mathbb{E}[|||X^{N,K}(t)-X^{N,K}(s)|||_1^4] \leqq c_{58}(t-s)^2. \tag{4.7}$$

Now let $\varphi(x)$ be any positive function in $C^2(\mathbb{R}^d)$ satisfying (3.22).

For such a function we obtain from (1.15)

$$
\begin{aligned}
\mathbb{E}[\langle X^N(t),\varphi\rangle|\mathscr{F}_s] &= \langle X^N(s),\varphi\rangle + \mathbb{E}\left[\int_s^t \langle X^N(u), (-\tfrac{1}{2}\nabla g^N(.,u))\cdot\nabla\varphi + \tfrac{1}{2}\Delta\varphi\rangle\,du|\mathscr{F}_s\right] \\
&\leqq \langle X^N(s),\varphi\rangle + c_{59}\,\mathbb{E}\left[\int_s^t \langle X^N(u), |\nabla g^N(.,u)|^2+1\rangle\,du|\mathscr{F}_s\right] \\
&\leqq \langle X^N(s),\varphi\rangle + c_{59}(\mathbb{E}[A^N(t)|\mathscr{F}_s]-A^N(s)) + c_{59}(t-s).
\end{aligned}
$$

Therefore the process

$$t\to\langle X^N(t),\varphi\rangle - c_{59}A^N(t) - c_{59}t$$

is a supermartingale. Similarly the process

$$t\to\langle X^N(t),\varphi\rangle + c_{59}A^N(t) + c_{59}t$$

is a submartingale.

Similar as in the proof of Proposition 3.1 (cf. (3.24), (3.25) and (3.29)) we may use these semimartingales and (4.7) to prove the relative compactness of the sequence $\mathscr{L}(X^{N,K})$, $N\in\mathbb{N}$, K fixed, in $\mathscr{P}(\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d)))$. By the remark following (4.6) we therefore also have proved the relative compactness of the sequence $\mathscr{L}(X^N)$, $N\in\mathbb{N}$.

As in the proof of Theorem 1 (cf. (3.1)) we may assume by the Theorem of Skorokhod, that there exists some process $X^\infty = X^\infty(t)$, $0 \leqq t \leqq T$, with trajectories in $\mathscr{C}([0,T], \mathscr{P}(\mathbb{R}^d))$, such that

$$\lim_{N\to\infty} \sup_{t\leqq T} |||X^N(t) - X^\infty(t)|||_1 = 0 \quad \mathbb{P}\text{-a.s.} \tag{4.8}$$

As next point we need the description of the dynamics governing the time evolution of the possible limit processes $X^\infty = X^\infty(t)$, $0 \leqq t \leqq T$.

For that aim we first remark, that there exists a positive (random) function $g_1^\infty(x,t)$ with

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |g_1^N(x,t) - g_1^\infty(x,t)|^2 \, dx \, dt\right] = 0. \tag{4.9}$$

(4.9) is proved completely analogous to (3.5) by using

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} |\lambda|^2 \, |\widetilde{g_1^N}(\lambda,t)|^2 \, d\lambda \, dt\right] = \mathbb{E}\left[\int_0^T \|\nabla g_1^N(.,t)\|_2^2 \, dt\right] < \infty$$

(uniformly in N), which follows from (4.5), instead of Proposition 3.2.

Since by (4.8), (1.13) and (1.14)

$$\lim_{N\to\infty} \sup_{t\leqq T} |||g_1^N(.,t) - X^\infty(t)|||_1 = 0 \quad \mathbb{P}\text{-a.s.}, \tag{4.10}$$

we have by (4.9)

$$\int_0^T \int_{\mathbb{R}^d} f(x,t) X^\infty(t)(dx) \, dt = \int_0^T \int_{\mathbb{R}^d} f(x,t) \, g_1^\infty(x,t) \, dx \, dt,$$
$$f \in C_b^2(\mathbb{R}^d \times [0,T]) \quad \mathbb{P}\text{-a.s.} \tag{4.11}$$

Therefore the measure $X^\infty(t)(dx)\,dt$ on $\mathbb{R}^d \times [0,T]$ is absolutely continuous with respect to Lebesgue measure and has the density

$$g_1^\infty(.,.) \in L^2(\mathbb{R}^d \times [0,T]) \quad \mathbb{P}\text{-a.s.}$$

Having established this regularity property of $X^\infty(t)$, we can identify its dynamics, and show, that $X^\infty(t)$ satisfies the following weak form of (2.13), (2.14).

$$\langle X^\infty(t), f\rangle - \langle \mu_0, f\rangle - \tfrac{1}{2}\int_0^t \langle g_1^\infty(.,s), (1 + \tfrac{1}{2} g_1^\infty(.,s)) \Delta f\rangle \, ds = 0,$$
$$f \in C_b^2(\mathbb{R}^d), \ 0 \leqq t \leqq T \quad \mathbb{P}\text{-a.s.} \tag{4.12}$$

