Method of integration of equations with partial derivatives of the second order with 2 dependent and 2 independent variables.

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1. Let it be given for integration a system of equations of the second order with 2 dependent variables z_1, z_2 and 2 independent x_1, x_2 :

$$(1) \quad \begin{cases} F_1(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2; p_{11}^1, p_{12}^1, p_{22}^1; p_{11}^2, p_{12}^2, p_{22}^2) = 0 \\ F_2(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2; p_{11}^1, p_{12}^1, p_{22}^1; p_{11}^2, p_{12}^2, p_{22}^2) = 0, \end{cases}$$

where is designated

$$p_i^k = \frac{\partial z_k}{\partial x_i}; \quad p_{ij}^k = \frac{\partial^2 z_k}{\partial x_i \partial x_j} \quad (i, j, k = 1, 2).$$

Let us add to the system (1) such a 3rd equation

$$(2) \quad \Phi(x_1, x_2; z_1, z_2; p_1^1, p_2^1; p_1^2, p_2^2; p_{11}^1, p_{12}^1, p_{22}^1, p_{11}^2, p_{12}^2, p_{22}^2) = \text{Const.},$$

that the system (1)-(2) be compatible. For this, designating

$$\begin{aligned} X_i &= \frac{\partial F_g}{\partial x_i} + \frac{\partial F_g}{\partial z_1} p_i^1 + \frac{\partial F_g}{\partial z_2} p_i^2 + \frac{\partial F_g}{\partial p_{11}^1} p_{1i}^1 + \frac{\partial F_g}{\partial p_{12}^1} p_{1i}^2 + \frac{\partial F_g}{\partial p_{21}^1} p_{2i}^1 + \frac{\partial F_g}{\partial p_{22}^1} p_{2i}^2 = \frac{dF_g}{dx_i} \\ P_{ij}^1 &= \frac{\partial F_g}{\partial p_{ij}^1}; \quad P_{ij}^2 = \frac{\partial F_g}{\partial p_{ij}^2} \\ X_i &= \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial z_1} p_i^1 + \frac{\partial \Phi}{\partial z_2} p_i^2 + \frac{\partial \Phi}{\partial p_{11}^1} p_{1i}^1 + \frac{\partial \Phi}{\partial p_{12}^1} p_{1i}^2 + \frac{\partial \Phi}{\partial p_{21}^1} p_{2i}^1 + \frac{\partial \Phi}{\partial p_{22}^1} p_{2i}^2 = \frac{d\Phi}{dx_i} \\ P_{ij}^1 &= \frac{\partial \Phi}{\partial p_{ij}^1}; \quad P_{ij}^2 = \frac{\partial \Phi}{\partial p_{ij}^2} \quad (g, i, j = 1, 2), \end{aligned}$$

we obtain that all the determinants of the 6th order of matrix

$$(3) \quad \begin{vmatrix} X_1 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{11}^2 & 0 & P_{12}^2 & P_{22}^2 \\ X_1 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{11}^2 & 0 & P_{12}^2 & P_{22}^2 \\ X_2 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{11}^2 & 0 & P_{12}^2 & P_{22}^2 \\ X_2 & 0 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{22}^2 & P_{11}^2 & P_{12}^2 \\ X_2 & 0 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{22}^2 & P_{11}^2 & P_{12}^2 \\ X_2 & 0 & P_{11}^1 & P_{12}^1 & P_{22}^1 & 0 & P_{22}^2 & P_{11}^2 & P_{12}^2 \end{vmatrix}$$

are null.

This matrix, as it is shown in my former papers ⁽¹⁾, may be obtained by means of slight generalization of the investigations of E. von WEBER ⁽²⁾.

If one of the determinants, Δ , of the 5th order of this matrix, is different from nought, then, in order that all the determinants of the 6th order be equal to null, it is necessary and sufficient to make null the 4 determinants, obtained from the addition of one of the remainings horizontal lines and one of the 4 remainings columns to the elements of the determinant Δ .

