On Holomorphically Subprojective Kählerian Manifolds. - III (*).

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Dedicated to Professor S. TACHIBANA on his 50th birthday

Summary. – *In two previous papers [9, 10] the present authors have established the differential geometric properties of holomorphically subprojective Kählerian manifolds. The purpose of this paper is to seek the necessary and sufficient conditions in order that a Kählerian manifold be the holomorphically subprojective Kählerian manifold of the first kind, corresponding to P. Rachevsky's conditions of a subprojective Riemannian manifold.*

1. – Introduction.

Let us consider an n complex dimensional Kählerian manifold M . If there exists a complex coordinate system such that every holomorphically planar curve is given with respect to this system by $n-2$ homogeneous linear equations and one other equation that need not be linear, then M is called a *holomorphically subprojective Kählerian manifold*. $M(n \geq 3)$ is holomorphically subprojective if and only if there exists a local complex coordinate system (z^λ) ($\lambda, \mu, \dots = 1, 2, \dots, n$) such that the Christoffel symbol may take the form

$$\Gamma_{\mu\nu}^\lambda = \varrho_\mu \delta_\lambda^\nu + \varrho_\nu \delta_\mu^\lambda + f_{\mu\nu} z^\lambda,$$

where ϱ_λ and $f_{\mu\lambda}$ are self-conjugate vector and symmetric tensor respectively.

The purpose of this paper is to get the necessary and sufficient conditions in order that a Kählerian manifold be the holomorphically subprojective Kählerian manifold of the first kind, corresponding to P. Rachevsky's conditions of a subprojective Riemannian manifold. In § 2 we shall give a short summary of Kählerian manifold with vanishing Bochner curvature tensor and holomorphically subprojective Kählerian manifold of the first kind. We consider some partial differential equations and the integrability conditions (I), (II) and (III) of those in § 3. The conditions (I), (II) and (III) are discussed in § 4 ~ § 6 and in § 7 we prove the main theorem.

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2. – Preliminaries.

At first we always agree to adopt the summation convention and the following convention: (ij) (resp. $[ij]$) for indices i and j means the symmetric (resp. skew-symmetric) part with respect to indices i and j , for example

$$u_{(ij)} = u_{ij} + u_{ji} \quad (\text{resp. } u_{[ij]} = u_{ij} - u_{ji}) \\ (i, j, \dots, r, s, \dots = 1, 2, \dots, 2n).$$

Let us consider a $2n$ real dimensional Kählerian manifold with complex structure J and Riemannian metric g which satisfy the following relations

$$J_j^r J_r^h = -\delta_j^h, \quad g_{ji} = J_j^r J_i^s g_{rs}, \quad \nabla_j J_i^h = 0,$$

∇ being the operator of covariant derivation with respect to Riemannian connection defined by g .

We consider the tensor field given by

$$(2.1) \quad K_{kji}^h = R_{kji}^h + \frac{1}{2(n+2)} (L_{i[k} \delta_{j]}^h + g_{i[k} L_{j]}^h \\ - M_{i[k} J_{j]}^h - J_{i[k} M_{j]}^h + 2M_{kj} J_i^h + 2J_{kj} M_i^h)$$

so called the Bochner curvature tensor, where we have put

$$(2.2) \quad L_{ji} = R_{ji} - \frac{R}{4(n+1)} g_{ji}, \quad M_{ji} = J_j^r L_{ri}.$$

In M with vanishing Bochner curvature tensor, the following relation which plays important roles in the following sections has been proved by M. MATSUMOTO [3]:

$$(2.3) \quad 4(n+1)\nabla_k R_{ji} = g_{k(i}\nabla_{j)}R + 2g_{ji}\nabla_k R - J_{k(j}J_{i)}^r \nabla_r R.$$

In what follows, we remember the properties of holomorphically subprojective Kählerian manifolds. It is known that in order that a $2n$ ($n \geq 3$) real dimensional Kählerian manifold M is holomorphically subprojective, it is necessary and sufficient that there exists a local real coordinate system (x^i) such that the Christoffel symbol of M may take the form

$$(2.4) \quad \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} = \varrho_{(i} \delta_{j)}^h + \tilde{\varrho}_{(i} J_{j)}^h + f_{ji} x^h - f_{jr} J_i^r \hat{x}^h,$$

$$(2.5) \quad f_{[ji]} = 0, \quad f_{r[i} J_{j]}^r = 0,$$

where ϱ_i and f_{ji} are covariant vector and tensor fields respectively, and we have put $\hat{x}^h = J_r^h x^r$ and $\tilde{\varrho}_h = -J_h^r \varrho_r$.

Although the original definition of the holomorphically subprojective Kählerian manifold of the first kind is characterized by some functions, as a new definition we may take the following

THEOREM A [10]. — *A $2n$ ($n \geq 3$) dimensional holomorphically subprojective Kählerian manifold M of the first kind has the following properties (I), (II) and (III):*

(I) *M has a vanishing Bochner curvature tensor which has the Ricci tensor of the form*

$$R_{jk} = \alpha g_{jk} + \beta(\sigma_j \sigma_k + \bar{\sigma}_j \bar{\sigma}_k),$$

where σ_i is a gradient vector, and α and β are some functions.