We have for fixed $f \in C_b^2(\mathbb{R}^d)$

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle X^\infty(t), f\rangle - \langle \mu_0, f\rangle - \tfrac{1}{2}\int_0^t \langle g_1^\infty(.,s), (1 + \tfrac{1}{2} g_1^\infty(.,s)) \Delta f\rangle \, ds\right|\right] \\
&\leqq \mathbb{E}[|\langle X^\infty(t), f\rangle - \langle X^N(t), f\rangle|] + \mathbb{E}[|\langle \mu_0, f\rangle - \langle X^N(0), f\rangle|] \\
&\quad + \tfrac{1}{2}\mathbb{E}\left[\int_0^t |\langle g_1^\infty(.,s), \Delta f\rangle - \langle X^N(s), \Delta f\rangle| \, ds\right]
\end{aligned}$$

$$+\tfrac{1}{4}\mathbb{E}\left[\int_0^t |\langle g_1^\infty(.,s), g_1^\infty(.,s)\Delta f\rangle - \langle g_1^N(.,s), g_1^N(.,s)\Delta f\rangle|\, ds\right]$$

$$+\tfrac{1}{2}\mathbb{E}\left[\left|\int_0^t (\langle g_1^N(.,s), \nabla g_1^N(.,s)\cdot\nabla f\rangle - \langle X^N(s), \nabla g^N(.,s)\cdot\nabla f\rangle)\, ds\right|\right]$$

$$+\mathbb{E}\left[\left|\frac{1}{N}\sum_{k=1}^N \int_0^t \nabla f(\bar{X}_k^N(s))\cdot dW_k(s)\right|\right]$$

$$+\mathbb{E}\left[\left|\langle X^N(t),f\rangle - \langle X^N(0),f\rangle - \tfrac{1}{2}\int_0^t \langle X^N(s), (-\nabla g^N(.,s))\cdot\nabla f + \Delta f\rangle\, ds\right.\right.$$

$$\left.\left. -\frac{1}{N}\sum_{k=1}^N \int_0^t \nabla f(\bar{X}_k^N(s))\cdot dW_k(s)\right|\right]$$

$$= \sum_{j=1}^7 I_j^N(t). \tag{4.13}$$

(In $I_5^N(t)$ we used integration by parts:

$$\langle g_1^N(.,s), g_1^N(s)\Delta f\rangle = -2\langle g_1^N(.,s), \nabla g_1^N(.,s)\cdot\nabla f\rangle.)$$

Let us estimate the terms on the right side of (4.13).

By (1.15)

$$I_7^N(t) = 0, \tag{4.14}$$

and by (1.8)

$$I_6^N(t) \leqq 2T^{1/2}\|\nabla f\|_\infty / N^{1/2}. \tag{4.15}$$

$$I_5^N(t) = \tfrac{1}{2}\mathbb{E}\left[\left|\int_0^t \langle X^N(s), W^N * (\nabla g_1^N(.,s)\cdot\nabla f) - (W^N * \nabla g_1^N(.,s))\cdot\nabla f\rangle\, ds\right|\right]$$

(by (2.9))

$$= \tfrac{1}{2}\mathbb{E}\left[\left|\int_0^t \left(\int_{\mathbb{R}^d} X^N(s)(dx)\int_{\mathbb{R}^d} W^N(x-y)\,\nabla g_1^N(y,s)\cdot(\nabla f(y)-\nabla f(x))\,dy\right) ds\right|\right]$$

$$\leqq \tfrac{1}{2} c_{60}\chi_N^{-1}\|D^2 f\|_\infty \mathbb{E}\left[\int_0^t \langle X^N(s) * W^N, |\nabla g_1^N(.,s)|\rangle\, ds\right]$$

$$(c_{60} = \operatorname{diam}(\operatorname{supp} W^1(.)),\ \|D^2 f\|_\infty = \sup_{i,j\leqq d}\|\partial_{ij}^2 f\|_\infty)$$

$$\leqq c_{60}\chi_N^{-1}\left(\mathbb{E}\left[\int_0^T \|g_1^N(.,s)\|_2^2\, ds\right]\right)^{1/2}\left(\mathbb{E}\left[\int_0^T \|\nabla g_1^N(.,s)\|_2^2\, ds\right]\right)^{1/2}$$

$$\leqq c_{61}\chi_N^{-1} \tag{4.16}$$

(by (4.5)).

$$I_4^N(t) \leqq \tfrac{1}{4}\|\Delta f\|_\infty \int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d} |g_1^N(x,t) - g_1^\infty(x,t)|\,|g_1^N(x,t)+g_1^\infty(x,t)|\,dx\right] dt$$

$$\leqq \tfrac{1}{4}\|\Delta f\|_\infty \left(\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} |g_1^N(x,t)-g_1^\infty(x,t)|^2\,dx\,dt\right]\right)^{1/2}$$

$$\cdot\left(\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} |g_1^N(x,t)+g_1^\infty(x,t)|^2\,dx\,dt\right]\right)^{1/2}.$$

By (4.5) and (4.9) we obtain from this inequality

$$\lim_{N\to\infty} I_4^N(t)=0. \tag{4.17}$$

(2.6) and (4.8) yield

$$\lim_{N\to\infty} \sum_{j=1}^{3} I_j^N(t)=0. \tag{4.18}$$

(4.13)–(4.18) imply

$$\langle X^\infty(t),f\rangle-\langle\mu_0,f\rangle-\tfrac{1}{2}\int_0^t\langle g_1^\infty(.,s),(1+\tfrac{1}{2}g_1^\infty(.,s))\Delta f\rangle\,ds=0\quad \mathbb{P}\text{-a.s.},$$
$$f\in C_b^2(\mathbb{R}^d),\ 0\leqq t\leqq T. \tag{4.19}$$

Finally (4.19), the fact, that $X^\infty(.)\in\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d))$, and the continuity of

$$C_b^2(\mathbb{R}^d)\times[0,T]\ni(f,t)\to\int_0^t\langle g_1^\infty(.,s),(1+\tfrac{1}{2}g_1^\infty(.,s))\Delta f\rangle\,ds,$$

which follows from

$$g_1^\infty(.,.)\in L^2(\mathbb{R}^d\times[0,T]), \tag{4.20}$$

imply the validity of (4.12).

