On a Certain Class of Set Theoretic Properties.

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Summary. - *One of the most useful results in introductory topology is the theorem that if* S *is connected and* C *is a connected subset such that* $S - C = X \cup Y$, *separate* (« $\cup$ » *means union), then* $X \cup C$ *and* $Y \cup C$ *are connected. In this paper a study is made of other set properties satisfying a similar theorem. That is, if* $P \subset S$ *both have a property* $\mathfrak{S}$ *and* $S - P = X \cup Y$ *separate, then* $P \cup X$ *and* $P \cup Y$ *have this property. Furthermore, many properties have corresponding localizations, for example local connectedness, and it is determined under what conditions the corresponding local property satisfies the above theorem if the original property does.*

Throughout the paper we assume the sets to be imbedded in a space that is at least HAUSDORFF. In those theorems involving homology properties, we shall assume the space is compact and that the cycles are those defined by CECH.

THEOREM A. - *If* $P \subset S$, P *and* S *both have the proprty* $\mathfrak{S}$ *aud* $S - P = X \cup Y$ *separate, then* $X \cup P$ *and* $Y \cup P$ *both have* $\mathfrak{S}$.

DEFINITION. - S *is said to be locally* $\mathfrak{S}(l\mathfrak{S})$ *if for every point* $p \in S$ *and open set* U *relative to* S, $p \in U$, *there exists an open set* V *relative to* S, *such that* $p \in V \subset U$, *and* V *has* $\mathfrak{S}$.

DEFINITION. - S is said to be *strongly locally* $\mathfrak{S}(sl\mathfrak{S})$ if for every $P \subset S$ such that P has $\mathfrak{S}$ and open set U relative to S such that $P \subset U$, there exists an open set V relative to S such that $P \subset V \subset U$, and such that V has $\mathfrak{S}$.

DEFINITION. - Any property $\mathfrak{S}$ that is enjoyed by every set consisting of a single point, is called a *point property.*

The following theorem is clear from the definitions.

THEOREM 1. - *If* $\mathfrak{S}$ *is a point property, and* S *is* $sl\mathfrak{S}$, *then* S *is* $l\mathfrak{S}$.

THEOREM 2. - *If* S *is* $l\mathfrak{S}$, $sl\mathfrak{S}$, *and* $\mathfrak{S}$ *satisfies theorem A, then* $l\mathfrak{S}$ *satisfies theorem A.*

Proof. Consider $P \subset S$ where P and S are $l\mathfrak{S}$, and $S - P = X \cup Y$ separate. Suppose $x \in X \cup P$ and $p \in U \subset X \cup P$ where U is open in $X \cup P$. There exists an open set U' relative to S such that $U = U' \cap (X \cup P)$. If $p \in X$, there exists an open set $V' \subset U'$ such that $p \in V' \subset X \cup P$ since X is separate from Y. Since S is $l\mathfrak{S}$ there exists an open set W' of S such that $p \in W' \subset V'$ and W' has $\mathfrak{S}$. Next suppose $p \in P$, and let $U'' = P \cup U'$. Since P is $l\mathfrak{S}$, there exists

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an open set V'' of P such that $p \in V'' \subset U''$, V'' has $\mathfrak{S}$, and there exists a set V' open in S such that $V'' = P \cap V'$. We can suppose $V' \subset U'$ since $P \cap V' = P \cap (V' \cap U')$ where $V' \cap U'$ is the required open set. Since S is $sl\mathfrak{S}$ and $V'' \subset V'$, there exists a set W' open in S such that $p \in V'' \subset W' \subset V'$ and W' has $\mathfrak{S}$. Furthermore $V'' = V' \cap P$ implies $V'' = W' \cap P$. If $W' \subset X \cup P$, we have found the desired set. If $W' \not\subset X \cup P$, then $W' - V'' = W_x' \cup W_y'$ separate, where $W_x' = W' \cap X$, $W_y' = W' \cap Y$. Since $\mathfrak{S}$ satisfies theorem A, and W' and V'' have $\mathfrak{S}$, it follows that $V'' \cup W_x'$ and $V'' \cup W_y'$ have $\mathfrak{S}$. But $W_x' \cup V'' = (W' \cap X) \cup (W' \cap P) = W' \cap (X \cup P)$; thus $W_x' \cup V''$ is open in $X \cup P$ and has $\mathfrak{S}$. Since the same arqument holds for $p \in Y$, we have shown that $l\mathfrak{S}$ satisfies theorem A.

COROLLARY 2.1. - *If* S *is* sl$\mathfrak{S}$ *were* $\mathfrak{S}$ *is a point property satisfying theorem A, then* l$\mathfrak{S}$ *satisfies theorem A.*

COROLLARY 2.2. - *If* $\mathfrak{S}$ *is a property satisfied only by open sets, and* $\mathfrak{S}$ *satisfies theorem A, then* l$\mathfrak{S}$ *satisfies theorem A.*

Proof. S is automatically $sl\mathfrak{S}$ for if $P \subset U$ has $\mathfrak{S}$, then P must be open and $V = P$ is an open set vith $\mathfrak{S}$ such that $P \subset V \subset U$, and the result follows from the theorem. Finally we note that it is possible for such a property $\mathfrak{S}$ to satisfy theorem A, since if P and S are both open, then $X \cup P$ and $Y \cup P$ are open (this is stated in theorem 5).

The following is a list of fundamental properties which have meaningful local properties. For the meaning of any property not defined here see G. T, WHYBURN, *Analytic Topology*, « Colloquium Series ».

$\mathfrak{S}$ 1. To be connected.

$\mathfrak{S}$ 2. To have a closure that is compact (i. e. every covering by open sets has a finite subcovering). (We state the property in terms of the closure so that the local property will have meaning).

$\mathfrak{S}$ 3. To have a boundary that is compact.

$\mathfrak{S}$ 4. To consist of a finite number of points.

$\mathfrak{S}$ 5. To consist of an at most countably infinite number of points.

$\mathfrak{S}$ 6. To consist of an uncountable number of points.

$\mathfrak{S}$ 7. To have a boundary consisting of a finite number of points.

$\mathfrak{S}$ 8. To have a boundary consisting of an at most countably infinite number of points.

$\mathfrak{S}$ 9. To have a particular point p as a non-cut point.

$\mathfrak{S}$ 10. To consist entirely of non-cut points.

$\mathfrak{S}$ 11. To be arc-wise connected.

$\mathfrak{S}$ 12. To be an open connected set whose closure is a dendrite (a dendrite is a locally connected continuum that contains no simple closed curve. Again we state the property so that the local property is significant).

