Nonlinear Boundary Value Problems and Orlicz Spaces (*).

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Sunto. - Usando metodi dell'analisi funzionale si ottengono teoremi di esistenza delle soluzioni di un'equazione della forma $Lx + Nx = 0$ dove L è un operatore differenziale ordinario lineare autoaggiunto di ordine pari, positivo, mentre N è un operatore non necessariamente lineare.

1. - Introduction.

In this paper we use functional analytic techniques to show the existence of solutions to boundary value problems of the form

$$(1.1) \quad Lx + Nx = 0$$

where L is an even order, positive, self-adjoint, linear ordinary differential operator and N is an operator, usually nonlinear.

The following theorem is an example of the type of problem we first considered.

THEOREM 1.1. - Let $f(t, x, y)$ be continuous from $D = \{(t, x, y) : 0 \leq t \leq 1, |x| \leq R, |y| < \infty\}$ into R^n and let f satisfy

- a) $x \cdot f(t, x, y) + |y|^2 \geq 0$ when $x \cdot y = 0$ and $|x| = R$;
- b) $|f(t, x, y)| \leq \varphi(|y|)$ where $\varphi : [0, \infty) \rightarrow (0, \infty)$ is continuous and $\int_0^\infty r dr / \varphi(r) = \infty$ (a Nagumo condition);
- c) $|f(t, x, y)| \leq 2K(x \cdot f(t, x, y) + |y|^2) + L$, K and L non-negative constants.

Then the problem

$$(1.2) \quad x'' = f(t, x, x'), \quad x(0) = x(1), \quad x'(0) = x'(1)$$

has at least one solution in D .

Here x and y are n -vectors, $x \cdot y$ is the dot product, and $|x| = (x \cdot x)^{\frac{1}{2}}$.

For the two point boundary value problem a theorem of this type was proved by NAGUMO [16] and, for the vector case, by HARTMAN [6]. The development of Theorem 1.1 may be followed in [6], [11], [17] and [2]. A resume is given in [7, Chap. XII].

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With $Lx = -x''$, and the boundary conditions incorporated into the domain of L , and with N the nonlinear Nemytskii operator generated by f , (1.2) can be written in the form (1.1).

Since the null space of L is non-trivial we choose to use the «method of alternative problems» (see CESARI [3], HALE [8], or MAWHIN [15]) to attack the problem. We can then give an operator-theoretic proof for a class of problems like (1.2) with regard to which the following are of interest:

1) Certain higher order operators L , with a finite dimensional null space, can be considered.

2) Condition (a) of Theorem 1.1 becomes a condition on vector fields to be used with Leray-Schauder degree theory.

3) The problem may be put into a natural Orlicz space setting and (b) of Theorem 1.1 is replaced by a growth condition on the N -function determining the space. For example, we can allow $|f(t, x, y)| \leq |y|^p$, $p > 1$ an integer, and prove our theorem.

Contents of the paper. Section 2: A list of notations and function spaces used in the paper. Section 3: An operator-theoretic proof of a version of Theorem 1.1. Section 4: An abstract version of the theorem.

2. - Notation and spaces.

We list here the several function spaces and some notation we use in subsequent sections.

a) Euclidean n -space will be denoted by R^n , the norm will be $|x| = (x \cdot x)^{\frac{1}{2}}$ where $x \cdot y = \sum_{i=1}^n x_i y_i$.

b) The space of continuous functions from $[0, 1]$ into R^n with norm $\|x(\cdot)\|_0 = \sup \{|x(t)| : 0 \leq t \leq 1\}$ will be denoted by $C[0, 1]$. The subset of absolutely continuous functions will be denoted by $AC[0, 1]$; the subset of functions with n continuous derivatives, by $C^n[0, 1]$; and the subset of functions with derivatives of every order, by $C^\infty[0, 1]$.

c) The Lebesgue spaces, with the usual norm, will be denoted by $L_p[0, 1]$ and $\|\cdot\|_p$, respectively, where $p \geq 1$. Often we shall write L_p for $L_p[0, 1]$ and $\|\cdot\|$ for $\|\cdot\|_2$. The inner product on L_2 will be denoted by $(\cdot, \cdot)$.