(II) *The vector x^h in (2.4) is contravariant analytic.*

(III) *There exist a gradient vector ϱ_i and a symmetric tensor f_{jk} such that for some functions ϕ and φ*

$$f_{r[i} J_{i]}^r = 0$$

and

$$\nabla_j \varrho_i = -\varrho_i \varrho_j + \tilde{\varrho}_j \tilde{\varrho}_i + f_{ji} + \phi g_{ji} + \varphi(\sigma_j \sigma_i + \bar{\sigma}_j \bar{\sigma}_i).$$

3. — Partial differential equations.

In the following we seek the necessary and sufficient conditions that a $2n$ ($n \geq 3$) dimensional Kählerian manifold M be the holomorphically subprojective Kählerian manifold of the first kind. To investigate these some partial differential equations must be considered in M and need seek the integrability conditions of them.

Let us consider now partial differential equations for a vector z_i

$$(3.1) \quad \nabla_j z_i = -\varrho_{(i} z_{j)} + \tilde{\varrho}_{(i} \tilde{z}_{j)} - f_{ji} z_r \xi^r + f_{js} J_i^s z_r \tilde{\xi}^r,$$

where ϱ_i is a gradient vector, f_{ji} a symmetric tensor satisfying $f_{r[i} J_{i]}^r = 0$ and ξ^h a certain vector, and we have put $\tilde{\xi}^h = J_r^h \xi^r$, $\tilde{\varrho}_i = -J_i^r \varrho_r$ and $\tilde{z}_i = -J_i^r z_r$.

As for the integrability conditions of (3.1), we shall consider the following three properties:

(I) A Kählerian manifold M has a vanishing Bochner curvature tensor which has the Ricci tensor of the form

$$R_{ji} = \alpha g_{ji} + \beta(\sigma_j \sigma_i + \bar{\sigma}_j \bar{\sigma}_i),$$

where σ_i is a gradient vector, and β and α are some functions.

(II) The vector $\xi^h = \kappa\sigma^h + \mu\sigma^h$ for some functions κ and μ is contravariant analytic, that is, ξ^h satisfies $\mathfrak{L}_{\xi} J_i^h = 0$, or, equivalently

$$\nabla_j \xi_i = J_j^r J_i^s \nabla_r \xi_s,$$

where $\mathfrak{L}_{\xi}$ denotes the operator of Lie derivation with respect to ξ^h .

(III) There exist a gradient vector ϱ_i and a symmetric tensor f_{ji} satisfying $f_{r[i} J_{j]}^r = 0$ and

$$\nabla_k \varrho_j = -\varrho_k \varrho_j + \tilde{\varrho}_k \tilde{\varrho}_j + f_{kj} + \phi g_{kj} + \varphi(\xi_k \xi_j + \tilde{\xi}_k \tilde{\xi}_j),$$

where ϕ and φ are some functions.

4. - Condition (I).

In this section, we try to determine the form of $\nabla_j \sigma_i$ under the condition (I). Then since we have

$$(4.1) \quad R_{ji} = \alpha g_{ji} + \beta(\sigma_j \sigma_i + \tilde{\sigma}_j \tilde{\sigma}_i),$$

where σ_i is a gradient vector, we have

$$(4.2) \quad R = 2n\alpha + 2\beta|\sigma|^2$$

and applying ∇_k to (4.1) and (4.2) respectively, it follows that

$$(4.3) \quad \nabla_k R_{ji} = \alpha_k g_{ji} + \beta_k(\sigma_j \sigma_i + \tilde{\sigma}_j \tilde{\sigma}_i) + \beta(\nabla_k \sigma_i \sigma_j + \sigma_i \nabla_k \sigma_j + \nabla_k \tilde{\sigma}_i \tilde{\sigma}_j + \tilde{\sigma}_i \nabla_k \tilde{\sigma}_j),$$

$$(4.4) \quad \nabla_k R = 2n\alpha_k + 2|\sigma|^2 \beta_k + 4\beta \sigma^r \nabla_k \sigma_r.$$

By contraction (4.3) with g^{kj} , we get

$$(4.5) \quad \nabla_i R = 2\alpha_i + 2(\beta_r \sigma^r \sigma_i + \beta_r \tilde{\sigma}^r \tilde{\sigma}_i) + 2\beta(\nabla_i \sigma_r \sigma^r + \nabla_r \sigma^r \sigma_i + \tilde{\sigma}^r \nabla_r \tilde{\sigma}_i).$$

On the other hand, by our assumption, that is, $K_{kji}^h = 0$, we may use (2.3). By virtue of (4.3) and (4.4), it follows from (2.3) that

$$\begin{aligned} (4.6) \quad & 2(n+1)[\alpha_k g_{ji} + \beta_k(\sigma_j \sigma_i + \tilde{\sigma}_j \tilde{\sigma}_i) + \beta(\nabla_k \sigma_i \sigma_j + \sigma_i \nabla_k \sigma_j + \nabla_k \tilde{\sigma}_i \tilde{\sigma}_j + \tilde{\sigma}_i \nabla_k \tilde{\sigma}_j)] \\ &= (n\alpha_i + |\sigma|^2 \beta_i + 2\beta \sigma^r \nabla_r \sigma_i) g_{jk} \\ &+ 2g_{ji} (n\alpha_k + |\sigma|^2 \beta_k + 2\beta \sigma^r \nabla_r \sigma_k) + J_{k(i} (n\tilde{\alpha}_{j)} + |\sigma|^2 \tilde{\beta}_{j)} + 2\beta_j \sigma^r \nabla_{|r|} \tilde{\sigma}_{i)} \end{aligned}$$

and if we contract (4.6) with σ^i , $\sigma^i \sigma^j$ and $\sigma^i \sigma^j \sigma^k$ respectively, then we can find