THEOREM 3. - *All of the properties* $\mathfrak{S}$ 1 *to* $\mathfrak{S}$ 12 *listed above satisfy theorem A.*

Proof. The result is well known for 1. It follows for 2 and 3 trivially, since any closed subset of a compact set is countably compact. The result is obvious for 4, 5, and 6.

The result follows for 7 and 8 as follows. Let P and S both have either property 7 or 8 and be such that $S - P = X \cup Y$ separate. It is readily shown that $B(X \cup P) \subset B(S) \cup B(P)$ where $B(A) =$ the boundary of A relative to a universal imbedding space; thus if $B(S)$ and $B(P)$ are both finite or at most countably infinite, then $B(X \cup P)$ enjoys the same property.

To prove the result for 9, we consider a particular point $p \varepsilon S$ and $P \subset S$ such that p is a non-cut point of P and of S, and such that $S - P = X \cup Y$ separate. Suppose p is a cut point of $X \cup P$, i. e. $(X \cup P) - p = X_1 \cup X_2$ separate. Since $P - p$ is connected, we can suppose $P - p \subset X_1$, and thus $X_2 \subset X$. Now $S - p = X_2 \cup (X_1 \cup Y)$ is a separation which contradicts p's being a non-cut point of S; therefore, p is a non-cut point of $X \cup P$ and likewise of $Y \cup P$. The result now follows for 10 by repeated application of 9.

To show the result for 11, let P and S be arc-wise connected, where $P \subset S$ and $S - P = X \cup Y$ separate. Consider the points x_1 and $x_2 \varepsilon X \cup P$ and any arc A of S with end points x_1 and x_2. Let $p_1 \varepsilon A$ be the last point on A such that the open subarc from x_1 to p_1 lies in X. Either $p_1 \varepsilon X$ or P, since no point of Y is a limit point of X. If $p_1 \boldsymbol{\varepsilon} P$ (*) then it follows from its definition that there exists a point q_1 in the arc from p_1 to x_2 such that $q_1 \varepsilon P$, and arc $p_1 q_1 \subset X \subset P$. If $p_1 \varepsilon p$, define $q_1 = p_1$, then in either case the arc x_1 to q_1 lies in $X \cup P$. Similary there exists a point $q_2 \varepsilon P$ such that the arc x_2 to q_2 lies in $X \cup P$. Since P is arcwise connected, there exists in P an arc from q_1 to q_2. We can now extract an arc in $X \cup P$ from the union of the three arcs x_1 to q_1, q_1 to q_2, and x_2 to q_2 which all lie in $X \cup P$. Thus $X \cup P$ and similary, $Y \cup P$ are arc-wise connected.

To show the result for 12, we observe that $X \cup P$ is an open connected subset of S by $\mathfrak{S}$ 1, and $\overline{X \cup P} \subset \bar{S}$ implies that $\overline{X \cup P}$ is a dendrite since $\bar{S}$ is.

The $l\mathfrak{S}$ properties corresponding to the 12 properties listed above are as follows:

$l\mathfrak{S}$ 1. To be locally connected.

$l\mathfrak{S}$ 2. To be locally compact.

$l\mathfrak{S}$ 3. To be locally peripherally compact (this is a concept introduced by R. L. WILDER).

$l\mathfrak{S}$ 4. To be locally finite.

$l\mathfrak{S}$ 5. To be locally countable infinite.

$l\mathfrak{S}$ 6. To consist entirely of condensation points.

$l\mathfrak{S}$ 7. To be regular curve.

$l\mathfrak{S}$ 8. To be a rational curve.

(*) ($\boldsymbol{\varepsilon}$) means « not contained in ».

$l\mathcal{S}$ 9. To have a particular point p as a local non-cut point.

$l\mathcal{S}$ 10. To consist entirely of local non-cut points.

$l\mathcal{S}$ 11. To be locally arc-wise connected.

$l\mathcal{S}$ 12. To be locally a dendrite.

THEOREM 4. - *If* S *is* $l\mathcal{S}$, *it is* $sl\mathcal{S}$ *for each of the above 12 properties.*

Proof. Let P be a subset having $\mathcal{S}$ of S having $l\mathcal{S}$ and let U be an open set relative to S such that $P \subset U$. Cover P with open sets $\{V_p\}$ such that V_p has $\mathcal{S}$ and $V_p \subset U$.

Proof. for 1. Let $V = \bigcup_p V_p$, then $P \subset V \subset U$, where V is open and connected.

2. There exists a finite subcovering $\{V_i\}$ since P is compact and $V = \bigcup_{i=1}^n V_i$ is an open set $\subset U$ such that $\bar{V}$ is compact.

3. There exists a finite subcovering $\{V_i\}$ of $B(P)$ (the boundary) since $B(P)$ is compact and $V = \bigcup_{i=1}^n V_i) \bigcup P \supset P$ is an open set $\subset U$ such that $B(V) \subset \bigcup_{i=1}^n B(V_i)$. Thus $B(V)$ is compact since it is a closed subset of a compact set.

4. There are only a finite number of $\{V_p\}$ since P is finite; therefore, $\bigcup_{i=1}^n V_i$ is an open finite set $\subset U$.

5. There at most a countable number of sets $\{V_p\}$ since P is countable; therefore, $\bigcup_{i=1}^\infty V_i$ is an open countable set $\subset U$.

6. $V = \bigcup_p V_p \supset P$ is uncountable since P is an open subset of U.

7. Since $B(P)$ is finite, there exists a finite sub-covering of it. Now $V = (\bigcup_{i=1}^n V_i) \bigcup P \subset U$ is an open set $\supset P$ with a finite boundary, since $\subset \bigcup_{i=1}^n B(V_i) =$ a finite set.

8. Since $B(P)$ is countable, there exists a countable subcovering of it. Now $V = (\bigcup_{i=1}^\infty V_i) \bigcup P \subset U$ is an open set $\supset P$ with a countable boundary.

9. This property implies local connectedness, for corresponding to each q (whether $= p$ or not) an open set $U \supset q$, there exists and open set V such that $q \subset V \subset U$ and such that p is not a cut point of V. Thus in any case (even if $p \not\in V$), V must be connected. Thus $V = \bigcup_q V_q \supset P$ is connected and $V \subset U$. Suppose $V - p$ is disconnected, then since $P - p$ is connected and $\subset V - p$, it lies in one component V_1 of $V - p$. Now $V_p - p$ is connected and $V_2 = V_p \bigcup V_1 \supset P$ is an open connected subset of U. $V_2 - p = (V_p - p) \bigcup V_1$ is connected since p is a limit point of V_1; thus $(V_p - p) \bigcap V_1 \neq 0$.