To finish the proof of Theorem 2 we now have to show, that any solution $X^\infty(.)\in\mathscr{C}([0,T],\mathscr{P}(\mathbb{R}^d))$ of (4.12) for some function $g_1^\infty(.,.)$ satisfying (4.11) and (4.20) has for any fixed $t\in[0,T]$ the density $p(.,t)$ with respect to Lebesgue measure, where the function $p(.,.)$ is the unique classical solution of (2.13), (2.14) (cf. Remark b). But to obtain this, we first need some regularity results for the function $g_1^\infty(.,.)$.

Let us remark, that (2.11) and (4.12) imply for any $f\in C_b^2(\mathbb{R}^d\times\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)X^\infty(t)(dy)f(x,y)\\&\quad-\tfrac{1}{2}\int_0^t\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x,s)(1+\tfrac{1}{2}g_1^\infty(x,s))g_1^\infty(y,s)\Delta_x f(x,y)\,dx\,dy\Big)ds\\&\quad-\tfrac{1}{2}\int_0^t\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x,s)g_1^\infty(y,s)(1+\tfrac{1}{2}g_1^\infty(y,s))\Delta_y f(x,y)\,dx\,dy\Big)ds\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_0(x)p_0(y)f(x,y)\,dx\,dy.\end{aligned} \tag{4.21}$$

(Δ_x (resp. Δ_y) is the Laplace operator with respect to x (resp. y)).

Next let

$$q_r(x,y)=q_r(x-y),\qquad r>0,$$

defined by

$$q_r(x)=\frac{2d}{r^2}((G_d(|x|)-G_d(r))\vee 0)=q_1(x/r)/r^d, \tag{4.22}$$

where $G_d(.)$ is the Greens function of the Laplace operator in $\mathbb{R}^d$, i.e.

$$G_d(u)=\begin{cases} -\frac{1}{2}u & \text{if } d=1 \\ -\dfrac{1}{2\pi}\log(u) & \text{if } d=2 \\ \dfrac{1}{d\,\omega_d(d-2)}u^{2-d} & \text{if } d\geqq 3. \end{cases}$$

$\left(\omega_d=\pi^{d/2}\Big/\Gamma\left(\dfrac{d+2}{2}\right)\text{ volume of the unit ball in }\mathbb{R}^d\right)$.

Note, that $q_r(x)\geqq 0$ for all $x\in\mathbb{R}^d$, and that

$$\int_{\mathbb{R}^d} q_r(x)\,dx=1 \qquad \text{for all } r>0. \tag{4.23}$$

It is easy to conclude from the definition of $q_r(.)$ the validity of the following equations in the sense of distributions.

$$\Delta q_r(x)=\frac{2d}{r^2}(\mu_r^d(x)-\delta_0(x)), \tag{4.24}$$

$$\lim_{r\to 0} q_r(.)=\delta_0(.), \tag{4.25}$$

where the distribution $\mu_r^d(.)$ is defined by

$$\int_{\mathbb{R}^d}\mu_r^d(x)\,\varphi(x)\,dx=\int_{S^{d-1}}\varphi(\theta r)\,d\theta, \qquad \varphi\in\mathscr{S}(\mathbb{R}^d),$$

where $d\theta$ is the normalized uniform distribution on the surface S^{d-1} of the unit ball in $\mathbb{R}^d$, i.e. $\mu_r^d(.)$ is the normalized uniform distribution on the surface of the sphere with radius r.

Let us define now for any $0<\delta<r/2$ and $\varepsilon>0$

$$q_{r,\delta,\varepsilon}(x)=\frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\Big(\int_{\mathbb{R}^d} q_\eta(x-y)\,\sigma_\varepsilon(y)\,dy\Big)\,d\eta. \tag{4.26}$$

From (4.24) we conclude

$$\Delta q_{r,\delta,\varepsilon}(x)=\frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\frac{2d}{\eta^2}\Big(\int_{\mathbb{R}^d}\mu_\eta^d(x-z)\,\sigma_\varepsilon(z)\,dz-\sigma_\varepsilon(x)\Big)\,d\eta. \tag{4.27}$$

Next we have

$$\begin{aligned}\tilde{q}_r(\lambda)&=-|\lambda|^{-2}(2\pi)^{-d/2}\int_{\mathbb{R}^d}(-\lambda^2)\,e^{i\lambda x}\,q_r(x)\,dx\\ &=-|\lambda|^{-2}(2\pi)^{-d/2}\int_{\mathbb{R}^d}(\Delta e^{i\lambda x})\,q_r(x)\,dx\\ &=-|\lambda|^{-2}(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{i\lambda x}(\Delta q_r(x))\,dx\\ &=-|\lambda|^{-2}(2\pi)^{-d/2}\,2d\int_{S^{d-1}}(e^{i\lambda\theta r}-1)\,r^{-2}\,d\theta \quad \text{(by (4.24))}\\ &=-|\lambda|^{-2}(2\pi)^{-d/2}\,2d\int_{S^{d-1}}(\cos(\lambda\theta r)-1)\,r^{-2}\,d\theta\geqq 0.\end{aligned} \tag{4.28}$$