Let us cancel the first horizontal line and the 1, 7, 8, 9 columns. By such means we obtain a determinant of the 5th order Δ_{1789} ; when he is not equal to null, this brings us to limitations

$$(4) \quad D\left(\frac{F_1, F_2}{p_{11}^1, p_{11}^2}\right) \neq 0; \quad D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{12}^1, p_{22}^1}\right) \neq 0,$$

where is designated

$$D\left(\frac{F_1, F_2}{p_{11}^1, p_{11}^2}\right) = \frac{\partial(F_1, F_2)}{\partial(p_{11}^1, p_{11}^2)}; \dots$$

Thus we obtain a system of 4 non linear equations of the first order with one unknown function Φ :

$$(5) \quad \begin{aligned} D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{22}^2}\right) \alpha &= D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{22}^2}\right) \gamma \\ D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{11}^2}\right) \alpha + D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{12}^2}\right) \beta &= D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right) \gamma \\ D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{12}^2}\right) \alpha + D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{22}^2}\right) \beta &= D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{12}^2}\right) \gamma \\ D\left[\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, x_2}\right] \alpha + D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, x_1}\right] \beta &= D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, x_2}\right] \gamma, \end{aligned}$$

where is

$$\alpha = D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{12}^1, p_{11}^2}\right); \quad \beta = D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{12}^1, p_{22}^1}\right); \quad \gamma = D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right),$$

⁽¹⁾ M. KOURENSKY, *Sur l'intégration des équations dif. aux dérivées partielles avec plusieurs variables dépendantes* (« Mémoires de la Cl. Phys.-Math., Académie des Sc. de l'Ukraine », t. V, livre 3, 1927, pp. 79-92); *Die Grundformeln zur Integration der Gleichungen mit partiellen Ableitungen 1. und 2. Ordnung mit mehreren abhändigen und unabhängigen Variablen* (« Sitzungsberichte der M.-N.-Ar. Section Sewcenko Gesellsch. », Lemberg, 1929).

⁽²⁾ E. VON WEBER, « Mathem. Annalen », Bd. 49, 1897; ss. 544-550.

so, that

$$\begin{aligned}
 & \left| \begin{array}{ccc} D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{11}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{12}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right) \\ D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{12}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{22}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{12}^2}\right) \\ D\left[\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, x_2}\right], & D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, x_1}\right], & D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, x_2}\right] \end{array} \right| = 0; \\
 (5') & \left| \begin{array}{ccc} D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{11}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{12}^2}\right) & D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{22}^2}\right) \\ D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{12}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{22}^2}\right) & \\ D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{12}^2}\right) & \\ D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{12}^2}\right), & D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{22}^2}\right) & \end{array} \right| = 0.
 \end{aligned}$$

Having found the particular integral $\Phi_1 = C_1$ of this system (in the case, certainly, if this system can be integrated, so that the POISSON-JACOBI's conditions be satisfied for each pair of these 4 equations (5)), we shall have a system of 3 equations

$$F_1 = 0; \quad F_2 = 0; \quad \Phi_1 = C_1,$$

compatible between themselves. If $\Phi_2 = C_2$ means some other particular integral of the system (5), then the system of three equations

$$F_1 = 0; \quad F_2 = 0; \quad \Phi_2 = C_2$$

will, also, be compatible. The integrals $\Phi_1 = C_1$, $\Phi_2 = C_2$ are obliged to satisfy the system (5').

The compatibility of the system of 4 equations

$$F_1 = 0; \quad F_2 = 0; \quad \Phi_1 = C_1; \quad \Phi_2 = C_2$$

is verified by the conditions of compatibility for each of the two systems:

$$\begin{aligned}
 \text{(I)} & \quad F_1 = 0; \quad \Phi_1 = C_1; \quad \Phi_2 = C_2 \\
 \text{(II)} & \quad F_2 = 0; \quad \Phi_1 = C_1; \quad \Phi_2 = C_2.
 \end{aligned}$$

Assume, that we have found such 4 integrals of the system (5):

$$\Phi_1 = C_1; \dots \quad \Phi_4 = C_4,$$

which make, together with the given system $F_1 = 0$, $F_2 = 0$, a compatible

system of 6 equations, then the derivatives p_{ij}^k , obtained from these 6 equations, satisfy the identity

$$\frac{dp_{ij}^k}{dx_h} = \frac{dp_{ih}^k}{dx_j} \quad (i, j, h, k = 1, 2)$$

(any 3 equations of all the 6 equations

$$F_1 = 0; \quad F_2 = 0; \quad \Phi_1 = C_1; \dots \quad \Phi_4 = C_4$$

will be compatible).