10. This follows by successive application of number 9.

11. Let $V = \bigcup_p V_p \supset P$, then V is a connected subset of U. Now consider points q and $r \in V$, then there exists a finite chain chosen from $\{V_p\}$ joining q to r; i. e. $\{V_i\}$ $i = 1, \ldots, n$ such that $q \in V_1$, $r \in V_n$ and $V_i \bigcap V_{i+1} \neq 0$. Choose $q_i \in V_i \bigcap V_{i+1}$ for $i = 1, \ldots, n-1$, then there exist arcs joining q to q_1 in V_1, q_1 to q_2 in V_2, ..., q_n to r in V_n. We can now extract a subarc from q to r in V from the union of these arcs; thus V is arc-wise connected.

12. Let $V = \bigcup_{i=1}^{n} V_i$ be a finite subcovering of the boundary of P, then $P \cup V \subset U$ is an open connected set such that $\overline{P \cup V}$ is a dendrite.

THEOREM 5. - *The properties* $1\mathfrak{F}_i$ (i = 1, ..., 12) *satisfy theorem* A.

Proof. This follows directly from theorem 2 after applying theorems 3 and 4.

The following is a list of fundamental properties which either have no meaningful corresponding local property or one that does not satisfy theorem 4.

$\mathfrak{F}$ 13. To be closed (relative to some fixed space S' in which both P and S of theorem A are imbedded).

$\mathfrak{F}$ 14. To be open (relative to some fixed space S' in which both P and S are imbedded).

$\mathfrak{F}$ 15. To be countably compact (i. e. every infinite subset of S has a limit point in S).

$\mathfrak{F}$ 16. To be compact.

$\mathfrak{F}$ 17. To be complete.

$\mathfrak{F}$ 18. To be everywhere dense.

$\mathfrak{F}$ 19. To be nowhere dense.

$\mathfrak{F}$ 20. To be perfect.

$\mathfrak{F}$ 21. To be separable (i. e. to have a countable dense subset).

$\mathfrak{F}$ 22. To be perfectly separable (i. e. to have a countable number of open sets equivalent to the collection of all open sets).

$\mathfrak{F}$ 23. To have (MENGER-URYSOHN) dimension n,

$\mathfrak{F}$ 24. To be cyclicly connected.

$\mathfrak{F}$ 25. To be a linear graph (i. e. the union of a finite number of arcs having at most end points in common).

THEOREM 6. *The properties* $\mathfrak{F}$ 13 *to* $\mathfrak{F}$ 25 *satisfy theorem* A.

Proof. The result follows for $\mathfrak{F}$ 13 to $\mathfrak{F}$ 20 with little or no proof.

For $\mathfrak{F}$ 21 let D and E be the countable dense subset of S and P respectively, where $S - P = X \cup Y$, separate, Let $D_1 = D \cap (X \cup P)$ and consider $x \varepsilon X$ if $x \not\varepsilon D$, then $x \varepsilon \overline{D}$. Now $X \cap \overline{Y} = 0$ implies $x \varepsilon \overline{D}_1$, and if $x \varepsilon P$, then $x \varepsilon \overline{E}$. Thus $\overline{D_1 \cup E} \supset X \cup P$ and $D_1 \cup E$ is a countable dense subset of $X \cup P$.

The result is trivial for $\mathfrak{F}$ 22 since any subset of a perfectly separable set is perfectly separable.

The result follows for $\mathfrak{F}$ 23 since $X \cup P \subset S$ implies $\dim (X \cup P) \leqq n$ and $\dim P = n$ implies $\dim (X \cup P) = n$.

To prove the result for $\mathfrak{F}$ 24, consider points x_1 and $x_2 \varepsilon X \cup P$, and a simple closed curve of $S \supset x_1$ and x_2. Let A and B be the two arcs from x_1 to x_2 which make up S. Now we can apply the argument of theorem 3 used in connection with $\mathfrak{F}$ 11 (arc-wise-connectedness) to each of the arcs A and B. This gives sub-arcs $x_1p_1q_1$ and $x_2p_2q_2$ of A and sub-arcs $x_1p_1'q_1'$ and $x_2p_2'q_2'$ of B, where no two arcs have points in common, q_1, q_2, q_1', and $q_2' \varepsilon P$, and p_1, p_2, p_1', p_2' are defined as before. Since P in cyclicly connected, there exists a simple closed curve in P through q_1 and q_2. Let r_1 and r_2 be the first points on arcs p_1q_1 and p_2q_2 that lie on this simple closed curve, then there exist arcs R_1 and R_2 from r_1 to r_2 lying in $(A \cup S) \cap P$ having only $r_1 \cup r_2$ in common. Now let R be the one of the arcs R_1, R_2 that does not pass through either p_1' or p_2' if such exists. If this is not possible, then R_1 must pass through p_1' but not p_2' and R_2 must pass through p_2', but not p_1', or visa versa, and in either case, of course, p_1' and $p_2' \varepsilon P$ (arc $x_1p_1') - p_1' \subset X$, (arc $x_2p_2') - p_2' \subset Y$. Now the arcs x_1 to r_1, r_1 to p_2' on R_2, p_2' to x_2 join to form an arc x_1 to x_2 that is disjoint except for x_1 and x_2 from the one formed by joining the arcs x_2 to r_2, r_2 to p_1' on R_1, and p_1' to x_1. Thus we have a simple closed curve in $X \cup P$ through x_1 and x_2. Next consider R defined above and let $A' =$ the arc from x_1 to x_2 in $X \cup P$ formed by joining the arcs x_1 to r_1, R, and r_2 to x_2. R may intersect arc p_1' to q_1' or p_2' to q_2' or both, but since p_1', $p_2' \,\overline{\varepsilon}\, B$, hence $\overline{\varepsilon}\, \overline{R}$, there exist points $r_1' \varepsilon P$ on arc p_1' to q_1' and $r_2' \varepsilon P$ on arc p_2' to q_2' such that arcs x_1 to r_1' and x_2 to r_2' do not intersect R. Since P is cyclicly connected, there exists a simple closed curve through r_1' and r_2' consisting of the arcs C and D from r_1' to r_2'. Let s and t be the first and last points, respectively, on the arc A' from x_1 to x_2 that lie on $C \cup D$. If s and t both lie on C (or D), then the arc formed by joining the arcs x_1 to s on A', s to t on C, and t to x_2 on A' is disjoint except for x_1 and x_2 from the arc formed by joining the arcs x_1 to r_1', D, and r_2' to x_2. Thus we have a simple closed curve through x_1 and x_2 in $X \cup P$. If s lies on C and t on D (or visa versa), then the arc formed by joining the arcs x_1 to s on A', s to r_2' on C, and r_2' to x_2 is disjoint except for x_1 and x_2 from the arc formed by joining the arcs x_1 to r_1', r_1' to t on D, and t to x_2 on A'. Thus in every case we obtain a simple closed curve in $X \cup P$ through x_1 and x_2.