(2.2), (4.23), (4.25), (4.26) and (4.28) yield

$$0 \leqq \frac{1}{2\delta} \int_{r-\delta}^{r+\delta} \widetilde{q_\eta}(\lambda) \exp(-\lambda^2 \varepsilon/2)\, d\eta = \widetilde{q_{r,\delta,\varepsilon}}(\lambda) \leqq (2\pi)^{-d/2}, \tag{4.29}$$

and

$$\lim_{\varepsilon,\delta\to 0} \widetilde{q_{r,\delta,\varepsilon}}(\lambda) = \widetilde{q_r}(\lambda), \tag{4.30}$$

resp.

$$\lim_{r\to 0} \widetilde{q_r}(\lambda) = (2\pi)^{-d/2}. \tag{4.31}$$

Inserting (4.26), (4.27) into (4.21) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)\, X^\infty(t)(dy)\, q_{r,\delta,\varepsilon}(x-y) \\
&\quad - \int_0^t \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (g_1^\infty(x,s) + \tfrac{1}{2} g_1^\infty(x,s)^2)\, \Delta q_{r,\delta,\varepsilon}(x-y)\, g_1^\infty(y,s)\, dx\, dy\Big)\, ds \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)\, X^\infty(t)(dy)\, q_{r,\delta,\varepsilon}(x-y) \\
&\quad - \int_0^t \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (g_1^\infty(x,s) + \tfrac{1}{2} g_1^\infty(x,s)^2)\right. \\
&\quad \left.\cdot \left(\frac{1}{2\delta}\int_{r-\delta}^{r+\delta} \frac{2d}{\eta^2} \int_{S^{d-1}} (g_1^\infty(x+\theta\eta-y,s) - g_1^\infty(x-y,s))\, \sigma_\varepsilon(y)\, d\theta\, d\eta\right) dx\, dy\right) ds \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_0(x)\, p_0(y)\, q_{r,\delta,\varepsilon}(x-y)\, dx\, dy.
\end{aligned} \tag{4.32}$$

We shall finally let ε and δ in (4.32) go to 0, but begin with rewriting (4.32) in the following way

$$\begin{aligned}
&\tfrac{1}{2}\int_0^T \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x,s)^2\, g_1^\infty(x-y,s)\, \sigma_\varepsilon(y)\, dx\, dy\Big)\, ds \left(\frac{d}{\delta}\int_{r-\delta}^{r+\delta} \eta^{-2}\, d\eta\right) \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_0(x)\, p_0(y)\, q_{r,\delta,\varepsilon}(x-y)\, dx\, dy \\
&\qquad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(T)(dx)\, X^\infty(T)(dy)\, q_{r,\delta,\varepsilon}(x-y) \\
&\qquad + \int_0^T \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (g_1^\infty(x,s) + \tfrac{1}{2} g_1^\infty(x,s)^2)\right. \\
&\qquad \left.\cdot \left(\frac{d}{\delta}\int_{r-\delta}^{r+\delta} \eta^{-2} \int_{S^{d-1}} g_1^\infty(x-y+\theta\eta,s)\, \sigma_\varepsilon(y)\, d\theta\, d\eta\right) dx\, dy\right) ds \\
&\qquad - \int_0^T \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x,s)\, g_1^\infty(x-y,s)\, \sigma_\varepsilon(y)\, dx\, dy\Big)\, ds \left(\frac{d}{\delta}\int_{r-\delta}^{r+\delta} \eta^{-2}\, d\eta\right).
\end{aligned} \tag{4.33}$$

By (2.11) $p_0(.) \in L^2(\mathbb{R}^d)$ and therefore by (2.1), (2.2) and (4.29)

$$\begin{aligned}
&\sup_{r,\delta,\varepsilon>0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_0(x)\, p_0(y)\, q_{r,\delta,\varepsilon}(x-y)\, dx\, dy \\
&\quad = \sup_{r,\delta,\varepsilon>0} (2\pi)^{d/2} \int_{\mathbb{R}^d} |\widetilde{p_0}(\lambda)|^2\, \widetilde{q_{r,\delta,\varepsilon}}(\lambda)\, d\lambda \\
&\quad \leqq \int_{\mathbb{R}^d} |\widetilde{p_0}(\lambda)|^2\, d\lambda = \|p_0(.)\|_2^2.
\end{aligned} \tag{4.34}$$

Since $g_1^\infty(.,t)$ is a probability density for almost all $t \in [0,T]$ (by (4.11)) and by (4.20), the third term on the right side of (4.33) is for fixed r, δ uniformly in $\varepsilon > 0$ less than

$$c(r,\delta)\int_0^T \Big(\int_{\mathbb{R}^d} \big(g_1^\infty(x,s) + \tfrac{1}{2} g_1^\infty(x,s)^2\big) \cdot \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(z-y,s)\,\sigma_\varepsilon(y)\,dz\,dy\Big)dx\Big)ds < \infty. \tag{4.35}$$

By (4.34) and (4.35), and since the second and the fourth term on the right side of (4.33) are strictly negative, we have (by fixing r, δ) uniformly in $\varepsilon > 0$

$$\int_0^T \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x,s)^2\, g_1^\infty(x-y,s)\,\sigma_\varepsilon(y)\,dx\,dy\Big)ds < \infty. \tag{4.36}$$

