Mean curvatures of a subspace in a Finsler space.

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Sunto. - *Il contenuto del presente lavoro trovasi riassunto nel seguente capoverso.*

In a recent paper [1] the author has generalized CALONGHI's theorem [2], concerning the mean curvature of a surface in ordinary space, to a three-dimensional FINSLER space. The geometrical definition of the mean curvature thus obtained is independent of the choice of the element of support of the space. The object of the present paper is to give an extension to an m-dimensional subspace F_m of an n-dimensional FINSLER space F_n. For clearness, we shall begin by investigating the mean curvature of a hypersurface, and then consider the general case.

1. The mean curvature of a hypersurface F_{n-1} in F_n.

Let F_n be an n-dimensional FINSLER space with the element of support (x, x'), along which the contravariant components of the unit vector are denoted by l^i $(i = 1, 2, ..., n)$. Let g_{ik} (x, x') $(i, k = 1, 2, ..., n)$ be the covariant components of the metric tensor. We consider a hypersurface F_{n-1} of equations

$$x^i = x^i(u^\lambda):$$

hereafter Latin indices $i, j, k, ...$ are in the range $1, 2, ..., n$ and Greek indices $\lambda, \mu, \nu, ...$ in the range $1, 2, ..., n-1$. For a tensor in F_{n-1} of components T^λ_μ, the components in F_n are given by the equations

$$T^i_j = T^\lambda_\mu \vartheta^i_\lambda \vartheta^\mu_j,$$

where

$$\vartheta^i_\lambda = \frac{\partial x^i}{\partial u^\lambda}, \qquad \vartheta^\mu_j = g_{ik} g^{\lambda\mu} \vartheta^k_\lambda .$$

We assume that all the functions throughout the following discussion are analytic.

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[1] ZHANG MING YNG, *Die mittlere Krümmung einer Fläche im dreidimensionalen Finslerschen Raum,* « Science Record, Academia Sinica », new series, vol. 1 (1950), n. 1.

[2] M. CALONGHI, *Sulla curvatura media delle superficie,* « Rendic. dei Lincei », (6) 11 (1930), 554-558.

It is known that the normal curvature N of a curve in F_{n-1} at a point P with the tangent vector t^i is given by

$$N = n^i \frac{Dt_i}{ds} = - t_i \frac{Dn^i}{ds} = \Omega_{\lambda\mu} t^\lambda t^\mu , \tag{1}$$

where D denotes covariant differentiation, s is the arc-length of the curve and n^i is the normal vector of F_{n-1}. Then the mean curvature M of F_{n-1} at P is

$$M = g^{\lambda\mu} \Omega_{\lambda\mu} \Omega^{\lambda}_{\lambda} . \tag{2}$$

In particular, when the direction of the element of support (x, x') coincides with n^i or with the tangential direction of F_{n-1}, we obtain the mean curvature of CARTAN (3) or that of BERWALD (4) respectively.

Consider in F_{n-1} a variable closed hypersurface F_{n-2} containing P and denote by $w^i(Q)$ the contravariant components of the normal unit vector of F_{n-2} in F_{n-1} at a generic point Q of F_{n-2}. If $w^i(Q/P)$ are the components of the vector obtained from $w^i(Q)$ by a parallel displacement with respect to F_n from Q to P, along an arbitrary arc C in F_{n-1}, and the complementary of the angle between $w^i(Q/P)$ and $n^i(P)$ is taken as the angle θ between $w^i(Q)$ and the tangential hyperplane E_{n-1} of F_{n-1} at P, then we have the following theorem:

THEOREM 1. - *The mean curvature* M *of* F_{n-1} *at* P *is given by*

$$M = \lim_{F_{n-2} \to P} \frac{\int_{F_{n-2}} \theta d\tau_{n-2}}{\int_{R_{n-1}} d\tau_{n-1}} ,$$

where R_{n-1} *denotes the domain of* F_{n-1} *enclosed by* F_{n-2}, *and* $d\tau_{n-1}$ *and* $d\tau_{n-2}$ *denote the volume elements of* R_{n-1} *and* F_{n-2} *respectively.*

PROOF. - Without loss of generality, we can assume that the coordinates of P and Q in F_{n-1} are $(0, 0, \ldots, 0)$ and $(u^1, u^2, \ldots, u^{n-1})$ respectively. Denoting the infinitesimal vector on C by $\delta x^i = \vartheta^i_\lambda \delta u^\lambda$, oriented by the sense

(3) E. CARTAN, *Les espace de Finsler*, « Actualités scientifiques et industrielles », 79 (1934), Hermann & Cie., Paris.