d) For Orlicz spaces we shall assume an acquaintance with KRASNOSEL'SKII and RUTICKII [13]. (Or, see [5].) To facilitate reading one can think of $\Phi(u) = u^p$ and $L_\Phi = L_p$, $p > 1$. Our theorems do allow functions that grow faster than u^p .

i) Let $\Phi(u)$ and $\Psi(u)$ be complementary N -functions and let $L_\Psi = L_\Phi[0, 1]$, with norm $\|\cdot\|_\Phi$, and $L_\Psi = L_\Psi[0, 1]$, with norm $\|\cdot\|_\Psi$, be the corresponding Orlicz spaces [13, pp. 11, 67].

ii) If $\Psi(u)$ satisfies a Δ_3 -condition, then the bounded, measurable functions, and hence the L_2 -functions, are dense in L_Ψ ; L_Ψ is separable; and $L_\Psi^* = L_\Phi$ (where $*$ denotes the dual space) [13, pp. 81, 130].

iii) If $\Psi(u) < u^2 < \Phi(u)$, then $L_\Phi \subset L_2 \subset L_\Psi$ and $\|\cdot\|_\Psi \leq k\|\cdot\|_\Phi$ for some constant $k > 0$ [13, pp. 110, 112]. And if $u^2 < \Phi(u)$, then $\Psi(u)$, its complementary N -function, satisfies $\Psi(u) < u^2$ (see [13, p. 16]).

iv) Hence if $\Psi(u) < u^2 < \Phi(u)$ and $\Psi(u)$ satisfies a Δ_2 -condition we can say: If $x \in L_\Phi = L_\Psi^*$, $y \in (L_\Psi \cap L_2)$, and $\langle x, y \rangle$ denotes the usual pairing between an element y in a space X and x in X^* , then $\langle x, y \rangle = (x, y)$ (the L_2 -inner product). When this conclusion and the conclusions of (ii) and (iii) hold, we say that (L_Ψ, L_2, L_Φ) are in normal position (see [1]).

v) If $\Phi(u)$ satisfies a Δ_3 -condition, then $\Psi(u)$ satisfies a Δ_2 -condition. The functions $\Phi(u) = u^p$ or $\Phi(u) = \exp(u^p)$, $p > 1$, satisfy Δ_3 -conditions [13, p. 35].

e) The Sobolev space $\{x(t): x, x', \dots, x^{(n-1)} \in AC[0, 1], x^{(n)} \in L_2\}$ with norm $\|\cdot\|_H = \|x\|_0 + \dots + \|x^{(n-1)}\|_0 + \|x^{(n)}\|$ will be denoted by $H^n[0, 1]$. With a different norm we could have $H^n[0, 1]$ a Hilbert space. But this norm makes one step in a later proof easier. For a discussion of this space see [14].

f) The Orlicz-Sobolev space $\{x(t): x, \dots, x^{(n-1)} \in AC[0, 1], x^{(n)} \in L_\Phi\}$ with norm $\|\cdot\|_{H_\Phi} = \|x\|_0 + \dots + \|x^{(n-1)}\|_0 + \|x^{(n)}\|_\Phi$ will be denoted by $H_\Phi^n[0, 1]$. For a discussion of this space see [5].

g) If A is an operator the domain, range, and null space A will be denoted by D_A , R_A , and N_A , respectively.

3. - A second order problem.

THEOREM 3.1. - Let $f(t, x, y): [0, 1] \times R^n \times R^n \rightarrow R^n$ be continuous and satisfy:

a) $|f(t, x, y)| \leq a + b|x| + c|y|^p$ for some positive constants a, b, c and some integer $p > 1$; and

b) $\|x'(\cdot)\|^2 + \int_0^1 \{x(t) \cdot f(t, x(t), x'(t))\} dt \geq 0$ on ∂B_r where $B_r = \{x \in H_\Phi[0, 1]: \|x\|_0^2 + \|x'\|_\Phi^2 \leq r^2\}$ for some $r > 0$. Here H_Φ is the Orlicz-Sobolev space with N -function $\Phi(s) = s^p$. (See Sections 2.f and 2.e for this terminology.)