Then, we come to the determination of the unknown functions x_1, x_2 by means of quadratures of the equations

$$\begin{aligned} dp_i^k &= p_{i1}^k dx_1 + p_{i2}^k dx_2 \\ dz_k &= p_1^k dx_1 + p_2^k dx_2 \end{aligned} \quad (i, k = 1, 2).$$

2. Let us show, how the integration of the system of non linear equations (5) can be brought to the integration of the system of linear equations of first order with one dependent variable Φ .

Let us introduce the significations

$$(6) \quad \frac{\alpha}{\gamma} = \lambda; \quad \frac{\beta}{\gamma} = \mu.$$

Then the system (5) will be as follows:

$$\begin{aligned} (7) \quad \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{12}^1, p_{11}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right)} &= \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{11}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{11}^2}\right)} - \mu \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{12}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{11}^2}\right)} = \\ &= \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{12}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{12}^2}\right)} - \mu \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, p_{22}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{12}^2}\right)} = \\ &= \frac{D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, x_2}\right]}{D\left[\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, x_2}\right]} - \mu \frac{D\left[\frac{F_1, F_2, \Phi}{p_{11}^1, p_{11}^2, x_1}\right]}{D\left[\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, x_2}\right]} = \frac{D\left(\frac{F_1, F_2, \Phi}{p_{11}^1, p_{22}^1, p_{22}^2}\right)}{D\left(\frac{F_1, F_2, \Phi}{p_{12}^1, p_{22}^1, p_{22}^2}\right)} = \lambda. \end{aligned}$$

By using the significations of the form

$$\begin{aligned} D\left(\frac{F_1, F_2}{p_{i_1 i_2}^k, p_{j_1 j_2}^l}\right) &= D_{i_1 i_2 j_1 j_2}^{k l}; \quad D\left[\frac{F_1, F_2}{p_{i_1 i_2}^k, x_j}\right] = D_{i_1 i_2 j}^{k o} \\ (i_1, i_2; j_1, j_2; j, k, l &= 1, 2), \end{aligned}$$

so that it will be

$$D_{i_1 i_2 j_1 j_2}^{k l} = -D_{j_1 j_2 i_1 i_2}^{l k}; \quad D_{i_1 i_2 j}^{k o} = -D_{j i_1 i_2}^{o k},$$

from the last relation (6), from the first, the last and the 3rd of the middle members of the equations (7), we shall obtain the following system of linear equations of the 1st order, equivalent to the system (5):