(4) L. BERWALD, *Über die Kauptkrümmungen einer Fläche im dreidimensionalen Finslerschen Raum*, « Monatsh. f. Math. u. Phys. », 43 (1936), 1-14.

from Q to P, we have

$$(3)\qquad w_i(Q/P) = w_i(Q) - \int_Q^P \overset{*}{\Gamma}{}^j_{ik}(x,\ x')w_j(x,\ x')\delta x^k$$

$$- \int_Q^P A^j_{ik}(x,\ x')w_j(x,\ x')\omega^k$$

$$= w_i(P) - \int_Q^P \frac{\partial w_i}{\partial x^k}\delta x^k - \int_Q^P \overset{*}{\Gamma}{}^j_{ih}(x,\ x')w_j(x,\ x')\delta x^k$$

$$- \int_Q^P A^j_{ik}(x,\ x')w_j(x,\ x')\omega^k$$

$$= w_i(P) + \int_Q^P Dw_i(x,\ x'),$$

where we have placed

$$w_i(P) = w_i(Q) - \int_Q^P \frac{\partial w_i}{\partial x^k}\delta x^k, \qquad \omega^h = Dl^k.$$

Since F_n is of a Euclidean connection, $w_i(Q/P)$ remain the covariant components of a unit vector. Hence from (3) it follows that

$$(4)\qquad \sin\theta = n^i(P)w_i(Q/P) = n^i(P)\left\{w_i(P) + \int_Q^P Dw_i\right\}.$$

Moreover, $w_i(P)$ is a covariant vector in F_{n-1}, so that

$$n^i(P)w^i(P) = 0, \quad n^i(P)Dw_i(P) = -\,w_i(P)Dn^i(P).$$

From (1) it is clear that

$$-\,\delta x^i Dn_i = \Omega_{\lambda\mu}\delta u^\lambda \delta u^\mu = \Omega_{\lambda\mu}\delta u^\lambda \delta x^i \vartheta^\mu_i,$$

and consequently

$$-\,Dn_i = \Omega_{\lambda\mu}\delta u^\lambda \vartheta^\mu_i,$$

so we get from (4)

$$(5)\qquad \sin\theta = \int_Q^P n^i(P)Dw_i = \int_Q^P n^i Dw_i + 0(u^\lambda)$$

$$= -\int_Q^P w_i Dn^i + 0(u^\lambda) = \int_Q^P \Omega^\lambda_\mu w_\lambda \delta u^\mu + 0(u^\lambda)$$

$$= \Omega^\lambda_\mu(Q)w_\lambda(Q)u^\mu + 0(u^\lambda) \qquad (u^\lambda \to 0).$$

Now suppose that F_{n-2} is defined by

$$u^\lambda = u^\lambda(v^\alpha) \qquad (\alpha = 1, 2, \dots, n-2);$$

then

(6) $$d\tau_{n-2} = \sqrt{\bar{g}}\, dv^1 dv^2 \dots dv^{n-2},$$

where

$$\bar{g} = \det(g_{\alpha\beta}), \qquad g_{\alpha\beta} = g_{\lambda\mu}\frac{\partial u^\lambda}{\partial v^\alpha}\frac{\partial u^\mu}{\partial v^\beta} \qquad (\alpha, \beta = 1, 2, \dots, n-2).$$

By changing the coordinates

(7) $$\underset{\lambda}{u}^\lambda = v^\alpha, \qquad \underset{\lambda}{u}^\lambda = (-1)^{\lambda-1}u^\lambda, \qquad \underset{\lambda}{u}^\beta = v^{\beta-1} \qquad (\alpha < \lambda < \beta),$$