Let $g_\varepsilon^\infty(x,t) = \int_{\mathbb{R}^d} g_1^\infty(x-y,t)\,\sigma_\varepsilon(y)\,dy$. Since

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} |g_1^\infty(x,t) - g_\varepsilon^\infty(x,t)|^2\,dx\,dt = 0, \tag{4.37}$$

there exists a subsequence $\varepsilon_1 > \varepsilon_2 > \ldots$, $\lim_{k\to\infty} \varepsilon_k = 0$, such that

$$\lim_{k\to\infty} g_{\varepsilon_k}^\infty(x,s) = g_1^\infty(x,s) \tag{4.38}$$

for almost all $(x,s) \in \mathbb{R}^d \times [0,T]$ with respect to Lebesgue measure $dx\,dt$ and with respect to $g_1^\infty(x,t)^2\,dx\,dt$, and so by Fatou's Lemma

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} g_1^\infty(x,s)^3\,dx\,ds &= \int_0^T \int_{\mathbb{R}^d} \big(\lim_{k\to\infty} g_{\varepsilon_k}^\infty(x,s)\big)\, g_1^\infty(x,s)^2\,dx\,ds \\ &\leqq \liminf_{k\to\infty} \int_0^T \int_{\mathbb{R}^d} g_{\varepsilon_k}^\infty(x,s)\, g_1^\infty(x,s)^2\,dx\,ds < \infty\end{aligned}$$

(by (4.36)), i.e.

$$g_1^\infty(.,.) \in L^3(\mathbb{R}^d \times [0,T]). \tag{4.39}$$

By

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} g_\varepsilon^\infty(x,s)^3\,dx\,ds &= \int_0^t \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} g_1^\infty(x-y,s)\,\sigma_\varepsilon(y)\,dy\Big)^3 dx\,ds \\ &\leqq \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_1^\infty(x-y,s)^3\,\sigma_\varepsilon(y)\,dx\,dy\,ds \\ &= \int_0^t \int_{\mathbb{R}^d} g_1^\infty(x,s)^3\,dx\,ds < \infty\end{aligned} \tag{4.40}$$

(by (4.39)) the functions $g_{\varepsilon_k}^\infty(.,.)$ are bounded uniformly in ε_k (and therefore weakly compact) in the space $L^3(\mathbb{R}^d \times [0,t])$. Therefore and by (4.38) their product with $g_1^\infty(.,.)^2$, which by (4.39) is an element of the dual space $L^{3/2}(\mathbb{R}^d$

$\times [0,t]$) converge as follows

$$\lim_{k\to\infty} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_1^\infty(x,s)^2\, g_1^\infty(x-y,s)\,\sigma_{\varepsilon_k}(y)\,dx\,dy\,ds = \int_0^t \int_{\mathbb{R}^d} g_1^\infty(x,s)^3\,dx\,ds. \tag{4.41}$$

Using (4.39) we may show in quite a similar way as (4.41)

$$\begin{aligned}\lim_{k\to\infty} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} &(g_1^\infty(x,s)+\tfrac{1}{2}g_1^\infty(x,s)^2)\\ &\cdot\left(\frac{d}{\delta_k}\int_{r-\delta_k}^{r+\delta_k} \eta^{-2} \int_{S^{d-1}} g_1^\infty(x+\theta\eta-y,s)\,\sigma_{\varepsilon_k}(y)\,d\theta\,d\eta\right)dx\,dy\,ds\\ = \int_0^t \int_{\mathbb{R}^d} &(g_1^\infty(x,s)+\tfrac{1}{2}g_1^\infty(x,s)^2)\left(\frac{2d}{r^2}\int_{S^{d-1}} g_1^\infty(x+\theta r,s)\,d\theta\right)dx\,ds\end{aligned} \tag{4.42}$$

for suitable sequences $\delta_1, \delta_2, \ldots, \lim_{k\to\infty}\delta_k=0$, and $\varepsilon_1, \varepsilon_2, \ldots, \lim_{k\to\infty}\varepsilon_k=0$, which can be assumed to be the same as (4.41). Now we conclude from (4.32), (4.34), (4.37), (4.41) and (4.42), that

$$\begin{aligned}&\limsup_{k\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)\,X^\infty(t)(dy)\,q_{r,\delta_k,\varepsilon_k}(x-y)\\ &\quad -\int_0^t\int_{\mathbb{R}^d} (g_1^\infty(x,s)+\tfrac{1}{2}g_1^\infty(x,s)^2)\left(\frac{2d}{r^2}\int_{S^{d-1}} (g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))\,d\theta\right)dx\,ds\\ &\leqq \|p_0(\cdot)\|_2^2.\end{aligned} \tag{4.43}$$

It is easily seen, that

$$\begin{aligned}&-\int_0^t\int_{\mathbb{R}^d} (g_1^\infty(x,s)+\tfrac{1}{2}g_1^\infty(x,s)^2)\left(\frac{2d}{r^2}\int_{S^{d-1}} (g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))\,d\theta\right)dx\,ds\\ &= +\frac{d}{r^2}\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} (g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))^2\,dx\,d\theta\Big)ds\\ &\quad+\frac{d}{2r^2}\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} (g_1^\infty(x+\theta r,s)+g_1^\infty(x,s))(g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))^2\,dx\,d\theta\Big)ds\\ &= d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} (1+g_1^\infty(x,s))\frac{1}{r^2}(g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))^2\,dx\,d\theta\Big)ds \geqq 0.\end{aligned}$$