$$\begin{aligned} & (D_{12}^{1 2 1} + \mu D_{11}^{2 2 1}) \frac{\partial \Phi}{\partial p_{11}^1} + D_{22}^{1 1 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{11}^{1 1 2} + \mu D_{11}^{1 1 2}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + \mu D_{22}^{1 1 1} \frac{\partial \Phi}{\partial p_{11}^2} = 0 \\ & (D_{12}^{1 1 2} + \lambda D_{11}^{2 2 1}) \frac{\partial \Phi}{\partial p_{11}^1} + D_{11}^{2 1 1} \frac{\partial \Phi}{\partial p_{12}^1} + \lambda D_{11}^{1 1 2} \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + (D_{11}^{1 1 2} + \lambda D_{22}^{1 1 1}) \frac{\partial \Phi}{\partial p_{11}^2} = 0 \\ & D_{22}^{1 2 2} \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{22}^{2 2 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{22}^{2 1 1} + \lambda D_{12}^{1 2 2}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + (D_{11}^{1 2 2} + \lambda D_{22}^{1 1 2}) \frac{\partial \Phi}{\partial p_{22}^2} = 0 \\ (8) \quad & (D_{22}^{1 1 2} + \mu D_{12}^{2 1 1}) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{11}^{2 2 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{11}^{2 1 1} + \lambda D_{12}^{1 1 2}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + (D_{11}^{1 2 2} + \lambda D_{22}^{1 1 2} + \mu D_{11}^{1 1 2}) \frac{\partial \Phi}{\partial p_{11}^2} + \mu D_{11}^{2 1 1} \frac{\partial \Phi}{\partial p_{12}^2} = 0 \\ & (D_{22}^{1 1 2} + \mu D_{22}^{2 1 1}) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{12}^{2 2 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{12}^{2 1 1} + \lambda D_{12}^{1 1 2}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + \mu D_{11}^{1 2 2} \frac{\partial \Phi}{\partial p_{11}^2} + (D_{11}^{1 2 2} + \lambda D_{22}^{1 1 2}) \frac{\partial \Phi}{\partial p_{12}^2} + \\ & \quad + \mu D_{11}^{2 1 1} \frac{\partial \Phi}{\partial p_{22}^2} = 0 \\ & (D_{22}^{1 2 0} + \mu D_{11}^{0 2 1}) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{22}^{0 2 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{22}^{0 1 1} + \lambda D_{12}^{1 2 0}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ & \quad + \mu D_{11}^{1 2 0} \frac{\partial \Phi}{\partial p_{11}^2} + \mu D_{11}^{2 1 1} \frac{d\Phi}{dx_1} + \\ & \quad + (D_{11}^{1 2 2} + \lambda D_{22}^{1 1 2}) \frac{d\Phi}{dx_2} = 0. \end{aligned}$$

The first and the second of these equations will be dependent on one another, if

$$\frac{D_{11}^{2 1 1}}{D_{22}^{1 1 1}} = \frac{\lambda D_{11}^{1 1 2}}{D_{11}^{1 1 2} + \mu D_{11}^{1 1 2}} = \frac{D_{11}^{1 1 2} + \lambda D_{22}^{1 1 1}}{\mu D_{22}^{1 1 1}} = \frac{D_{12}^{1 1 2} + \lambda D_{11}^{2 2 1}}{D_{12}^{1 2 2} + \mu D_{14}^{2 2 1}},$$

or

$$(9) \quad \begin{aligned} D_{11}^{2 \ 1 \ 1} (D_{22}^{1 \ 1 \ 1} \lambda + D_{11}^{1 \ 1 \ 2} \mu + D_{11}^{1 \ 1 \ 1}) &= 0 \\ D_{11}^{1 \ 2 \ 1} (D_{22}^{1 \ 1 \ 1} \lambda + D_{11}^{1 \ 1 \ 2} \mu + D_{11}^{1 \ 1 \ 1}) &= 0 \\ D_{22}^{1 \ 1 \ 2} (D_{22}^{1 \ 1 \ 1} \lambda + D_{11}^{1 \ 1 \ 2} \mu + D_{11}^{1 \ 1 \ 1}) &= 0; \end{aligned}$$

the last of these equations was obtained because of the identity

$$D_{11}^{1 \ 1 \ 1} D_{11}^{2 \ 1 \ 1} + D_{11}^{2 \ 1 \ 1} D_{12}^{1 \ 2 \ 1} + D_{11}^{2 \ 1 \ 1} D_{22}^{1 \ 1 \ 1} \equiv 0.$$

When we determine λ and μ from the equations (9), then, on the basis of the 1^{rst} of the limitations (4), we shall satisfy only the equation

$$(10) \quad D_{22}^{1 \ 1 \ 1} \lambda + D_{11}^{1 \ 1 \ 2} \mu + D_{11}^{1 \ 1 \ 1} = 0.$$