$$\begin{aligned}
 (4.7) \quad & 2(n+1)[\alpha_k \sigma_j + \beta_k \sigma_j |\sigma|^2 + \beta(\sigma^r \nabla_k \sigma_r \sigma_j + |\sigma|^2 \nabla_k \sigma_j + \sigma^r \nabla_k \tilde{\sigma}_r \tilde{\sigma}_j)] \\
 & = (n\alpha_r + |\sigma|^2 \beta_r + 2\beta \sigma^s \nabla_r \sigma_s) \sigma^r g_{kj} + \sigma_k (n\alpha_j + |\sigma|^2 \beta_j + 2\beta \sigma^r \nabla_j \sigma_r) \\
 & + 2\sigma_j (n\alpha_k + |\sigma|^2 \beta_k + 2\beta \sigma^r \nabla_k \sigma_r) + J_{kj} (n\tilde{\alpha}_i + |\sigma|^2 \tilde{\beta}_i + 2\beta \sigma^r \nabla_r \tilde{\sigma}_i) \sigma^i \\
 & - \tilde{\sigma}_k (n\tilde{\alpha}_j + |\sigma|^2 \tilde{\beta}_j + 2\beta \sigma^r \nabla_r \tilde{\sigma}_j),
 \end{aligned}$$

$$\begin{aligned}
 (4.8) \quad & (n+1)|\sigma|^2(\alpha_k + |\sigma|^2 \beta_k + 2\beta \sigma^r \nabla_k \sigma_r) \\
 & = \sigma_k (n\alpha_j + |\sigma|^2 \beta_j + 2\beta \sigma^r \nabla_j \sigma_r) \sigma^j + |\sigma|^2 (n\alpha_k + |\sigma|^2 \beta_k + 2\beta \sigma^r \nabla_k \sigma_r) \\
 & - \tilde{\sigma}_k (n\tilde{\alpha}_r + \tilde{\beta}_r + 2\beta \sigma^s \nabla_s \tilde{\sigma}_r) \sigma^r,
 \end{aligned}$$

$$(4.9) \quad \alpha_r \sigma^r = (|\sigma|^2 \beta_r + 2\beta \sigma^s \nabla_r \sigma_s) \sigma^r.$$

Moreover, contracting (4.7) (resp. (4.8)) with J^{kj} (resp. $\tilde{\sigma}^k$), we find

$$(4.10) \quad \alpha_r \tilde{\sigma}^r = (|\sigma|^2 \beta_r + 2\beta \sigma^s \nabla_r \sigma_s) \tilde{\sigma}^r,$$

$$(4.11) \quad \alpha_r \tilde{\sigma}^r = 0,$$

from which

$$(4.12) \quad (|\sigma|^2 \beta_r + 2\beta \sigma^s \nabla_r \sigma_s) \tilde{\sigma}^r = 0$$

and therefore by making use of (4.9), (4.11) and (4.12), the equation (4.7) can be rewritten as follows:

$$\begin{aligned}
 (4.13) \quad & 2\sigma_j [\alpha_k + n|\sigma|^2 \beta_k + (n-1)\beta \sigma^r \nabla_k \sigma_r] + 2(n+1)\beta(|\sigma|^2 \nabla_k \sigma_j + \sigma^r \nabla_k \tilde{\sigma}_r \tilde{\sigma}_j) \\
 & = (n+1)\alpha_r \sigma^r g_{kj} + \sigma_k (n\alpha_j + |\sigma|^2 \beta_j + 2\beta \sigma^r \nabla_j \sigma_r) - \tilde{\sigma}_k (n\alpha_j + |\sigma|^2 \tilde{\beta}_j + 2\beta \sigma^r \nabla_r \tilde{\sigma}_j).
 \end{aligned}$$

Multiplying $J_m^k J_n^j$ to (4.13), it holds that

$$\begin{aligned}
 & 2\tilde{\sigma}_n [\tilde{\alpha}_m + n|\sigma|^2 \beta_m - (n-1)\beta J_m^k \sigma^r \nabla_k \sigma_r] + 2(n+1)\beta(|\sigma|^2 J_m^k J_n^j \nabla_k \sigma_j + J_m^k \nabla_k \tilde{\sigma}_r \sigma^r \sigma_j) \\
 & = (n+1)\alpha_r \sigma^r g_{mn} + \tilde{\sigma}_m (n\tilde{\alpha}_n + |\sigma|^2 \tilde{\beta}_n - 2\beta J_n^j \nabla_j \sigma_r \sigma^r) - \sigma_m (n\alpha_n + |\sigma|^2 \beta_n + 2\beta \sigma^r \nabla_r \sigma_n)
 \end{aligned}$$

and consequently, by taking account of (4.13) and the equation above we obtain

$$\begin{aligned}
 (4.14) \quad & \sigma_j [\alpha_k + n|\sigma|^2 \beta_k + (n-1)\beta \sigma^r \nabla_k \sigma_r + (n+1)\beta J_k^s \nabla_s \tilde{\sigma}_r \sigma^r] \\
 & + \tilde{\sigma}_j [\tilde{\alpha}_k + n|\sigma|^2 \tilde{\beta}_k + (n-1)\beta \sigma^r \nabla_r \tilde{\sigma}_k + (n+1)\beta \sigma^r \nabla_k \tilde{\sigma}_r] \\
 & + (n+1)\beta |\sigma|^2 (\nabla_k \sigma_j + J_k^r J_j^s \nabla_r \sigma_s) = (n+1)\alpha_r \sigma^r g_{jk}.
 \end{aligned}$$

Hence, let us consider the open subset M' in M such that

$$M' = \{P \in M : \beta \neq 0 \text{ and } |\sigma|^2 \neq 0 \text{ at } P\}.$$

Then M' is nonempty. In fact, if M' is empty, then M is a manifold of constant holomorphic sectional curvature. Also, we can easily know that a manifold of constant holomorphic sectional curvature is holomorphically subprojective. So we shall always consider the our discussions in M' in the following.