Then the boundary value problem

$$(3.1) \quad x'' = f(t, x, x'), \quad x(0) = x(1), \quad x'(0) = x'(1)$$

has at least one solution in $B_r \cap C^2[0, 1]$.

REMARK. — Other N -functions are admissible—see Theorem 4.1.

PROOF. — Let L be the positive, self-adjoint, linear differential operator defined by $D_L = \{x \in H^2[0, 1]: x(0) = x(1), x'(0) = x'(1)\}$, $Lx = -x''$ and let N be the non-linear Nemytskii operator defined by $Nx = f(\cdot, x(\cdot), x'(\cdot))$ where D_N will be selected later in such a way that R_N will lie in the domains of operators applied to N .

Then (3.1) can be rewritten as

$$(3.2) \quad Lx + Nx = 0$$

where a solution is in $D_L \cap D_N \cap C^2[0, 1]$.

Define the projection operator P by

$$(3.3) \quad Px = \int_0^1 x(t) dt, \quad D_P = L_1[0, 1].$$

If we let $P(L_1) = T_0$ and $(I - P)(L_1) = T_1$, then $L_1[0, 1] = T_0 \oplus T_1$. We shall often restrict P to subspaces of $L_1[0, 1]$, e.g., L_2 or H^2 , without comment and we note that T_0 is the constant functions and $N_L = T_0$.

Also, if $L_2[0, 1] = S$ and $(S \cap T_i) = S_i$ ($i = 1, 2$), then $S = S_1 \oplus S_0$ and in this case we have $S_1 \perp S_0$.

Define the operator $\tilde{H}$ by

$$(3.4) \quad D_{\tilde{H}} = S_1, \quad \tilde{H}x = \int_0^1 (1-t)sx(s)ds + \int_t^1 t(1-s)x(s)ds + x(0).$$

If $H = \tilde{H}|_{R_L}$, then

$$H = (L|(D_L \cap S_1))^{-1} \quad \text{and} \quad R_H = (D_L \cap S_1).$$

If we only require that a solution of (3.2) be in $D_L \cap D_N$, then following the method of alternative problems, see [4], (3.2) is equivalent to

$$(3.5) \quad x + H(I - P)Nx = x_0, \quad x_0 \in S_0, \quad x \in S$$

$$(3.6) \quad PNx = 0.$$

But a solution x to (3.5), (3.6) will be in $D_L \subset H^2$. Then $Nx \in C[0, 1]$ and hence $H(I - P)Nx \in C^2[0, 1]$ and $x \in C^2[0, 1]$, so x is a true solution of (3.1).

We now rewrite (3.5), (3.6) by using a natural decomposition of $H(I - P)$ into the form J^*J . This technique has been studied in detail by KANNAN and LOCKER [9]; here we shall simply outline the method.

Define the operators T and T^* by

$$D_T = \{x \in H^1[0, 1]: x(0) = x(1)\}, \quad Tx = x' \quad \text{and} \quad T^* = -T.$$

The operator L equals TT^* ; T^* is the adjoint of T ; and $N_T = (D_T \cap S_0) = N_{T^*}$.

Let $J = [T|D_T \cap S_1]^{-1}$ and $J^* = [T^*|D_L \cap S_1]^{-1}$. Then $J: R_L \rightarrow (D_T \cap S_1)$, $J^*: (D_T \cap S_1) \rightarrow (D_L \cap S_1)$, and $H = J^*J$. And we can represent J and J^* as

$$(3.7) \quad Jx = \int_0^t x(s) ds - P \left[\int_0^t x(s) ds \right] = -J^*x.$$

With $x - x_0 = x_1$, (3.5), (3.6) can be rewritten as

$$\begin{aligned} x_1 + J^*J(I - P)N(x_1 + x_0) &= 0 \\ PN(x_1 + x_0) &= 0. \end{aligned}$$