The result follows immediately for $\mathcal{S}$ 25 since any closed connected subset of a linear graph is another graph.

We shall now consider some higher dimensional properties involving CECH cycles and homology. As previously stated, we shall apply these propertes only to compact spaces. For a statement of most of the concepts connected with CECH cycles to be used here see, for example, E. G. BEGLE, *Locally Connected Spaces and Generalized Manifolds*, « American Journal of Mathematics », vol. 64 (1940), pp. 553-574. Before stating these properties, we give five lemmas that will appear in R. L. WILDER's Colloquium book that will be useful in the following.

LEMMA 7. - *If* L *is a closed subset of* S *(compact) and* $\mathfrak{U}$ *is a covering of* S, *then there exists a covering* $\mathcal{W}$, *a refinement of* $\mathfrak{U}$, *such that if the nucleus of a cell (the intersection of the open sets which are it vertices) ofs* $\mathcal{W}$ *meets both* L *and* S — L, *then it meets* F(L), *the boundary of* L.

LEMMA 8. - *If* z^r *is a cycle mod* K *on* M, *then collection* $\{\partial z^r(\mathfrak{U})\}$ *is an* $(r-1)$-*cycle on* K, *which we denote by* ∂z^r. *Evidently* $\partial z^r \sim 0$ *on* M.

LEMMA 9. - *If* z^r *is a cycle on* K *such that* $z^r \sim 0$ *on* M, *then there exists a cycle* z^{r+1} *mod* K *on* M *such that* $\partial z^{r+1} \sim z^r$.

LEMMA 10. - *If* z^r *is a cycle mod* K *on* M *such that* $\partial z^r \sim 0$ *on* M, *then there exists a cycle* γ^r *on* M *such that* $z^r \sim \gamma^r$ *mod* K.

LEMMA 11. - *If* z^r is *a cycle mod* K *such that* $z^r \sim 0$ *mod* M, *then there exists a cycle* γ^r *mod* K *on* M *such that* $z^r \sim \gamma^r$ *mod* K.

$\mathfrak{F}$ 26. To be *simply*-i-*connected*; i. e. S has $\mathfrak{F}$ 26 if every i-dimensional CECH cycle on S is ~ 0 on S. (S is also compact by the above stated assumption).

$\mathfrak{F}$ 27. To have the point p as a *non*-i-*cut point relative to* S: i. e. every i-dimensional cycle on a closed subset of $S-p$ is ~ 0 on a closed subset of $S-p$ (where p does not necessarily bslong to S; i. e. if $p \varepsilon S$, this means that S is simply-i-connected).

$\mathfrak{F}$ 28. To consist entirely of non-i-cut points.

$\mathfrak{F}$ 29. To be *locally*-i-*connected at a point* p; i. e. corresponding to every open est U such that $p \varepsilon U$, there exists an open set V such that $p \varepsilon V \subset U$ and such that every i-cycle on V is ~ 0 on U. (Again p need not belong to S, in which case the condition is vacuously satisfied since no U exists with $p \varepsilon U$).

$\mathfrak{F}$ 30. To be *locally-i-connected*; i. e. to be locally-i-connected at each point.

$\mathfrak{F}$ 31. To have a point p as a *local non*-i-*cut point*; i. e. corresponding to every neighborhood U such that $p \varepsilon U$, there exists an open set V such that $p \varepsilon V \subset U$ and such that every i-cycle on a closed subest of $V-p$ is ~ 0 on a closed subset of $U-p$.

$\mathfrak{F}$ 32. To consist entirely of non-i-cut points.

$\mathfrak{F}$ 33. To be *co-locally*-i-*connected at* p; i. e. corresponding to any open set U such that $p \varepsilon U$, there exists an open set V such that $p \varepsilon V \subset U$ and such that if z^i is any i-cycle mod $S-U$, then $z^i \sim 0$ mod $S-V$.

$\mathfrak{F}$ 34. To be *co-locally*-i-*connected*; i. e. to be co-locally-i-connected at each point.

$\mathfrak{F}$ 35. To have *local* i-*betti number* 1 *at a point* p *of the interior of* S (relative to some fixed imbedding space S'); i. e. S is not co-locally-i-connected at p, and corresponding to any open set U such that $p \varepsilon U$, there exists an open set V such that $p \varepsilon V \subset U$ and such that if z_1^i and z_2^i are cycles mod $S-U$ on S, then there exist integers n_1 and n_2, not both 0, such that $n_1 z_1^i \sim n_1 z_1^i$ mod $S-V$ on S.

$\mathfrak{F}$ 36. To have local i-betti number 1 at such interior point.

$\mathcal{S}$ 37. To be the *closure of an open generalized-n-manifold* (imbedded is a space S'); i. e. is the closure of an open generalized-n-manifold if

1) dim $S = n$
2) S is locally-i-connected $(0 \leq i \leq n)$
3. S is co-locally-i-connected $(0 \leq i \leq n-1)$
4) S has local n-betti number 1 at every point p interior to S relative to S'.

(This definition has more significance if we require S' to be a space with respect to which S has interior points although this is not necessary in the following proof.) (see E. G. BEGLE, *Duality Theorems for Generalized Manifolds*, « American Journal of Mathematics », vol. 67 (1945), pp. 59-70).

THEOREM 12. - *The properties* $\mathcal{S}$ i (i = 26,..., 37) *satisfy theorem* A.