Transvecting (4.14) with $\tilde{\sigma}^k$ and $\tilde{\sigma}^k \sigma^j$ respectively and regarding to (4.9) and (4.10), we have

$$(4.15) \quad (\nabla_r \sigma_s + J_r^a J_s^b \nabla_a \sigma_b) \sigma^r \sigma^s \tilde{\sigma}_j = |\sigma|^2 (\sigma^r \nabla_r \tilde{\sigma}_j - \sigma^r \nabla_j \tilde{\sigma}),$$

$$(4.16) \quad |\sigma|^2 (\alpha_k + n |\sigma|^2 \beta_k + 2n \beta \sigma^r \nabla_k \sigma_r) = (n+1) \alpha_r \sigma^r \sigma_k.$$

By the way, we take the skew-symmetric part with respect to indices k and j in (4.13), then

$$(4.17) \quad [(n+2) \alpha_{[k} + (2n+1) |\sigma|^2 \beta_{[k} + 2n \beta \sigma^r \nabla_r \sigma_{[k}] \sigma_{j]} \\ = [n \tilde{\alpha}_{[k} + |\sigma|^2 \tilde{\beta}_{[k} + 2\beta \sigma^r \nabla_r \tilde{\sigma}_{[k} - 2(n+1) \beta \sigma^r \nabla_{[k} \tilde{\sigma}_{|r|]}] \tilde{\sigma}_{j]},$$

which yields that

$$(4.18) \quad |\sigma|^2 (\alpha_k + |\sigma|^2 \beta_k) = (|\sigma|^2 \sigma_r \beta^r + \alpha_r \sigma^r) \sigma_k + |\sigma|^2 \beta_r \tilde{\sigma}^r \tilde{\sigma}_k,$$

where we have contracted (4.17) with σ^j and made use of (4.12) and (4.16), and subtracting (4.4) from (4.5), it follows that

$$(4.19) \quad (n-1) \alpha_i + |\sigma|^2 \beta_i + \beta (\nabla_i \sigma_r + J_i^s J_r^t \nabla_s \sigma_t) \sigma^r = (\beta_r \sigma^r + \beta \nabla_r \sigma^r) \sigma_i + \beta_r \tilde{\sigma}^r \tilde{\sigma}_i,$$

which means that

$$(4.20) \quad (n-1) \alpha_r \sigma^r + \beta (\nabla_r \sigma_s + J_r^i J_s^j \nabla_i \sigma_j) \sigma^r \sigma^s = \beta |\sigma|^2 \nabla_r \sigma^r.$$

If we take account of (4.15) and (4.19), then we get

$$(4.21) \quad (n-1) |\sigma|^2 \alpha_i + |\sigma|^4 \beta_i = |\sigma|^2 \beta_r \tilde{\sigma}^r \tilde{\sigma}_i \\ + [|\sigma|^2 (\beta_r \sigma^r + \beta \nabla_r \sigma^r) - \beta (\nabla_r \sigma_s + J_r^i J_s^j \nabla_i \sigma_j) \sigma^r \sigma^s] \sigma_i,$$

from which, by making use of (4.18) and (4.20)

$$(4.22) \quad |\sigma|^2 \alpha_i = \alpha_r \sigma^r \sigma_i$$

and therefore this means from (4.18)

$$(4.23) \quad |\sigma|^2 \beta_k = \beta_r \sigma^r \sigma_k + \beta_r \tilde{\sigma}^r \tilde{\sigma}_k.$$

It follows from (4.16) that

$$2\beta |\sigma|^2 \sigma^r \nabla_r \sigma_k = (\alpha_r \sigma^r - |\sigma|^2 \beta_r \sigma^r) \sigma_k - |\sigma|^2 \beta_r \tilde{\sigma}^r \tilde{\sigma}_k,$$

or, equivalently

$$(4.24) \quad 2\beta |\sigma|^2 \sigma^r \nabla_r \tilde{\sigma}_j = (\alpha_r \sigma^r - |\sigma|^2 \beta_r \sigma^r) \tilde{\sigma}_j + |\sigma|^2 \beta_r \tilde{\sigma}^r \sigma_j$$

and by means of (4.20) and (4.24), from (4.15) we have

$$(4.25) \quad 2\beta |\sigma|^2 \sigma^r \nabla_j \tilde{\sigma}_r = [(2n-1)\alpha_r \sigma^r - |\sigma|^2 \beta_r \sigma^r - 2|\sigma|^2 \beta_r \nabla_r \sigma^r] \tilde{\sigma}_j + |\sigma|^2 \beta_r \tilde{\sigma}^r \sigma_j.$$