With $x_1 = J^*y_1$, $y_1 \in S_1$ and using that J is one-to-one, the above system is equivalent to

$$(3.8) \quad y_1 + J(I - P)N(J^*y_1 + x_0) = 0$$

$$(3.9) \quad PN(J^*y_1 + x_0) = 0 \quad y_1 \in S_1, \quad x_0 \in S_0.$$

The first part of the plan for solving (3.8), (3.9) is based on hypothesis (a) which tells us that if $x \in H_\Phi^1[0, 1]$, with $\Phi(s) = s^p$, then

$$\int_0^1 |f(t, x(t), x'(t))| dt \leq M + \int_0^1 |x'(t)|^p dt < \infty,$$

i.e., $Nx \in L_1$. (See Section 2 for the definition of spaces.)

We shall restrict J^* to $(L_\Phi \cap S_1)$. Then

$$(L_\Phi \cap S_1) \xrightarrow{J^*} H_\Phi^1 \xrightarrow{(I-P)N} (L_1 \cap T_1) \xrightarrow{J} (AC \cap S_1) \subset (L_\Phi \cap S_1).$$

And we shall show that $J(I - P)NJ^*$ is compact. But we need an adjustment since $L_1 \notin D_J$.

Let K^* have the same representation as J^* in (3.7) and let $D_{K^*} = L_\Phi \cap S_1$; let K have the same representation as J in (3.7) and let $D_K = L_1 \cap T_1$. If $x \in D_K$, then $Kx \in S_1 \cap AC[0, 1]$.

We shall solve the problem

$$(3.10) \quad z_1 + K(I - P)N(K^*z_1 + x_0) = 0, \quad z_1 \in L_\Phi \cap S_1, \quad x_0 \in S_0$$

$$(3.11) \quad PN(K^*z_1 + x_0) = 0.$$

The solution z_1 (we need not worry about the constant functions in the discussion of this point) will be in $L_\Phi \cap S_1$. Then $N(K^*z_1 + x_0)$ will be in L_1 and indeed z_1 will be absolutely continuous. Repeating this procedure we will have z_1

in C^1 and thus z_1 is in the domain of J^* , $(I - P)N(J^*z_1 + x_0)$ is in the domain of J , and z_1 is a solution of (3.8), (3.9).

To show: $K(I - P)NK^*: L_\Phi \rightarrow L_\Phi$ is continuous and compact.

To show that $K: L_\Phi \rightarrow H_\Phi^1 \cap S_1$ is continuous and bounded is straightforward; we shall show that $N: H_\Phi^1 \rightarrow L_1$ is continuous and bounded; $(I - P)$ is a projection; and to show that $K: L_1 \rightarrow L_\Phi$ is continuous and compact is again straightforward.

$N: H_\Phi^1 \rightarrow L_1$ is bounded:

$$\|Nx\|_1 \leq a + b \int_0^1 |x(t)| dt + c \int_0^1 |x'(t)|^p dt \leq a + d \|x\|_{H_\Phi}.$$

$N: H_\Phi^1 \rightarrow L_1$ is continuous:

$$\begin{aligned} \|Nx - Ny\|_1 &\leq \int_0^1 |f(t, x(t), x'(t)) - f(t, y(t), x'(t))| dt \\ &\quad + \int_0^1 |f(t, y(t), x'(t)) - f(t, y(t), y'(t))| dt. \end{aligned}$$

The first integral: With $x'(t) \in L_p$,

$$g(t, c) = f(t, c, x'(t))$$

is continuous in c and measurable in t . Hence by Kransnosel'skii's version of Lusin's theorem [12] we may divide $[0, 1]$ into disjoint subsets I_1 and I_2 such that

I_1 is closed and $g|_{I_1 \times R^n}$ is continuous;

I_2 has arbitrarily small measure.

We have just seen that the integrands are bounded so we can choose I_2 such that the integral over I_2 is less than $\varepsilon/4$. On $I_1 \times \{x: |x - y(t)|_0 \leq 1\}$, f is uniformly continuous, hence if $|x(t) - y(t)|_0$ is sufficiently small, the integral over I_1 is less than $\varepsilon/4$.