λ being not summed, we get

$$\underset{\lambda}{w}_\alpha = w_\nu \frac{\partial u^\nu}{\partial \underset{\lambda}{u}_\alpha} = w_\nu \frac{\partial u^\nu}{\partial v^\alpha} = 0,$$

$$\underset{\lambda}{w}_\beta = w_\nu \frac{\partial u^\nu}{\partial \underset{\lambda}{u}^\beta} = w_\nu \frac{\partial u^\nu}{\partial v^{\beta-1}} = 0,$$

since $\dfrac{\partial u^\nu}{\partial v^\alpha}$ are the contravariant components of a vector in F_{n-2}, and w_ν the covariant components of a normal unit vector to F_{n-2}. Thus we are led to the relation

(8) $$\underset{\lambda}{w}_\lambda = \frac{1}{\sqrt{\underset{\lambda}{g}^{\lambda\lambda}}} = \frac{\sqrt{\underset{\lambda}{\bar{g}}}}{\sqrt{\underset{\lambda}{g}}} \qquad (\lambda \text{ not summed}),$$

where $\underset{\lambda}{g}^{\mu\nu}$, $\underset{\lambda}{\bar{g}}$ and $\underset{\lambda}{g}$ denote the transformed expression of $g^{\mu\nu}$, $\bar{g}$ and $g = \det(g_{\lambda\mu})$ respectively, as it is easily seen from the assumption

$$\underset{\lambda}{g}^{\mu\nu}\underset{\lambda}{w}_\mu \underset{\lambda}{w}_\nu = \underset{\lambda}{g}^{\lambda\lambda}\underset{\lambda}{w}_\lambda^2 = 1 \qquad (\lambda \text{ not summed}).$$

Therefore from (6), (7) and (8) we deduce

$$\begin{aligned} w_\lambda d\tau_{n-2} &= w_\lambda \sqrt{\bar{g}}\, dv^1 dv^2 \dots dv^{n-2} \\ &= (-1)^{\lambda-1}\underset{\lambda}{w}_\lambda \sqrt{\underset{\lambda}{\bar{g}}}\, d\underset{\lambda}{u}^1 d\underset{\lambda}{u}^2 \dots d\underset{\lambda}{u}^{\lambda-1} d\underset{\lambda}{u}^{\lambda+1} \dots d\underset{\lambda}{u}^{n-1} \\ &= (-1)^{\lambda-1}\sqrt{\underset{\lambda}{g}}\, d\underset{\lambda}{u}^1 d\underset{\lambda}{u}^2 \dots d\underset{\lambda}{u}^{\lambda-1} d\underset{\lambda}{u}^{\lambda+1} \dots d\underset{\lambda}{u}^{n-1}. \end{aligned}$$

By applying the generalized GREEN'S integral theorem, we obtain from (5) and (9) that

$$\int_{F_{n-2}} \theta d\tau_{n-2} = \int_{F_{n-2}} \Omega^{\lambda}_{\mu} u^{\mu} w_{\lambda} d\tau_{n-2} + 0(u^{\lambda^{n-1}})$$

$$= \int_{F_{n-2}} \sum_{\lambda=1}^{n-1} \Omega^{\lambda}_{\mu} u^{\mu} \sqrt{\underset{\lambda}{g}} (-1)^{\lambda-1} \underset{\lambda}{du^1} \underset{\lambda}{du^2} \dots \underset{\lambda}{du^{\lambda-1}} \underset{\lambda}{du^{\lambda+1}} \dots \underset{\lambda}{du^{n-1}} + 0(u^{\lambda^{n-1}})$$

$$= \sum_{\lambda=1}^{n-1} \int_{F_{n-2}} \Omega^{\lambda}_{\mu} u^{\mu} \sqrt{g} (\underset{\lambda}{du^1} \underset{\lambda}{du^2} \dots \underset{\lambda}{du^{n-2}} - \underset{\lambda}{du^1} \underset{\lambda}{du^2} \dots \underset{\lambda}{du^{n-3}} \underset{\lambda}{du^{n-1}}$$