Therefore we obtain from (4.43)

$$\begin{aligned}&\liminf_{r\to 0}\limsup_{k\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^\infty(t)(dx)\,X^\infty(t)(dy)\,q_{r,\delta_k,\varepsilon_k}(x-y)\\ &\quad+\liminf_{r\to 0}\left(d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} r^{-2}(g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))^2\,dx\,d\theta\Big)ds\right)\\ &\quad+\liminf_{r\to 0}\left(d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} g_1^\infty(x,s)\,r^{-2}(g_1^\infty(x+\theta r,s)-g_1^\infty(x,s))^2\,dx\,d\theta\Big)ds\right)\\ &\leqq \|p_0(\cdot)\|_2^2.\end{aligned} \tag{4.44}$$

This inequality first yields

$$\|p_0(.)\|_2^2 \geqq \liminf_{r\to 0} \limsup_{k\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} X^{\varkappa}(t)(dx)\, X^{\varkappa}(t)(dy)\, q_{r,\delta_k,\varepsilon_k}(x-y)$$

$$= \liminf_{r\to 0} \limsup_{k\to\infty} \int_{\mathbb{R}^d} |\widetilde{X^{\infty}(t)}(\lambda)|^2\, \widetilde{q_{r,\delta_k,\varepsilon_k}}(\lambda)(2\pi)^{d/2}\, d\lambda$$

(by (2.1) and (2.2))

$$\geqq \int_{\mathbb{R}^d} |\widetilde{X^{\infty}(t)}(\lambda)|^2 (\liminf_{r\to 0} \liminf_{k\to\infty} \widetilde{q_{r,\delta_k,\varepsilon_k}}(\lambda))(2\pi)^{d/2}\, d\lambda$$

(by Fatou's Lemma and (4.29))

$$= \int_{\mathbb{R}^d} |\widetilde{X^{\infty}(t)}(\lambda)|^2\, d\lambda \tag{4.45}$$

(by (4.30) and (4.31)).

This implies, that for any fixed $t\in[0,T]$ $X^{\infty}(t)$ has a density $g^{\infty}(.,t)\in L^2(\mathbb{R}^d)$ with respect to Lebesgue measure, such that

$$\|g^{\infty}(.,t)\|_2 \leqq \|p_0(.)\|_2. \tag{4.46}$$

Therefore we may choose now

$$g_1^{\infty}(x,t) = g^{\infty}(x,t), \qquad x\in\mathbb{R}^d,\ 0\leqq t\leqq T. \tag{4.47}$$

We will continue now by searching lower bounds for the remaining "liminf's" on the left side of (4.44).

At first we have

$$\liminf_{r\to 0}\left(d\int_0^t \left(\int_{S^{d-1}}\int_{\mathbb{R}^d} r^{-2}(g^{\infty}(x+r\theta,s)-g^{\infty}(x,s))^2\, dx\, d\theta\right) ds\right)$$

$$= \liminf_{r\to 0}\left(d\int_0^t \left(\int_{S^{d-1}}\int_{\mathbb{R}^d} |\widetilde{g^{\infty}(.,s)}(\lambda)|^2 \left|\frac{e^{-i\theta\cdot\lambda r}-1}{r}\right|^2 d\lambda\, d\theta\right) ds\right)$$

(by (2.2))

$$\geqq \liminf_{r\to 0}\left(d\int_0^t \left(\int_{S^{d-1}}\int_{\{|\lambda|\leqq r^{-1/4}\}} \widetilde{g^{\infty}(.,s)}(\lambda)|^2 (|\theta\lambda|^2 - r)\, d\lambda\, d\theta\right) ds\right)$$

$$\left(\text{since } \left|\frac{e^{-i\theta\lambda r}-1}{r}\right|^2 \geqq |\theta\lambda|^2 - \frac{|\theta\lambda|^4 r^2}{2} \geqq |\theta\lambda|^2 - r \text{ for } |\lambda|\leqq r^{-1/4}\right)$$

$$= d\int_0^t \left(\int_{S^{d-1}}\int_{\mathbb{R}^d} |\widetilde{g^{\infty}(.,s)}(\lambda)|^2\, |\theta\lambda|^2\, d\lambda\, d\theta\right) ds$$

(by (4.20) and (4.47))

$$= \int_0^t \left(\int_{\mathbb{R}^d} |\widetilde{g^{\infty}(.,s)}(\lambda)|^2\, |\lambda|^2\, d\lambda\right) ds. \tag{4.48}$$

Here we used the formula

$$d\int_{S^{d-1}} |\lambda\cdot\theta|^2\, d\theta = |\lambda|^2, \qquad \lambda\in\mathbb{R}^d. \tag{4.49}$$

(4.48) and (2.3) show the existence of $(x,t)\to\nabla g^{\infty}(x,t)$ as a $L^2(\mathbb{R}^d\times[0,t])$-function with

$$\infty>\liminf_{r\to 0}\left(d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} r^{-2}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))^2\,dx\,d\theta\Big)\,ds\right)$$
$$\geqq\int_0^t\Big(\int_{\mathbb{R}^d}|\nabla g^{\infty}(x,s)|^2\,dx\Big)\,ds,\qquad 0\leqq t\leqq T. \tag{4.50}$$