The second integral is handled in a similar manner.

Now we use hypothesis (b) and degree theory to solve (3.10), (3.11).

Let $L_\Phi = R_1 \oplus R_0$ where $R_i = L_\Phi \cap T_i$ ($i = 1, 2$), see (3.3), and define $T: R_1 \oplus R_0 \rightarrow R_1 \oplus R_0$ by

$$T \begin{pmatrix} z_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} K(I - P)N(K^*z_1, x_0) \\ -x_0 + PN(K^*z_1, x_0) \end{pmatrix}, \quad N(K^*z_1, x_0) = N(K^*z_1 + x_0).$$

Then T is continuous and compact and (3.10), (3.11) is equivalent to

$$(3.12) \quad (I + T)(z_1, x_0) = 0.$$

Let $L_\phi^* = R_1^* \oplus R_0^*$ where $R_i^* = L_\phi^* \cap T_i$ ($i=1, 2$). If $w = z_1 + x_0 \in L_\phi$, $\tilde{w} = \tilde{z}_1 + \tilde{x}_0 \in L_\phi^*$, the usual pairing $\langle \tilde{w}, w \rangle = \langle \tilde{z}_1, z_1 \rangle + \langle \tilde{x}_0, x_0 \rangle$ since

$$\langle z_1, x_0 \rangle = \int_0^1 \{[\tilde{w}(t) - P\tilde{w}][Pw]\} dt = [P(I - P)\tilde{w}][Pw] = 0$$

and likewise $\langle \tilde{x}_0, z_1 \rangle = 0$.

Since $L_\phi \subset L_\phi^*$ we can form the pairing $\langle (I + \lambda T)(z_1, x_0), (z_1, x_0) \rangle$ for $0 < \lambda < 1$ and this is greater than $\lambda \langle (I + T)(z_1, x_0), (z_1, x_0) \rangle$ if $\langle z_1, z_1 \rangle + \langle x_0, x_0 \rangle = r^2 > 0$. Thus if $\langle (I + T)(z_1, x_0), (z_1, x_0) \rangle \geq 0$ on $\partial B_r = \{(z_1, x_0) : \langle z_1, z_1 \rangle + \langle x_0, x_0 \rangle = r^2 > 0\}$, then $\deg(I + T, B_r, 0) = \deg(I, B_r, 0) = 1$ and (3.12) has a solution in $(B_r \cup \partial B_r)$. And

$$\begin{aligned} \langle (I + T)(z_1, x_0), (z_1, x_0) \rangle &= \langle z_1, z_1 \rangle + \langle x_0, x_0 \rangle \\ &+ \langle K(I - P)N(K^*z_1, x_0), z_1 \rangle + \langle -x_0 + PN(K^*z_1, x_0), x_0 \rangle \quad (\text{see (i) below}) \\ &= \langle z_1, z_1 \rangle + \langle (I - P)N(J^*z_1, x_0), J^*z_1 \rangle + \langle PN(K^*z_1, x_0), x_0 \rangle \\ &= \langle z_1, z_1 \rangle + \langle N(J^*z_1, x_0), J^*z_1 + x_0 \rangle - \langle PN(K^*z_1, x_0), J^*z_1 \rangle \\ &- \langle (I - P)N(K^*z_1, x_0), x_0 \rangle = (\text{see (ii), below}) \int_0^1 |x'(t)|^2 dt \\ &+ \int_0^1 \{x(t) \cdot f(t, x(t), x'(t))\} dt \geq 0 \end{aligned}$$

on B_r by hypothesis (b). So we are done.

i) Here $K: (L_1 \cap S_1) \rightarrow (L_\phi \cap S_1)$ and $z_1 \in L_\phi$. So the dual operator $K': (L_\phi \cap S_1) \rightarrow L^\infty$. But on $(L_\phi \cap S_1) \subset (L_2 \cap S_1)$, $K' = J^*$. The second pairing here is between an element in L_1 and one in L^∞ .

ii) With $J^*z_1 + x_0 = x$, $z_1 = x'$. The last two pairings vanish by computation since again the integral of the pairing acts like P .