$$+ \dots + (-1)^{\lambda-1} \underset{\lambda}{du^1} \underset{\lambda}{du^2} \dots \underset{\lambda}{du^{\lambda-1}} \underset{\lambda}{du^{\lambda+1}} \dots \underset{\lambda}{du^{n-1}}$$

$$+ \dots + (-1)^{n-2} \underset{\lambda}{du^2} \underset{\lambda}{du^3} \dots \underset{\lambda}{du^{n-1}}) + 0(u^{\lambda^{n-1}})$$

$$= \sum_{\lambda=1}^{n-1} \int_{R_{n-1}} \Omega^{\lambda}_{\lambda} d\tau_{n-1} + 0(u^{\lambda^{n-1}})$$

$$= \Omega^{\lambda}_{\lambda}(P) \int_{R_{n-1}} d\tau_{n-1} + 0(u^{\lambda^{n-1}}), \qquad u^{\lambda} \to 0.$$

Thus we have completed the proof.

2. Mean curvatures of an m-dimensional subspace F_m in F_n.

There is n · difficulty in extending the same arguments to the general case where an m-dimensional subspace F_m is considered instead of F_{n-1}. For a normal vector $\overrightarrow{n_{\sigma 1}}$ to F_m, we can similarly define the mean curvature M_σ of F_m at a point P. Denote, in fact, by F_{m-1} a variable closed hypersurface in F_m containing P and by θ_σ the angle between the normal vector w_i to F_{m-1} in F_m and the tangential $(n-1)$-dimensional affine space $E_{\sigma 1, n-1}$ orthogonal to $\overrightarrow{n_{\sigma 1}}$. Then we have

THEOREM 2. - *The mean curvature* M_σ *of* F_m *at* P *corresponding to* $\overrightarrow{n_{\sigma 1}}$ *is given by*

$$M_\sigma = \lim_{F_{m-1} \to P} \frac{\int_{F_{m-1}} \theta_\sigma d\tau_{m-1}}{\int_{R_m} d\tau_m},$$

where R_m *denotes the domain of* F_m *enclosed by* F_{m-1}, *and* $d\tau_m$ *and* $d\tau_{m-1}$ *denote the volume elements of* R_m *and* F_{m-1} *respectively.*

COROLLARY 1. - If, in particular, F_m is « gespannt », i. e., if the $(n-m)$-dimensional affine space E_n^{n-m} orthogonal to E_m at a point P of F_m can be determined uniquely, then there exists a normal vector $\overrightarrow{n_1}$ which we shall call the mean curvature normal, such that the mean curvature M of F_m at P for $\overrightarrow{n}$ is given by the equation

$$M = \lim_{F_{m-1} \to P} \frac{\int_{F_{m-1}} \theta d\tau_{m-1}}{\int_{R_m} d\tau_m} = \sqrt{\sum_{\sigma=m+1}^{n} M_\sigma^2},$$

where θ denotes the angle between w_i and the tangential affine space E_{n-1} orthogonal to $\overrightarrow{n}$, and the components of $\overrightarrow{n}$ are

$$n^i = \sum_{\sigma=m+1}^{n} \Omega_{\sigma 1 \lambda\mu} g^{\lambda\mu} n_{\sigma 1}{}^i,$$

independent of the choice of the orthogonal ennuple $\{\overrightarrow{n}_{\sigma_1}\}$, so that $\overrightarrow{n}$ has the length M.

We shall call M the mean curvature of the « gespannt » subspace F_m.

COROLLARY 2. - If F_m is « gespannt », then its mean curvature is zero if its mean curvature for any normal vector is zero.

COROLLARY 3. - The mean curvature of a « gespannt » subspace F_m, for any normal vector orthogonal to the mean curvature normal, is zero.

These corollaries are evident generalizations of some known results in Riemannian geometry ([5]).

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([5]) Cf., for example, L. P. EISENHART, *Riemannian geometry*, « Princeton Univ. Press. », (1925), 169-170.