Finally, substituting (4.25) into (4.14), it holds that

$$(4.26) \quad \nabla_k \sigma_j = a g_{kj} + b_k \sigma_j + c_k \tilde{\sigma}_j,$$

or, equivalently

$$(4.27) \quad \nabla_k \tilde{\sigma}_j = a J_{kj} + b_k \tilde{\sigma}_j - c_k \sigma_j,$$

where we have put

$$(4.28) \quad \begin{cases} a = \frac{\alpha_r \sigma^r}{2\beta |\sigma|^2}, & b_j = -\frac{1}{2\beta |\sigma|^2} (\beta_r \sigma^r \sigma_j + \beta_r \tilde{\sigma}^r \tilde{\sigma}_j), \\ c_j = -\frac{1}{2\beta |\sigma|^4} [|\sigma|^2 \beta_r \tilde{\sigma}^r \sigma_j + (2n\alpha_r \sigma^r - |\sigma|^2 \beta_r \sigma^r - 2|\sigma|^2 \beta_r \nabla_r \sigma^r) \tilde{\sigma}_j]. \end{cases}$$

In the next place, we calculate some identities under the condition (I) which are necessary for what follows.

Now we put

$$(4.29) \quad H_{ji} = -\frac{1}{2(n+1)} \left[\frac{(n+2)\alpha - \beta |\sigma|^2}{n+1} g_{ji} + \beta (\sigma_j \sigma_i + \tilde{\sigma}_j \tilde{\sigma}_i) \right],$$

then we have by the condition (I)

$$(4.30) \quad \begin{aligned} -R_{kji}{}^r u_r &= u_{[k} H_{j]i} - \tilde{u}_{[j} H_{k]r} J_i{}^r - 2\tilde{u}_i H_{kr} J_j{}^r \\ &\quad + \frac{\beta}{2(n+1)} [g_{ik} (\sigma_j \sigma_r u^r + \tilde{\sigma}_j \tilde{\sigma}_r u^r) - J_{ik} (\sigma_j \tilde{\sigma}_r u^r - \tilde{\sigma}_j \sigma_r u^r) \\ &\quad + 2J_{kj} (\sigma_i \tilde{\sigma}_r u^r - \tilde{\sigma}_i \sigma_r u^r)] \end{aligned}$$

for any vector field w^h . Operating ∇_k to (4.29) and taking account of (4.26) and (4.27), we can get

$$(4.31) \quad -2(n+1)\nabla_{[k}H_{j]i} = \frac{\alpha_r \sigma^r}{2|\sigma|^2} (g_{i[l} \sigma_{k]} + \tilde{\sigma}_{[l} J_{k]i} + 2\tilde{\sigma}_i J_{kj}).$$

Let us calculate now $\nabla_i a = a_i$. We obtain with the aid of (4.22) and (4.28)

$$(4.32) \quad \alpha_i = 2a\beta\sigma_i,$$

which means that

$$\nabla_i \alpha_i = 2(a_i \beta \sigma_i + a \beta_j \sigma_i + a \beta \nabla_i \sigma_i)$$

and take the skew-symmetric part in this, then we get

$$\beta(a_j \sigma_i - a_i \sigma_j) + a(\beta_j \sigma_i - \beta_i \sigma_j) = 0.$$

At last, contracting this with σ^i and according to (4.23), it follows that

$$(4.33) \quad a_j = \frac{1}{\beta|\sigma|^2} (\beta a_r \sigma^r \sigma_j - a \beta_r \tilde{\sigma}^r \tilde{\sigma}_j).$$

5. - Condition (II).

Let us consider a vector field ξ^h which is given by

$$(5.1) \quad \xi^h = \kappa \sigma^h + \mu \tilde{\sigma}^h$$

for some functions, from which we obtain

$$(5.2) \quad \sigma^i = \frac{\kappa \xi^i - \mu \tilde{\xi}^i}{\kappa^2 + \mu^2}, \quad \tilde{\sigma}^i = \frac{\kappa \tilde{\xi}^i + \mu \xi^i}{\kappa^2 + \mu^2}.$$

Let us try to rewrite a_j , b_i and c_j by virtue of (5.2). Then we have

$$(5.3) \quad b_j = -\frac{1}{2\beta|\xi|^2} (\beta' \xi_j + \beta'' \tilde{\xi}_j),$$

$$(5.4) \quad c_j = \frac{1}{2\beta|\xi|^2} [\{2\beta\mu(\nabla_r \sigma^r - 2na) - \beta''\} \xi_j + \{2\beta\kappa(\nabla_r \sigma^r - 2na) + \beta'\} \tilde{\xi}_j],$$

$$(5.5) \quad a_j = \frac{1}{\beta|\xi|^2(\kappa^2 + \mu^2)} [\{\kappa\beta(\kappa a' - \mu a'') - \mu a(\kappa\beta'' + \mu\beta')\} \xi_j - \{\mu\beta(\kappa a' - \mu a'') + \kappa a(\kappa\beta'' + \mu\beta')\} \tilde{\xi}_j],$$

where we have put

$$(5.6) \quad a' = a_r \xi^r, \quad a'' = a_r \xi^r, \quad \beta' = \beta_r \xi^r, \quad \beta'' = \beta_r \xi^r.$$