4. - A general theorem.

In this section we will extend Theorem 3.1 by considering higher order, more general operators L and more general boundary conditions. To simplify notation we shall assume that all functions are real valued.

Let $\tau x(t) = \sum_{i=0}^n a_i(t)x^{(i)}(t)$ be a formal differential operator with coefficients $a_i(t) \in C^\infty[0, 1]$ and $a_n(t) \neq 0$ on $[0, 1]$, and let

$$B_i(x) = \sum_{j=0}^{n-1} [a_{ij}x^{(j)}(0) + \beta_{ij}x^{(j)}(1)], \quad i = 1, \dots, k$$

be a set of k linearly independent boundary values with the α_{ij}, β_{ij} being real constants and $0 \leq k \leq 2n$. Let τ^* be the formal adjoint of τ and let $B_i^*(x)$, $1 \leq i \leq 2n - k$, be a set of linearly independent, adjoint boundary values. Define the operators T , T^* , and L by

$$\begin{aligned} D_T &= \{x \in H^n[0, 1]: B_i(x) = 0, i = 1, \dots, k\}, & Tx &= \tau x; \\ D_{T^*} &= \{x \in H^n[0, 1]: B_i^*(x) = 0, i = 1, \dots, 2n - k\}, & T^*x &= \tau^*x; \end{aligned}$$

and

$$\begin{aligned} D_L &= \{x \in H^{2n}[0, 1]: B_i^*(x) = 0 = B_j(\tau^*x), i = 1, \dots, 2k - 1, j = 1, \dots, k\}; \\ Lx &= \tau\tau^*x. \end{aligned}$$

The following hold:

A) If $\alpha_{ij} = 0 = \beta_{ij}$ ($1 \leq i \leq k$, $0 \leq j \leq n - 1$), then T is a closed, densely-defined linear operator in $S = L_2[0, 1]$ with $\dim N_T = n$, $R_T = S$. Further, $T|(D_T \cap N_T^\perp)$ is a closed, 1-1 operator and $[T|(D_T \cap N_T^\perp)]^{-1}$ is a 1-1 completely continuous operator, in fact an integral operator, defined on all of S . Similar statements can be made with respect to T^* and L . For a reference see LOCKER [14].

B) L admits a decomposition, $L = TT^*$, where T^* is the adjoint of T ; $N_{T^*} = N_L (= N_T)$, $R_T = R_L$, and $L_2 = S = S_1 \oplus S_0$ where $S_1 = R_L$, $S_0 = N_L$;

$$T^*: (D_L \cap S_1) \rightarrow (D_T \cap S_1), \quad T: (D_T \cap S_1) \rightarrow R_L$$

and thus restricted these mappings are 1-1 and onto; and the inverses of T and T^* , which we shall denote by J and J^* , respectively, exist, are completely continuous, and in fact are integral operators.

THEOREM 4.1. - Let $f(t, x_0, \dots, x_n)$ be continuous on $[0, 1] \times R \times \dots \times R$ and let $Nx = f(\cdot, x(\cdot), \dots, x^{(n)}(\cdot))$. If a) $|f(t, x_0, \dots, x_n)| \leq a + b_0|x_0| + \dots + b_n|x_{n-1}| + \Phi(|x_n|)$, where $\Phi(u)$ is an N -function with complementary N -function $\Psi(u)$ such that $\Psi < u^2 < \Phi(u)$ and $\Psi(u)$ satisfies a Δ_2 -condition (see Section 2.d); and if

$$b) \quad \|x^{(n)}(\cdot)\|^2 + \int_0^1 \{x(t)f(t, x(t), \dots, x^{(n)}(t))\} dt \geq 0 \text{ on } \partial B_r, \text{ where}$$

$$B_r = \{x \in H_\Phi^n[0, 1]: \|x\|_{H_\Phi} \leq r\}$$

for some $r > 0$ (see Section 2.f), then the boundary value problem (4.1) $LX + Nx = 0$ has at least one solution in $D_L \cap C^{2n}[0, 1] \cap B_r$.