Proof. $\mathcal{S}$ 26. Consider $P \subset S$ each with property $\mathcal{S}$ 26 such that $S - P = X \cup Y$, separate. Let z^i be any i-cycle on $X \cup P$, then $z^i \sim 0$ on S. By lemma 9, there exists a cycle z^{i+1} mod $(X \cup P)$ on S such that $\partial z^{i+1} \sim z^i$. By lemma 7 we can construct a cofinal family of coverings with the property that if the nucleus of any cell formed from one of these coverings intersects both Y and $S - Y$, then it intersects. $F(Y)$. We shall confine the coordinates of all chains to this family. Let $z^{i+1}(\mathfrak{U}) = z_1{}^{i+1}(\mathfrak{U}) + z_2{}^{i+1}(\mathfrak{U})$ for each covering $\mathfrak{U}$, such that $z_2{}^{i+1}(\mathfrak{U})$ is the collection of cells of $z^{i+1}(\mathfrak{U})$ on Y and $z_1{}^{i+1}(\mathfrak{U})$ is the remainder of $z^{i+1}(\mathfrak{U})$; hence $z_1{}^{i+1}(\mathfrak{U})$ is on $X \cup P = S - Y$. Now $\partial z_2{}^{i+1}(\mathfrak{U}) = \gamma^i(\mathfrak{U})$ is a cycle on Y since $z_2{}^{i+1}(\mathfrak{U})$ is on Y. By lemma 8, $\{\partial z^{i+1}(\mathfrak{U})\}$ is a cycle on $X \cup P$; hence $\partial z_2{}^{i+1}(\mathfrak{U}) = \partial z^{i+1}(\mathfrak{U}) - \partial z_1{}^{i+1}(\mathfrak{U})$ is on $X \cup P = S - Y$. The choice of the cofinal family then implies that $\gamma^i(\mathfrak{U})$ is on $F(Y)$ for all $\mathfrak{U}$ of the family. To show that $\{\gamma^i(\mathfrak{U})\}$ is a CECH-cycle, we recall that z^{i+1} is a CECH cycle mod $X \cup P$; i. e. for any $\mathfrak{U} > \mathfrak{V}$ there exists $C^{i+2}(\mathfrak{V})$ such that $\partial C^{i+2}(\mathfrak{V}) = \pi^{\mathfrak{V}}_{\mathfrak{U}} z^{i+1}(\mathfrak{U}) - z^{i+1}(\mathfrak{V}) + \gamma^{i+1}(\mathfrak{V})$ where $\gamma^{i+1}(\mathfrak{V})$ is on $X \cup P$. This gives $\partial C^{i+2}(\mathfrak{V}) = \pi^{\mathfrak{V}}_{\mathfrak{U}} z_2{}^{i+1}(\mathfrak{U}) - z_2{}^{i+1}(\mathfrak{V}) + {}+ \pi^{\mathfrak{V}}_{\mathfrak{U}} z_1{}^{i+1}(\mathfrak{U}) - z_1{}^{i+1}(\mathfrak{V}) + \gamma^{i+1}(\mathfrak{V})$ where the last three terms of the right hand side are all on $X \cup P$. Let $C^{i+2}(\mathfrak{V}) = C_1{}^{i+2}(\mathfrak{V}) + C_2{}^{i+2}(\mathfrak{V})$ where $C_2{}^{i+2}(\mathfrak{V})$ consists of the cells of $C^{i+2}(\mathfrak{V})$ on Y and $C_1{}^{i+2}(\mathfrak{V})$ of those not on Y (hence in $S - Y$). This leads to the following: $\partial C_2{}^{i+2}(\mathfrak{V}) - (\pi^{\mathfrak{V}}_{\mathfrak{U}} z_2{}^{i+1}(\mathfrak{U}) - z_2{}^{i+1}(\mathfrak{V})) = -\partial C_1{}^{i+2}(\mathfrak{V}) + (\pi^{\mathfrak{V}}_{\mathfrak{U}} z_1{}^{i+1}(\mathfrak{U}) - z_1{}^{i+1}(\mathfrak{V})) + \gamma^{i+1}(\mathfrak{V})$ where the left hand side is on Y and the right hand side is on $S - Y$; thus both sides = a chain $X^{i+1}(\mathfrak{V})$ on $F(Y)$. By taking boundaries, we obtain $\partial X^{i+1}(\mathfrak{V}) = \partial\partial C^{i+2}(\mathfrak{V}) - (\pi^{\mathfrak{V}}_{\mathfrak{U}} \partial z_2{}^{i+1}(\mathfrak{U}) - \partial z_2{}^{i+1}(\mathfrak{V}))$, or $\partial[- X^{i+1}(\mathfrak{V})] = \pi^{\mathfrak{V}}_{\mathfrak{U}} \gamma^{i+1}(\mathfrak{U}) - \gamma^{i+1}(\mathfrak{V})$; thus, $\{\gamma^{i+1}(\mathfrak{U})\}$ is a CECH cycle on $F(Y) \subset P$. Since P has $\mathcal{S}$ 26, $\gamma^{i+1} \sim 0$ on P; hence, there exists a chain $z_3{}^{i+1}(\mathfrak{U})$ on P for each $\mathfrak{U}$ such that $\partial z_3{}^{i+1}(\mathfrak{U}) = \gamma^i(\mathfrak{U}) = \partial z_2{}^{i+1}(\mathfrak{U})$. It follows that $z^i(\mathfrak{U}) = \partial z^{i+1}(\mathfrak{U}) = \partial(z_1{}^{i+1}(\mathfrak{U}) + z_2{}^{i+1}(\mathfrak{U})) = \partial(z_1{}^{i+1}(\mathfrak{U}) + z_3{}^{i+1}(\mathfrak{U}))$; i. e. $z^i \sim 0$ on $X \cup P$. From this we conclude that $X \cup P$ and similarly $Y \cup P$ both have $\mathcal{S}$ 26.

$\mathcal{S}$ 27. The proof for $\mathcal{S}$ 27 is entirely analogous to that for $\mathcal{S}$ 26. Here we consider a cycle z^i on C_1, a closed subset of $(X \cup P) - p$. C_1 is also a closed subset of $S - p$; hence, $z^i \sim 0$ on C_2, a closed subset of $S - p$. The cycle γ^i is now established on $F(Y) \cap C_2$ which is a closed subest of $P - p$ and is ~ 0 on C_3, a closed subset of $P - p$. Finally we show that $z^i \sim 0$ on $C_2 \cap (X \cup P) \cup C_3$, which is a closed subset of $(X \cup P) - p$. This shows that $X \cup P$ and similarly $Y \cup P$ have $\mathcal{S}$ 27.

$\mathcal{S}$ 28. The result follows by repeated application of the result for $\mathcal{S}$ 27.