If we contract (5.5) with ξ^j , then we have

$$(5.7) \quad \beta(\mu a' + \kappa a'') + a(\kappa \beta'' + \mu \beta') = 0,$$

which implies from (5.5) that

$$(5.8) \quad a_j = \frac{1}{|\xi|^2} (a' \xi_j + a'' \xi_j).$$

Furthermore the equations (4.29) $\sim$ (4.31) can be rewritten as follows:

$$(5.9) \quad -R_{kji}{}^r u_r = u_{[k} H_{j]i} - \tilde{u}_{[j} H_{k]r} J_i{}^r - 2\tilde{u}_i H_{kr} J_j{}^r \\ + \frac{\beta}{2(n+1)(k^2 + \mu^2)} [(u' \xi_{[j} + u'' \xi_{[j}) g_{k]i} + (u'' \xi_{[j} - u' \xi_{[j}) J_{k]i} + 2(u'' \xi_i - u' \xi_i) J_{kj}],$$

$$(5.10) \quad H_{ki} = -\frac{1}{2(n+1)} \left[\frac{(n+2)\alpha - \beta|\sigma|^2}{n+1} g_{ki} + \frac{\beta}{\kappa^2 + \mu^2} (\xi_k \xi_i + \xi_k \xi_i) \right],$$

$$(5.11) \quad \nabla_{[k} H_{j]i} = -\frac{a\beta}{2(n+1)(\kappa^2 + \mu^2)} [\kappa(g_{i[j} \xi_{k]} + \xi_{[j} J_{k]i} + 2\xi_i J_{kj}) \\ - \mu(g_{i[j} \xi_{k]} - \xi_{[j} J_{k]i} - 2\xi_i J_{kj})],$$

where we put $u' = u_r \xi^r$ and $u'' = u_r \xi^r$.

Next, differentiating (5.1) covariantly, it follows from (4.26), (4.27) and (5.2) that

$$(5.12) \quad \nabla_j \xi_i = \kappa a g_{ji} + \mu a J_{ji} + (b_j + v_j) \xi_i + (c_j + w_j) \xi_i,$$

or, equivalently

$$(5.13) \quad \nabla_j \xi_i = \kappa a J_{ji} - \mu a g_{ji} + (b_j + v_j) \xi_i - (c_j + w_j) \xi_i,$$

where we put

$$(5.14) \quad v_j = \frac{\kappa \kappa_j + \mu \mu_j}{\kappa^2 + \mu^2}, \quad w_j = \frac{\kappa \mu_j - \mu \kappa_j}{\kappa^2 + \mu^2}.$$

Hence we assume that ξ^h is contravariant analytic. Then we have from the condition (II)

$$(5.15) \quad \nabla_j \xi_i - J_j{}^r J_i{}^s \nabla_r \xi_s = (b_j + v_j) \xi_i + (c_j + w_j) \xi_i - (\tilde{b}_j + \tilde{v}_j) \xi_i + (\tilde{c}_j + \tilde{w}_j) \xi_i = 0,$$

which implies that

$$(5.16) \quad b_j + \tilde{c}_j + v_j + \tilde{w}_j = 0.$$

In the next section, we deal with (5.16) under the condition (III) and we can see that (5.16) plays important roles in the proof of main theorem.

6. - Condition (III).

Let us now partial differential equations for vector z_i

$$(6.1) \quad \nabla_j z_i = -\varrho_{(i} z_{j)} + \tilde{\varrho}_{(i} \tilde{z}_{j)} - f_{ji} z_r \xi^r + f_{js} J_i^s z_r \xi^r,$$

where ϱ_i is gradient and f_{ji} a symmetric tensor satisfying

$$(6.2) \quad f_{[ij} J_{i]}^r = 0$$

and ξ^h is given by (5.1).

Hereafter, we choose κ and μ in (5.1) as follows:

$$(6.3) \quad \varrho' - \kappa a = -1, \quad \varrho'' + \mu a = 0,$$

where we set $\varrho' = \varrho_r \xi^r$ and $\varrho'' = \varrho_r \tilde{\xi}^r$.

In the first place, we differentiate (6.1) covariantly. Then

$$\begin{aligned} \nabla_k \nabla_j z_i = & -\nabla_k \varrho_i z_j - \nabla_k \varrho_j z_i - \varrho_i \nabla_k z_j - \varrho_j \nabla_k z_i \\ & + \nabla_k \tilde{\varrho}_i \tilde{z}_j + \nabla_k \tilde{\varrho}_j \tilde{z}_i + \tilde{\varrho}_i \nabla_k \tilde{z}_j + \tilde{\varrho}_j \nabla_k \tilde{z}_i \\ & - \nabla_k f_{ji} z_r \xi^r - f_{ji} \nabla_k (z_r \xi^r) + \nabla_k f_{js} J_i^s z_r \xi^r + f_{js} J_i^s \nabla_k (z_r \xi^r), \end{aligned}$$

from which

$$(6.4) \quad \nabla_{[k} \nabla_{j]} z_i = z_{[k} H'_{j]i} + \tilde{z}_{[j} J_{k]}^r H'_{ir} + \tilde{z}_i J_{[k}^r H'_{j]r} - z_r \xi^r Q_{kji} + z_r \tilde{\xi}^r Q_{kjr} J_i^r,$$

where we set

$$(6.5) \quad H'_{kj} = \nabla_k \varrho_j + \varrho_k \varrho_j - \tilde{\varrho}_k \tilde{\varrho}_j - f_{ki},$$

$$(6.6) \quad Q_{kji} = \nabla_{[k} f_{j]i} - (-f'_{[j} + b_{[j} + v_{[j}) f_{k]i} - (f''_{[j} + c_{[j} + w_{[j}) f_{k]r} J_i^r,$$

$$(6.7) \quad f'_j = f_{jr} \xi^r, \quad f''_j = f_{jr} \tilde{\xi}^r.$$

By our assumption (III), we may choose ϱ_i and f_{jk} such that

$$(6.8) \quad H_{jk} = H'_{jk},$$

or, equivalently

$$(6.9) \quad \nabla_k \varrho_i + \varrho_k \varrho_i - \tilde{\varrho}_k \tilde{\varrho}_i - f_{ki} = -\frac{1}{2(n+1)} \left[\frac{(n+2)\alpha - \beta|\sigma|^2}{n+1} g_{ki} + \frac{\beta}{\kappa^2 + \mu^2} (\xi_k \xi_i + \tilde{\xi}_k \tilde{\xi}_i) \right].$$