PROOF. - For the operator N we state the following, with the proofs to be given later.

C) $N: H_\Phi^n[0, 1] \rightarrow L_1[0, 1]$ is continuous and bounded.

D) $\|x_1\|^2 + (N(J^*x_1 + x_0), J^*x_1 + x_0) \geq 0$ for $\|x_1\|^2 + \|x_0\|^2 = r^2 > 0$ where $x_1 \in S_1$, $x_0 \in S_0$. (We note: i) This computation is made in $L_2[0, 1]$; ii) If $T = -T^*$, then $(J^*x_1 + x_0, x_1) = 0$. An assumption like $(J^*x_1 + x_0, x_1) = 0$ appears in Theorem 1, Hypothesis *a*.)

Statements (*A*)-(*D*) contain the key elements of the proof. We now just follow the proof of Theorem 3.1, making a few changes.

Statements (*A*) and (*D*) give the properties which carry us from (3.2) to (3.8), (3.9). This time

$$Px = \sum_{k=1}^k z_k \int_0^1 x(t) z_k(t) dt \quad \text{where } z_1, \dots, z_k$$

is an orthonormal basis for $S_0 = N_L \subset L_2$. But also $z_i \in L_\infty$, $1 \leq i \leq k$, so we can again, for convenience in later definition, regard P as a projection on L_1 .

As stated in (*A*), the operators H , J and J^* are completely continuous integral operators, defined on $L_2[0, 1]$ —we shall not worry about their representations. If $Nx \in C[0, 1]$, then $H(I - P)Nx \in C^{2n}[0, 1]$. And since J and J^* are integrals with L^∞ kernels we can again define K and K^* .

That $N: H_\Phi^n \rightarrow L_1$ is continuous and bounded is the same.

Again the mapping T (see (3.12)) has

$$T(z_1, x_0) \in AC[0, 1] \quad \text{if} \quad (z_1, x_0) \in R_1 \oplus R_0 = L_\Phi$$

and again we regard this mapping on two levels:

$T: L_\Phi \rightarrow L_\Phi$ for fixed point theory, and

$T: L_\Phi \rightarrow L_\Phi$ for computing $\langle (z_1, x_0), (I + T)(z_1, x_0) \rangle$, since $L_\Psi^* = L_\Phi$.

By hypothesis (*a*), (L_Ψ, L_2, L_Φ) are in normal position (see Section 2.d.iv) so this pairing equals the L_2 inner product $((z_1, x_0), (I + T)(z_1, x_0))$ (The direct sum decomposition of $L_2 = S_1 \oplus S_0$ can be made a Hilbert space.)

Finally, the pairings that vanished by computation now vanish by orthogonality in L_2 and the proof is complete.

The following is an application of Theorem 4.1. Since $\Phi(u) = \exp(u^9)$ satisfies a Δ_3 -condition, its complementary N -function $\Psi(u)$ satisfies a Δ_2 -condition. Also, $\Psi(u) < u^2 < \Phi(u)$. (See Sections 2.d.v and 2.d.iii.)

THEOREM 4.2. — Let $f: [0, 1] \times R \times R \times R \rightarrow R$ be continuous and assume:
a) $|f(t, x, y, z)| \leq a + b|x| + c|y| + d \exp |z|^9$; b) there exists an $r > 0$ such that

$$\int_0^1 \{[x''(t)]^2 + x(t)f(t, x(t), x'(t), x''(t))\} dt \geq 0 \quad \text{on } \partial B_r$$

where $B_r = \{x \in H_\Phi^2[0, 1]: \|x\|_{H_\Phi} \leq r\}$. ($\Phi(u) = \exp(u^9)$).

Then the boundary value problem

$$\begin{aligned}x^{(iv)} + f(t, x, x', x'') &= 0 \\ x'(0) = x'(1) = x'''(0) = x'''(1) &= 0\end{aligned}$$

has at least one solution in $C^4[0, 1] \cap B_r$.

A version of this theorem was announced in [10].

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