Let us study now the second "liminf" in (4.44). By Fatou's Lemma we have

$$\infty>\liminf_{r\to 0}\left(d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} g^{\infty}(x,s)\,r^{-2}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))^2\,dx\,d\theta\Big)\,ds\right)$$
$$\geqq d\int_{S^{d-1}}\liminf_{r\to 0}\left(\int_0^t\int_{\mathbb{R}^d} g^{\infty}(x,s)\,r^{-2}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))^2\,dx\,ds\right)d\theta. \tag{4.51}$$

Consequently for almost all $\theta\in S^{d-1}$

$$\liminf_{r\to 0}\left(\int_0^t\int_{\mathbb{R}^d} g^{\infty}(x,s)\,r^{-2}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))^2\,dx\,ds\right)<\infty,$$

and so we may take now for a fixed $\theta\in S^{d-1}$ some sequence $r_1>r_2>\ldots>0$, such that the above "liminf" is the limit for this sequence. Then the sequence of functions

$$(x,s)\to\frac{1}{r_i}(g^{\infty}(x+\theta r_i,s)-g^{\infty}(x,s)),\qquad i=1,2,\ldots$$

is weakly relatively compact in the space $L^2(\mathbb{R}^d\times[0,t];g^{\infty}(x,s)\,dx\,ds)$ of all functions f: $\mathbb{R}^d\times[0,t]\to\mathbb{R}$, which are square integrable with respect to the measure $g^{\infty}(x,s)\,dx\,ds$.

Any weak limit $(x,s)\to v(x,s)$ of that sequence satisfies

$$\int_0^t\int_{\mathbb{R}^d} g^{\infty}(x,s)\,|v(x,s)|^2\,dx\,ds$$
$$\leqq\lim_{i\to\infty}\int_0^t\int_{\mathbb{R}^d} g^{\infty}(x,s)\frac{1}{r_i^2}(g^{\infty}(x+\theta r_i,s)-g^{\infty}(x,s))^2\,dx\,ds. \tag{4.52}$$

To identify $v(.,.)$ we take some $g\in C_b^2(\mathbb{R}^d\times[0,t])$. Then

$$\int_0^t\Big(\int_{\mathbb{R}^d} g^{\infty}(x,s)\,r^{-1}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))\,g(x,s)\,dx\Big)\,ds$$
$$=\tfrac{1}{2}\int_0^t\Big(\int_{\mathbb{R}^d}(g^{\infty}(x+\theta r,s)+g^{\infty}(x,s))\,r^{-1}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))\,g(x,s)\,dx\Big)\,ds$$
$$-\tfrac{1}{2}\int_0^t\Big(\int_{\mathbb{R}^d} r^{-1}(g^{\infty}(x+\theta r,s)-g^{\infty}(x,s))^2\,g(x,s)\,dx\Big)\,ds$$
$$=A(1,r)+A(2,r). \tag{4.53}$$

For the first term $A(1,r)$ we have

$$A(1,r)=\tfrac{1}{2}\int_0^t\Big(\int_{\mathbb{R}^d} r^{-1}(g^\infty(x+\theta r,s)^2-g^\infty(x,s)^2)\,g(x,s)\,dx\Big)\,ds$$
$$=\tfrac{1}{2}\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)^2\,r^{-1}(g(x-\theta r,s)-g(x,s))\,dx\,ds.$$

By (4.20) and (4.47) this expression tends as $r\to 0$ to

$$-\tfrac{1}{2}\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)^2\,\nabla_\theta g(x,s)\,dx\,ds$$

($\nabla_\theta g(x,s)=\theta\cdot\nabla g(x,s)$ is the derivative of $x\to g(x,s)$ in direction θ)

$$=\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)\,\nabla_\theta g^\infty(x,s)\,g(x,s)\,dx\,ds \tag{4.54}$$

(by (4.50)).

The absolute value of the second term $A(2,r)$ in (4.53) can be estimated by

$$\frac{r}{2}\|g(.,.)\|_\infty\int_0^t\int_{\mathbb{R}^d} r^{-2}(g^\infty(x+\theta r,s)-g^\infty(x,s))^2\,dx\,ds$$
$$=\frac{r}{2}\|g(.,.)\|_\infty\int_0^t\int_{\mathbb{R}^d}|\widetilde{g^\infty(.,s)}(\lambda)|^2\left|\frac{e^{-i\theta\cdot\lambda r}-1}{r}\right|^2 d\lambda\,ds$$
$$\leqq\frac{r}{2}\|g(.,.)\|_\infty\int_0^t\int_{\mathbb{R}^d}|\widetilde{g^\infty(.,s)}(\lambda)|^2\,|\lambda|^2\,d\lambda\,ds.$$

By (4.50) we therefore have

$$\lim_{r\to 0}|A(2,r)|=0. \tag{4.55}$$

(4.54) and (4.55) show, that for fixed $\theta\in S^{d-1}$ the weak limit of the functions $(x,s)\to r_i^{-1}(g^\infty(x+\theta r_i,s)-g^\infty(x,s))$ in $L^2(\mathbb{R}^d\times[0,t];g^\infty(x,s)\,dx\,ds)$ is the function $(x,s)\to\theta\cdot\nabla g^\infty(x,s)$.