$\mathcal{S}$ 29. Let U be any open set in $X \cup P$ containing p (if one exists). $U \cap P$ is an open set in P; hence, since P has $\mathcal{S}$ 29, there exists an open set $V \subset U \cap P$ such that $p \in V$ and such that every i-cycle on V is ~ 0 on $U \cap P$. We can find an open set V' of S such that $V = V' \cap P$, and the choice can be made such that $V' \cap (X \cup P) \subset U$. Since S has $\mathcal{S}$ 29, there exists an open set W' of S such that $p \in W' \subset V'$ and such that every i-cycle on W' is ~ 0 on V'. (Of course if $p \in X$, then W' can be chosen at once without first choosing V). Let $W = W' \cap (X \cup P)$, then W is open in $X \cup P$. Now consider any i-cycle z^i on W. Since $W \supset W'$ $z^i \sim 0$ on V' and there exists a cycle z^{i+1} on V' mod W such that $\partial z^{i+1} \sim z^i$ on W. As in the proof for $\mathcal{S}$ 26 we can choose a cofinal family of coverings, such that z^{i+1} intersects $V' \cap F(P) \supset V$ in a cycle γ^i. Now $\gamma^i \sim 0$ on $U \cap P$ and we can use the chains in this homology, the part of z^{i+1} on $X \cup P$, and the chains in the homology between ∂z^{i+1} and z^i to form chains on U bounded by z^i; thus, $z^i \sim 0$ on U and $X \cup P$ has $\mathcal{S}$ 29.

$\mathcal{S}$ 30. The result follows by repeated application of the result for $\mathcal{S}$ 29.

$\mathcal{S}$ 31. The proof is almost exactly like that for $\mathcal{S}$ 29. Here V is chosen so that every i-cycle on a closed subset of $V - p$ is ~ 0 on a closed subset of $V - p$ is ~ 0 on a closed subset of $(U \cap P) - p$. Similarly W' is chosen so that every i-cycle on a closed subset of $W' - p$ is ~ 0 on a closed subset of $V' - p$. Now an i-cycle on a closed subset of $W - p$ is chosen and the procedure previously followed leads to an homology on a closed subset of $U - p$; thus p is a local non-i-cut point of $X \cup P$.

$\mathcal{S}$ 32. The result follows by repeated application of the result for 31.

$\mathcal{S}$ 33. Suppose $P \subset S$, both P and S have $\mathcal{S}$ 30 relative to p, and $S - P = X \cup Y$ separate. Suppose $p \in X \cup P$ and let U be any open set of $X \cup P$ such that $p \in U$. There exists an open set U' of S such that $U' \cap (X \cup P) = U$. Since S has $\mathcal{S}$ 30, there exists an open set V' of S such that $p \in V' \subset U'$ and such that every i-cycle mod $S - U'$ is ~ 0 mod $S - V'$. Now consider an i-cycle z^i of $X \cup P$ mod $(X \cup P) - U$; z^i is also a cycle of S mod $S - U'$ and is, therefore, ~ 0 on S mod $S - V'$. By lemma 11, there exists a cycle γ^i mod $(X \cup P) - U$ on $S - V'$ such that $z^i \sim \gamma^i$ mod $(X \cup P) - U$ on S. As before we will use lemma 7, which allows us to restrict the coordinates of all cycles to coverings with the property that a cell on both Y

and $S-Y$ is on $F(Y)$. Let $\mathfrak{U}$ be any such covering, then $z^i(\mathfrak{U})-\gamma^i(\mathfrak{U})+$ $+x^i(\mathfrak{U})=\partial C^{i+1}(\mathfrak{U})$ where $x^i(\mathfrak{U})$ is on $(X\cup P)-U$ and $C^{i+1}(\mathfrak{U})$ is on S. Let $C^{i+1}(\mathfrak{U})=C_1^{i+1}(\mathfrak{U})+C_2^{i+1}(\mathfrak{U})$ where $C_2^{i+1}(\mathfrak{U})$ consists of all cells of $C^{i+1}(\mathfrak{U})$ on Y and $C_1^{i+1}(\mathfrak{U})$ is the remainder; hence, on $S-X$. Similary we can write $\gamma^i(\mathfrak{U})=\gamma_1^i(\mathfrak{U})+\gamma_2^i(\mathfrak{U})$ where $\gamma_2^i(\mathfrak{U})$ consists of the cells on Y and $\gamma_1^i(\mathfrak{U})$ consists of the remainder of $\gamma^i(\mathfrak{U})$. Since $\partial C^{i+1}(\mathfrak{U})=\partial C_1^{i+1}(\mathfrak{U})+$ $+\partial C_2^{i+1}(\mathfrak{U})=z^i(\mathfrak{U})-\gamma_1^i(\mathfrak{U})-\gamma_2^i(\mathfrak{U})+x^i(\mathfrak{U})$, we can write $\partial C_2^{i+1}(\mathfrak{U})+$ $+\gamma_2^i(\mathfrak{U})=-\gamma_1^i(\mathfrak{U})+z^i(\mathfrak{U})+x^i(\mathfrak{U})-\partial C_2^{i+1}(\mathfrak{U})$ where the left side is on Y and the right side is on $S-Y$. Thus each side is on $F(Y)$ and we will let $\gamma_3^i(\mathfrak{U})=\partial C_2^{i+1}(\mathfrak{U})+\gamma_2^i(\mathfrak{U})$ for all $\mathfrak{U}$. We will show that $\{\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{U})\}$ is a cycle mod $(X\cup P)-U$ on $X\cup P$, for consider any $\mathfrak{V}>\mathfrak{U}$. We have $\pi_{\mathfrak{V}}^{\mathfrak{U}}(\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{V}))-(\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{U})=\pi_{\mathfrak{V}}^{\mathfrak{U}}z^i(\mathfrak{V})-z^i(\mathfrak{U})+\pi_{\mathfrak{V}}^{\mathfrak{U}}x^i(\mathfrak{V})-x^i(\mathfrak{U})-$ $-\partial(\pi_{\mathfrak{V}}^{\mathfrak{U}}C_1^{i+1}(\mathfrak{V})-C_1^{i+1}(\mathfrak{U}))$, but $z^i(\mathfrak{U})$ is a cycle on $X\cup P$ mod $(X\cup P)-U$, $\{x^i(\mathfrak{U})\}$ is on $(X\cup P)-U$, and $\pi_{\mathfrak{V}}^{\mathfrak{U}}C_1^{i+1}(\mathfrak{V})-C_1^{i+1}(\mathfrak{U})$ is on $X\cup P$. These combined facts tell us that the left side is ~ 0 on $X\cup P$ mod $(X\cup P-U$. Finally $\partial(\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{U}))=\partial(z^i(\mathfrak{U})+x^i(\mathfrak{U})-\partial C_1^{i+1}(\mathfrak{U}))=\partial z^i(\mathfrak{U})+\partial x^i(\mathfrak{U})$ which is on $(X\cup P)-U$. We also note that $(\gamma_1^i+\gamma_3^i)$, which we shall now call γ_4^i, is $\sim z^i$ mod $(X\cup P)-U$ on $X\cup P$. This follows since $z^i(\mathfrak{U})-$ $-(\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{U}))+x^i(\mathfrak{U})=\partial C_1^{i+1}(\mathfrak{U})$ for all $\mathfrak{U}$ where $x^i(\mathfrak{U})$ is on $(X\cup P)-U$ and $C_1^{i+1}(\mathfrak{U})$ is on $X\cup P$. We must now consider two cases.