Applying ∇_k to (6.5), it follows that

$$\nabla_k H'_{ji} = \nabla_k \nabla_j \varrho_i + \nabla_k \varrho_j \varrho_i + \varrho_j \nabla_k \varrho_i - \nabla_k \tilde{\varrho}_j \tilde{\varrho}_i - \tilde{\varrho}_j \nabla_k \tilde{\varrho}_i - \nabla_k f_{ji},$$

from which we get

$$(6.10) \quad \begin{aligned} \nabla_{[k} H'_{j]i} = & -R_{kji}{}^r \varrho_r + \varrho_j \nabla_k \varrho_i - \varrho_k \nabla_j \varrho_i - \tilde{\varrho}_i (\nabla_k \tilde{\varrho}_j - \nabla_j \tilde{\varrho}_k) \\ & - \tilde{\varrho}_j \nabla_k \tilde{\varrho}_i + \tilde{\varrho}_k \nabla_j \tilde{\varrho}_i - \nabla_{[k} f_{j]i} \end{aligned}$$

and eliminating $\nabla_{[k} f_{j]i}$ from (6.6) and (6.10) and owing to (5.9), (5.11), (6.5) and (6.8), it follows that

$$(6.11) \quad \begin{aligned} & \frac{\beta}{2(n+1)(\kappa^2 + \mu^2)} (g_{ilj} \xi_{kl} + \xi_{lij} J_{kl} + 2\xi_{li} J_{kj}) \\ & = Q_{kji} + (-f'_{ij} - \varrho_{ij} + b_{ij} + v_{ij}) f_{kji} + (f''_{ij} - \tilde{\varrho}_{ij} + c_{ij} + w_{ij}) f_{kji} J_i{}^r. \end{aligned}$$

On the other hand, differentiating (6.3) covariantly and regarding to (5.12), (5.14), (6.3) and (6.9), we find

$$\kappa_k a + \kappa a_k = \nabla_k \varrho' = \nabla_k \varrho_r \xi^r + \varrho^r \nabla_k \xi_r = \varrho_k + f'_k - A \xi_k + \varrho'(b_k + v_k) + \varrho''(c_k + w_k)$$

and

$$\mu_k a + \mu a_k = -\nabla_k \varrho'' = -\nabla_k \varrho_r \xi^r - \varrho^r \nabla_k \xi_r = \tilde{\varrho}_k - f''_k + A \tilde{\xi}_k - \varrho''(b_k + v_k) + \varrho'(c_k + w_k),$$

from which

$$(6.12) \quad \kappa a_k = \varrho_k + f'_k - A \xi_k + \varrho' b_k + \varrho'' c_k - v_k$$

and

$$(6.13) \quad \mu \tilde{a}_k = -\varrho_k - f''_k - A \tilde{\xi}_k - \varrho'' \tilde{b}_k + \varrho' \tilde{c}_k - \tilde{w}_k$$

and consequently, we have

$$(6.14) \quad \kappa a_k + \mu a_k = -2A \xi_k + \varrho'(b_k + c_k) + \varrho''(c_k - \tilde{b}_k) - v_k - \tilde{w}_k,$$

where we put

$$A = \frac{(n+2)\alpha + n\beta|\sigma|^2}{2(n+1)^2}.$$

Now if we put

$$(6.15) \quad \left\{ \begin{aligned} E &= \frac{1}{2\beta|\xi|^2} [2\beta|\xi|^2 A + a(\kappa\beta' - \mu\beta'') + 2\beta\{\kappa a' + a\mu^2(\nabla_r \sigma^r - 2na)\}], \\ F &= \frac{1}{2\beta|\xi|^2} [a(\kappa\beta'' + \mu\beta') + 2\beta\{\kappa a'' + a\kappa\mu(\nabla_r \sigma^r - 2na)\}], \\ G &= \frac{1}{2\beta|\xi|^2} [a(\kappa\beta'' + \mu\beta') + 2\beta\{\mu a' - a\kappa\mu(\nabla_r \sigma^r - 2na)\}], \\ H &= \frac{1}{2\beta|\xi|^2} [-2\beta|\xi|^2 A + a(\mu\beta'' - \kappa\beta') + 2\beta\{\mu a'' - a\kappa^2(\nabla_r \sigma^r - 2na)\}], \end{aligned} \right.$$

and substitute (6.12) and (6.13) into (6.11), then we have

$$(6.16) \quad Q_{kji} = \frac{\beta}{2(n+1)(\kappa^2 + \mu^2)} (g_{ilj}\xi_{kl} + \xi_{lj}J_{kl} + 2\xi_{li}J_{kj}) \\ + (E\xi_{lk} + F\xi_{lk})f_{jli} + (G\xi_{lk} + H\xi_{lk})f_{jlr}J_i^r,$$

because of (5.3), (5.4) and (5.5).

By the way, it follows from (6.15) that

$$(6.17) \quad F + G = 0$$

by virtue of (5.7).