So we obtain from (4.51) and (4.52)

$$\liminf_{r\to 0}\Big(d\int_0^t\Big(\int_{S^{d-1}}\int_{\mathbb{R}^d} g^\infty(x,s)\,r^{-2}(g^\infty(x+\theta r,s)-g^\infty(x,s))^2\,dx\,d\theta\Big)\,ds\Big)$$
$$\geqq d\int_{S^{d-1}}\Big(\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)\,|\theta\cdot\nabla g^\infty(x,s)|^2\,dx\,ds\Big)\,d\theta$$
$$=\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)\,|\nabla g^\infty(x,s)|^2\,dx\,ds \tag{4.56}$$

(by (4.49)).

(4.44), (4.45), (4.50) and (4.56) imply

$$\|g^\infty(.,t)\|_2^2\leqq\|p_0(.)\|_2^2-\int_0^t\int_{\mathbb{R}^d}|\nabla g^\infty(x,s)|^2\,dx\,ds$$
$$-\int_0^t\int_{\mathbb{R}^d} g^\infty(x,s)\,|\nabla g^\infty(x,s)|^2\,dx\,ds. \tag{4.57}$$

(4.50), i.e. the fact, that $(x,t)\to\nabla g^{\infty}(x,t)\in L^2(\mathbb{R}^d\times[0,T])$ allows us to write (4.12) in the following "less weak" form

$$\langle g^{\infty}(.,t),f\rangle-\langle p_0(.),f\rangle+\int_0^t\tfrac{1}{2}\langle 1+g^{\infty}(.,s),\nabla g^{\infty}(.,s)\cdot\nabla f\rangle\,ds=0,$$
$$f\in C_b^1(\mathbb{R}^d),\ 0\leqq t\leqq T,$$

or slightly generalized

$$\langle g^{\infty}(.,t),f(.,t)\rangle-\langle p_0(.),f(.,0)\rangle+\int_0^t\Big(\tfrac{1}{2}\langle g^{\infty}(.,s)+1,\nabla g^{\infty}(.,s)\cdot\nabla f(.,s)\rangle$$
$$-\Big\langle g^{\infty}(.,s),\frac{\partial}{\partial s}f(.,s)\Big\rangle\Big)\,ds=0,\qquad f\in C_b^1(\mathbb{R}^d\times[0,T]),\ 0\leqq t\leqq T. \tag{4.58}$$

Let us consider now the function $U(x,t)=g^{\infty}(x,t)-p(x,t)$, where $p(.,.)$ is the classical solution of (2.13), (2.14). (4.58) and (2.13) imply

$$\langle g^{\infty}(.,t),p(.,t)\rangle-\langle p_0(.),p_0(.)\rangle+\tfrac{1}{2}\int_0^t(\langle 1+g^{\infty}(.,s),\nabla g^{\infty}(.,s)\cdot\nabla p(.,s)\rangle$$
$$+\langle 1+p(.,s),\nabla p(.,s)\cdot\nabla g^{\infty}(.,s)\rangle)\,ds=0, \tag{4.59}$$

and

$$\langle p(.,t),p(.,t)\rangle-\langle p_0(.),p_0(.)\rangle+\int_0^t\langle 1+p(.,s),\nabla p(.,s)\cdot\nabla p(.,s)\rangle\,ds=0. \tag{4.60}$$

(4.57), (4.59) and (4.60) yield

$$\begin{aligned}\langle U(.,t),U(.,t)\rangle&=\langle g^{\infty}(.,t),g^{\infty}(.,t)\rangle-2\langle g^{\infty}(.,t),p(.,t)\rangle+\langle p(.,t),p(.,t)\rangle\\
&\leqq-\int_0^t(\langle 1+g^{\infty}(.,s),\nabla g^{\infty}(.,s)\cdot\nabla g^{\infty}(.,s)\rangle\\
&\quad-\langle 1+g^{\infty}(.,s),\nabla g^{\infty}(.,s)\cdot\nabla p(.,s)\rangle\\
&\quad-\langle 1+p(.,s),\nabla p(.,s)\cdot\nabla g^{\infty}(.,s)\rangle\\
&\quad+\langle 1+p(.,s),\nabla p(.,s)\cdot\nabla p(.,s)\rangle)\,ds\\
&=-\int_0^t(\langle 1+g^{\infty}(.,s),(\nabla g^{\infty}(.,s)-\nabla p(.,s))^2\rangle\\
&\quad+\langle\nabla g^{\infty}(.,s)-\nabla p(.,s),(g^{\infty}(.,s)-p(.,s))\,\nabla p(.,s)\rangle)\,ds\\
&\leqq-\int_0^t(\langle 1+g^{\infty}(.,s),(\nabla g^{\infty}(.,s)-\nabla p(.,s))^2\rangle\\
&\quad-\langle\nabla g^{\infty}(.,s)-\nabla p(.,s),\nabla g^{\infty}(.,s)-\nabla p(.,s)\rangle\\
&\quad-\langle g^{\infty}(.,s)-p(.,s),g^{\infty}(.,s)-p(.,s)\rangle\,\|\nabla p\|_{\infty}^2)\,ds,\end{aligned}$$

i.e.

$$\|U(.,t)\|_2^2\leqq\|\nabla p\|_{\infty}^2\int_0^t\|U(.,s)\|_2^2\,ds.$$

Gronwall's Lemma yields $\|U(.,t)\|_2=0$ for all $t\in[0,T]$. This shows, that for fixed $t\in[0,T]$ $g^\infty(x,t)=p(x,t)$ for almost all $x\in\mathbb{R}^d$, and therefore the proof of Theorem 2 is finished.

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