The second equation for λ and μ we shall draw up so, that we can simplify as much as possible the system (8):

$$(11) \quad D_{22}^{1 \ 1 \ 1} \lambda + D_{11}^{1 \ 2 \ 1} = 0.$$

On the basis of the equation (11) and the identity

$$D_{22}^{2 \ 1 \ 1} D_{22}^{1 \ 1 \ 1} + D_{12}^{1 \ 2 \ 2} D_{22}^{1 \ 1 \ 1} + D_{22}^{2 \ 1 \ 1} D_{12}^{1 \ 1 \ 1} \equiv 0,$$

the first equation of our system will change as follows:

$$D_{22}^{1 \ 2 \ 2} \left(D_{22}^{1 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{11}^1} + D_{11}^{1 \ 2 \ 1} \frac{\partial \Phi}{\partial p_{12}^1} + D_{12}^{1 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{22}^1} \right) = 0.$$

On account of this, if

$$D \left(\frac{F_1}{p_{22}^1}, \frac{F_2}{p_{22}^2} \right) \neq 0,$$

then the integration of the system (5) comes to the integration of the system of 5 linear equations of the 1^{rst} order with one unknown function Φ :

$$(12) \quad \begin{aligned} D_{12}^{1 \ 2 \ 1} \frac{\partial \Phi}{\partial p_{11}^1} + D_{22}^{1 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{12}^1} + D_{11}^{1 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{22}^1} &= 0 \\ (D_{12}^{1 \ 1 \ 2} + \lambda D_{11}^{2 \ 2 \ 1}) \frac{\partial \Phi}{\partial p_{11}^1} + D_{11}^{2 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{12}^1} + \lambda D_{11}^{1 \ 1 \ 2} \frac{\partial \Phi}{\partial p_{22}^1} + \\ &+ (D_{11}^{1 \ 1 \ 1} + \lambda D_{22}^{1 \ 1 \ 1}) \frac{\partial \Phi}{\partial p_{11}^2} = 0 \\ (D_{22}^{1 \ 1 \ 2} + \mu D_{12}^{2 \ 1 \ 1}) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{11}^{2 \ 2 \ 1} \frac{\partial \Phi}{\partial p_{12}^1} + (D_{11}^{2 \ 1 \ 1} + \lambda D_{12}^{1 \ 1 \ 2}) \frac{\partial \Phi}{\partial p_{22}^1} + \\ &+ \mu D_{11}^{1 \ 1 \ 2} \frac{\partial \Phi}{\partial p_{11}^2} + \\ &+ \mu D_{11}^{2 \ 1 \ 1} \frac{\partial \Phi}{\partial p_{12}^2} = 0 \end{aligned}$$

$$\begin{aligned}
 (12) \quad & (D_{22}^1{}_{12}{}^2 + \mu D_{22}^2{}_{11}{}^2) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{12}^2{}_{22}{}^4 \frac{\partial \Phi}{\partial p_{12}^1} + (D_{12}^2{}_{11}{}^4 + \lambda D_{12}^4{}_{12}{}^2) \frac{\partial \Phi}{\partial p_{22}^1} + \\
 & + \mu D_{11}^4{}_{22}{}^2 \frac{\partial \Phi}{\partial p_{11}^2} + \\
 & + \mu D_{11}^2{}_{11}{}^4 \frac{\partial \Phi}{\partial p_{22}^2} = 0 \\
 & (D_{22}^4{}_{2}{}^0 + \mu D_{11}^0{}_{11}{}^2) \frac{\partial \Phi}{\partial p_{11}^1} + \lambda D_{22}^0{}_{22}{}^4 \frac{\partial \Phi}{\partial p_{12}^1} + (D_{22}^0{}_{11}{}^4 + \lambda D_{12}^4{}_{22}{}^0) \frac{\partial \Phi}{\partial p_{22}^1} + \\
 & + \mu D_{11}^4{}_{11}{}^0 \frac{\partial \Phi}{\partial p_{11}^2} + \\
 & + \mu D_{11}^2{}_{11}{}^4 \frac{d\Phi}{dx_1} = 0;
 \end{aligned}$$

but if

$$D\left(\frac{E_1}{p_{22}^4}, \frac{F_2}{p_{22}^2}\right) = 0,$$

then the system (12) is substituted by the system only of the 4 last equations.

If we take another, not equal to null, determinant Δ of the 5th order, then we shall come to one or two limitations, different from the limitations (4).