Case 1). If $p\in X$, then V' could have been chosen such that $V'\subset X$, then $\gamma_3^i(\mathfrak{U})$ is on $F(Y)\subset(X\cup P-V'$ for all $\mathfrak{U}$. Also $\gamma_1^i(\mathfrak{U})$ is on $S-V'$ and on $X\cup P$; therefore, it is on $(X\cup P)-V'$. This shows that γ_4^i is on $(X\cup P)-V'$; hence z^i is ~ 0 mod $(X\cup P)-V'$ on $X\cup P$ since $z^i\sim\gamma_4^i$ mod $[(X\cup P)-U]\subset[(X\cup P)-V']$ on $X\cup P$.

Case 2). If $p\in P$, then let $V=V'\cap P$ and there exists an open set W on P such that $p\in W\subset V$ and such that every i-cycle mod $P-V$ on P is ~ 0 mod $P-W$ on P. Let W' be an open set in $V\cup P$ such that $p\in W'\subset V'$ and such that $W'\cap P=W$. We now show that $\{\gamma_3(\mathfrak{U})\}$ is a cycle mod $P-V$ on P. We first recall that $\gamma_3(\mathfrak{U})$ was shown to be on $F(Y)\subset P$ for all $\mathfrak{U}$. Furthermore $\partial\gamma_3(\mathfrak{U})=\partial(\partial C_2^{i+1}(\mathfrak{U})+\gamma_2^i(\mathfrak{U}))=\partial\gamma_2^i(\mathfrak{U})$, where $\gamma_2^i(\mathfrak{U})$ is part of $\gamma_1^i(\mathfrak{U})$ on $S-V'$; thus, $\partial\gamma_2^i(\mathfrak{U})$ is on both P and $S-V'$ and, thus, on $P\cap S-P\cup V'=P-V$. Since $\{\gamma_1(\mathfrak{U})+\gamma_3(\mathfrak{U})\}$ is a cycle on $X\cup P$ mod $(X\cup P)-U$, we have for any $V>U$ that $\pi_{\mathfrak{V}}^{\mathfrak{U}}(\gamma_1^i(V)+\gamma_3^i(V))-$ $-(\gamma_1^i(\mathfrak{U})+\gamma_3^i(\mathfrak{U}))+y^i(\mathfrak{U})=\partial K^{i+1}(\mathfrak{U})$, where $y^i(\mathfrak{U})$ is on $(X\cup P)-U$ and $K^{i+1}(\mathfrak{U})$ is on $X\cup P$. Let $K^{i+1}(\mathfrak{U})=K_1^{i+1}(\mathfrak{U})+K_2^{i+1}(\mathfrak{U})$ where $K_2^{i+1}(\mathfrak{U})$ is on V' and $K_1^{i+1}(\mathfrak{U})$ is on $(X\cup P)-V'$. Then $\partial K^{i+1}(\mathfrak{U})=\partial K_1^{i+1}(\mathfrak{U})+\partial K_2^{i+1}(\mathfrak{U})=$ $=(\pi_{\mathfrak{V}}^{\mathfrak{U}}\gamma_1^i(V)-\gamma_1^i(\mathfrak{U}))+(\pi_{\mathfrak{V}}^{\mathfrak{U}}\gamma_3^i(\mathfrak{V})-\gamma_3^i(\mathfrak{U}))+y^i(\mathfrak{U})$; hence $\partial K_1^{i+1}(\mathfrak{U})-$ $-(\pi_{\mathfrak{V}}^{\mathfrak{U}}\gamma_1^i(\mathfrak{V})-\gamma_1^i(\mathfrak{U}))-y^i(\mathfrak{U})=\pi_{\mathfrak{V}}^{\mathfrak{U}}\gamma_3^i(\mathfrak{V})-\gamma_3(\mathfrak{U})-\partial K_2^{i+1}(\mathfrak{U})$ where the left hand side is on $(X\cup P)-V'$ and the right hand side is on P. This tells us that the right hand side is on $[(X\cup P)-V']\cap P=P-V$ (if we restrict ourselves by lemma 7 to a suitable cofinal family of coverings) and is, therefore, equal to a chain $k^i(\mathfrak{U})$ on $P-V$; thus, $(\pi^{\mathfrak{U}}_{\mathfrak{V}}\gamma_3{}^i(\mathfrak{V})-\gamma_3{}^i(\mathfrak{U}))-k^i(\mathfrak{U})=$ $=\partial K_2{}^{i+1}(\mathfrak{U})$ which implies $\pi^{\mathfrak{U}}_{\mathfrak{V}}\gamma_3(\mathfrak{V})\backsim\gamma_3(\mathfrak{U})$ mod $(P-V)$ on P. This completes the proof that $\{\gamma_3{}^i(\mathfrak{U})\}=\gamma_3{}^i$ is a cycle on P mod $P-V$. By the choice of W, $\gamma_3{}^i\backsim 0$ mod $P-W$ on P; hence by lemma 11 there exists a cycle $\gamma_5{}^i$ mod $P-V$ on $P-W$ such that $\gamma_3{}^i\backsim\gamma_5{}^i$ mod $P-V$ on P. By an entirely analogous argument, we can show that $\{\gamma_1{}^i(\mathfrak{U})\}$ is a cycle on $(X\cup P)-V'$ mod $(X\cup P)-U$; thus $\gamma_1{}^i+\gamma_5{}^i$ is a relative cycle and it is on $(X\cup P)-W'$. Since $z^i\backsim\gamma_1{}^i+\gamma_3{}^i$ mod $(X\cup P)-U$ on $X\cup P$ and $\gamma_3{}^i\backsim\gamma_5{}^i$ mod $P-V$ on P, we have $z^i\backsim\gamma_1{}^i+\gamma_5{}^i$ mod $(X\cup P)-V'$ on $X\cup P$. Putting these facts together, we conclude that $z^i\backsim 0$ mod $X\cup P)-W'$.

$\mathcal{S}$ 34. The result follows by repeated application of the result for $\mathcal{S}$ 33.