3. In a particular case, when we have an equation of the 2nd order with only one function z :

$$(13) \quad F(x, y, z; p, q, r, s, t) = 0,$$

the addition to it of the equation

$$\Phi(x, y, z; p, q, r, s, t) = \text{const},$$

brings to the conditions of compatibility under the form of the equality to null of 2 determinants of the 4th order of the matrix

$$\left\| \begin{array}{ccccc}
 \frac{dF}{dx} & \frac{\partial F}{\partial r} & \frac{\partial F}{\partial s} & \frac{\partial F}{\partial t} & 0 \\
 \frac{d\Phi}{dx} & \frac{\partial \Phi}{\partial r} & \frac{\partial \Phi}{\partial s} & \frac{\partial \Phi}{\partial t} & 0 \\
 \frac{dF}{dy} & 0 & \frac{\partial F}{\partial r} & \frac{\partial F}{\partial s} & \frac{\partial F}{\partial t} \\
 \frac{d\Phi}{dy} & 0 & \frac{\partial \Phi}{\partial r} & \frac{\partial \Phi}{\partial s} & \frac{\partial \Phi}{\partial t}
 \end{array} \right\|.$$

It represent a particular case of the matrix (3). The limitations (4) are sub-

stituted by the following:

$$\frac{\partial F}{\partial r} \neq 0; \quad D\left(\frac{F}{r}, \frac{\Phi}{s}\right) \neq 0,$$

if we chose a minor of the 3rd order Δ_{15}^2 , not equal to null, cancelling the 2nd horizontal line and the 1st, 5th column. Adding to Δ_{15}^2 one of the horizontal lines, which is left, and one of the 1st, 5th column, we obtain a known system of the two equations with one unknown function Φ ; it will be the particular case of the system (5).

Determining λ from the square equation

$$\frac{\partial F}{\partial r} \lambda^2 - \frac{\partial F}{\partial s} \lambda + \frac{\partial F}{\partial t} = 0,$$

we come to the integration of the system of 2 linear equations

$$\begin{aligned} \frac{\partial F}{\partial r} \frac{d\Phi}{dx} + \left(\frac{\partial F}{\partial s} - \lambda \frac{\partial F}{\partial r} \right) \frac{d\Phi}{dy} + \left(\lambda \frac{dF}{dy} - \frac{dF}{dx} \right) \frac{\partial \Phi}{\partial r} - \frac{dF}{dy} \frac{\partial \Phi}{\partial s} &= 0 \\ \lambda \frac{\partial F}{\partial t} \frac{\partial \Phi}{\partial r} - \frac{\partial F}{\partial t} \frac{\partial \Phi}{\partial s} + \left(\frac{\partial F}{\partial s} - \lambda \frac{\partial F}{\partial r} \right) \frac{\partial \Phi}{\partial t} &= 0, \end{aligned}$$

by which I replace ⁽¹⁾ the known system ⁽²⁾ of the form

$$\begin{aligned} \frac{d\Phi}{dx} + \lambda_i \frac{d\Phi}{dy} - \frac{\frac{dF}{dx} \partial \Phi}{\frac{\partial F}{\partial r} \partial r} - \lambda_i \frac{\frac{dF}{dy} \partial \Phi}{\frac{\partial F}{\partial t} \partial t} &= 0 \\ \lambda_j^2 \frac{\partial \Phi}{\partial r} - \lambda_j \frac{\partial \Phi}{\partial s} + \frac{\partial \Phi}{\partial t} &= 0. \end{aligned}$$

The above stated method of integration of the system (1) represents, thus, a generalization of the DARBOUX'S method of integration of the equation (13), stated in cours of A. R. FORSYTH.

⁽¹⁾ M. KOURENSKY, *Sur la méthode d'intégration de l'équation aux dérivées partielles du second ordre avec une seule fonction inconnue et deux variables indépendantes*. (« Rendiconti della R. Accad. Naz. dei Lincei », vol. X, 1929, pp. 148-154).

⁽²⁾ A. R. FORSYTH, *Theory of differ. equations*, vol. VI, Cambridge, 1906, pp. 314-317.