$\mathcal{S}$ 35. Consider $P\subset S\subset S'$ such that $S-P=X\cup Y$ separate and $p\in X\cup P$. Since S is not co-locally connected at p, there exists an open set U such that for any open set V such that $p\in V\subset U$ there exists a cycle z^i mod $S-U$ on S not $\backsim 0$ mod $S-V$ on S. If $p\in X$, we can suppose $U\subset X$. If $p\in P$, then p is in the interior of P and of S relative to S'; hence, we can suppose $U\subset P$. In either case, $V\subset U\subset X\cup P$. There is no loss in assuming z^i is on U, then z^i is not $\backsim 0$ mod $(X\cup P)-X$ on $X\cup P$ since it is not $\backsim 0$ on the larger set $S-V$. This tells us that $(X\cup P)$ is not co-locally connected at p. To show the second part of the definition holds on $X\cup P$, we apply the argument of $\mathcal{S}$ 35 to $n_1x_1{}^i-n_2x_2{}^i$ were $z_1{}^i$ and $z_2{}^i$ are cycles mod $(X\cup P)-U$ and obtain an open set $W\subset U$ such that $n_1z_1{}^i-n_2z_2{}^i\backsim 0$ or $n_1z_1{}^i\backsim n_2z_2{}^i$ mod $(X\cup P)-W$ on $X\cup P$.

$\mathcal{S}$ 36. The result follows by repeated application of the result for $\mathcal{S}$ 35.

$\mathcal{S}$ 37. Let P and S be the closures of open generalized-n-manifolds imbeded in S' such that $P\subset S$ and $S-P=X\cup Y$ separate. We see that $X\cup P$ and $Y\cup P$ are also both closures of open generalized-n-manifolds by applying theorem A for the properties $\mathcal{S}$ 23, 30, 34, and 36.

DEFINITION. - We shall say that a property $\mathcal{S}$ *Satisfies theorem A strengthened by* $\mathcal{S}_1$ if in the hypothesis of theorem A, we require P to have $\mathcal{S}_1$ as well as $\mathcal{S}$ while S still just has $\mathcal{S}$.

The following properties satisfy theorem A strengthened by certain other properties.

38. *To be semi-*i*-connected at* p; i. e. corresponding to any open set U such that $p\in U$, there exists an apen set V such that $p\in V\subset U$ and that such that every i-cycle on V is $\backsim 0$ on S.

39. *To be* i*-avoidable at* p; i. e. corresponding to any open set U such that $p\in U$, there exists an open set V such that $p\in V\subset U$ and such that every i-cycle on $F(U)$ is $\backsim 0$ on $S-V$.

40. *To be locally-*i*-avoidable at* p; i. e. corresponding to any open set U

such that $p \in U$, there exist open sets V and W such that $p \in W \subset V \subset U$, and such that every i-cycle on $F(V)$ is ~ 0 on $S - W$.

41. *To be completely-*i*-avoidable at* p; i. e. corresponding to any open set U such that $p \in U$, there exists open sets V and W such that $p \in W \subset V \subset U$, and such that every i-cycle on $F(V)$ is ~ 0 on $U - W$.

THEOREM 13. - *Properties* $\mathcal{S}$ 38, 39. *and* 40 *satisfy theorem* A *strengthened by simple-*i*-connectedness and* $\mathcal{S}$ 37 *satisfies theorem* A *strengthened by semi-*i*-connectedness.*

Proof for $\mathcal{S}$ 38. Let U be any open set of $X \cup P$ such that $p \in U$. There exists an open set U' of S such that $U' \cap (X \cup P) = U$. Since S has $\mathcal{S}$ 38, there exists an open set V' of S such that $p \in V' \subset U'$ and such that every cycle z^i on V' is ~ 0 on S. Let z^i be any cycle of $X \cup P$ on $V = V' \cap (X \cup P)$, then $z^i \sim 0$ on S. By lemma 9, there exists a cycle z^{i+1} mod V on S such that $\partial z^{i+1} \sim z^i$ on V. By methods entirely analogous to ones previously used, we can show that z^{i+1} intersects $F(Y)$ in a cycle γ^i on P which is ~ 0 on P. This allows us to replace z^{i+1} by a chain γ^{i+1} on $X \cup P$ such that $\partial \gamma^{i+1} \sim z_i$ on V; hence $z^i \sim 0$ on $X \cup P$, and $X \cup P$ is semi-i-connected at p.

$\mathcal{S}$ 39. Let U be any open set such that $p \in U$, let U' be open is S such that $U' \cap (X \cup P) = U$. Since S has $\mathcal{S}$ 39 at p, there exist V' such that $p \in V' \subset U'$ and such that every i-cycle on $F(U')$ is ~ 0 on $S - V'$. Let $V = V' \cap (X \cup P)$ and z^i be any cycle of $X \cup P$ on $F(U)$, then $z^i \sim 0$ on $S - V'$. If $p \in X$, we can assume $U \subset X$ and the method of $\mathcal{S}$ 38 leads to an homology $z^i \sim 0$ on $(X \cup P) - V$. If $p \in P$, we proceed as in $\mathcal{S}$ 38 obtaining γ^i on P ~ 0 on P. However this homology may intersect V; therefore, we let $V'' = V \cap P$ and choose W'', open in P, such that $p \in W'' \subset V''$ and such that every i-cycle on $F(V'')$ is ~ 0 on $P - W''$. Now the homology $\gamma^i \sim 0$ on P intersects $F(V'')$ in a cycle $\gamma_1{}^i$ which is ~ 0 on $P - W''$. If we choose an open set W of $X \cup P$ such that $W \cap P = W''$ and $p \in W \subset V$, then piecing these homologies together gives $z^i \sim 0$ on $(X \cup P) - W$.

$\mathcal{S}$ 40. This is proved by an argument entirely analogous to the one used in $\mathcal{S}$ 39.

$\mathcal{S}$ 41. Consider the case where $p \in P$ and let U be any open set of $X \cup P$ such that $p \in U$. There exists an open set V' of P such that $p \in V'$ and such that any i-cycle of P on V' is ~ 0 on P. Let V be any open set of S such that $V \cap (X \cup P) \subset U$, and such that $p \in V$. Since S has $\mathcal{S}$ 41 at p, there exist open sets W and Q of S such that $p \in Q \subset W \subset V$ and such that any i-cycle of S on $F(W)$ is ~ 0 on $V - Q$. Since P has $\mathcal{S}$ 41 at p, there exist open sets R and T of $(X \cup P)$ such that $p \in T \subset R \subset Q \subset (X \cup P)$ and such that any i-cycle of P on $F(R \cap P)$ is ~ 0 on $(Q \cap P) - (T \cap P)$. Now any i-cycle of $X \cup P$ on $F[W \cap (X \cup P)]$ is ~ 0 on $U - [T \cap (X \cup P)]$. This follows from a proof entirely analogous to the ones above. The case $p \in X$ is also handled as before.