In the next place, we shall investigate deep into (5.15). Making use of (6.15), we have

$$\nabla_j \xi_i - J_j^r J_i^s \nabla_r \xi_s = (H - E)(\xi_j \xi_i - \xi_j \xi_i) \\ = 0,$$

which means that

$$(6.18) \quad H - E = 0.$$

Therefore if we can prove

$$(6.19) \quad H + E = 0 \quad \text{and} \quad F - G = 0,$$

then we know $E = F = G = H = 0$ by virtue of (6.17) and (6.18). In the rest of this section, let us show (6.19).

On the other hand, although M. MATSUMOTO in [3] has proved that $\nabla^i R (= g^{ir} \nabla_r R)$ is contravariant analytic in a compact Kählerian manifold with vanishing Bochner curvature tensor, by means of the equation in the middle of page 27, we can easily prove this fact without compact. So, in the following we can take account of this fact.

Making use of (4.23) and (4.28), it holds that

$$(6.20) \quad \beta_j = -2\beta b_j,$$

and we have by means of (4.26), and (4.32) and (6.20)

$$\nabla_j R = 2(n+1)\alpha_j.$$

Since $\nabla^j R$ is contravariant analytic, we get

$$(6.21) \quad \nabla_j \alpha_i = J_j^r J_i^s \nabla_r \alpha_s.$$

To make use of (6.21), let us calculate $\nabla_j \alpha_i$. We differentiate (4.32) covariantly and use (4.26). Then

$$\nabla_j \alpha_i = 2[\beta a_j \sigma_i - a\beta b_j \sigma_i + a^2 \beta g_{ji} + a\beta c_j \tilde{\sigma}_i],$$

and substituting this into (6.21) and owing to (6.20), we get

$$(\beta a_j - a\beta b_j + a\beta \tilde{c}_j)\sigma_i - (\beta \tilde{a}_j - a\beta \tilde{b}_j - a\beta c_j)\tilde{\sigma}_i = 0,$$

which yields that

$$\beta a_j - a\beta b_j + a\beta \tilde{c}_j = 0.$$

By virtue of (5.3), (5.4) and (5.8), the equation above becomes

$$\{a' - a\kappa(\nabla_r \sigma^r - 2na)\}\xi_j + \{a'' + a\mu(\nabla_r \sigma^r - 2na)\}\tilde{\xi}_j = 0,$$

which imply that

$$a' - a\kappa(\nabla_r \sigma^r - 2na) = 0, \quad a'' + a\mu(\nabla_r \sigma^r - 2na) = 0.$$

Finally, by virtue of the above equations, it follows from (6.15) that (6.19). Then from (6.16) we have

$$Q_{kji} = \frac{\beta}{2(n+1)(\kappa^2 + \mu^2)} (g_{i[lj}\xi_{k]l} + \xi_{i[lj}J_{k]l} + 2\xi_{i[l}J_{k]l}),$$

which means that

$$\nabla_k \nabla_j z_i - \nabla_j \nabla_k z_i = -R_{kji}{}^r z_r,$$

making use of (5.9), (6.4) and (6.8) and consequently, the partial differential equations (6.1) are completely integrable under the conditions (I), (II) and (III).

7. – Main theorem.

Since (6.1) is completely integrable, there exist $2n$ linearly independent solutions $z_i^{(h)}$ and any solution is expressible as a linear combination with constant coefficients of these particular solutions. Since f_{ji} and $f_{jr}J_i^r$ are symmetric, $z_i^{(h)}$ are gradient vectors. Hence if we set

$$(7.1) \quad z_i^{(h)} = \frac{\partial \bar{x}^h}{\partial x^i},$$

then $\bar{x}^h = \bar{x}^h(x^i)$ are $2n$ independent functions of x^h and consequently may be considered to define a transformation of local coordinate system.

Now the Christoffel symbols of the second kind in the coordinate system $(\bar{x}^h)$ have the following transformation law under a coordinate transformation

$$\left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} = \frac{\partial x^a}{\partial \bar{x}^j} - \frac{\partial x^b}{\partial \bar{x}^i} \left(\left\{ \begin{matrix} c \\ a \ b \end{matrix} \right\} \frac{\partial \bar{x}^h}{\partial x^c} - \frac{\partial^2 \bar{x}^h}{\partial x^a \partial x^b} \right),$$

from which we have by (6.1) and (7.1)

$$\begin{aligned}
 \overline{\left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\}} &= \frac{\partial x^a}{\partial \bar{x}^j} \frac{\partial x^b}{\partial \bar{x}^i} \left(\left\{ \begin{smallmatrix} c \\ a \ b \end{smallmatrix} \right\} z_c^{(h)} - \frac{\partial z_b^{(h)}}{\partial x^c} \right) \\
 &= - \frac{\partial x^a}{\partial \bar{x}^j} \frac{\partial \bar{x}^b}{\partial \bar{x}^i} \nabla_a z_b^{(h)} \\
 (7.2) \qquad &= \bar{\varrho}_{(j} \delta_{i)}^h + \varrho_{(j} \bar{J}_{i)}^h + \bar{f}_{ji} \tilde{\xi}^h - \bar{f}_{jr} \bar{J}_i{}^r \tilde{\xi}^h,
 \end{aligned}$$

where $\bar{\varrho}_j$, $\bar{f}_{ji}$, $\bar{J}_j{}^h$ and $\tilde{\xi}^h$ are components in the local coordinate system $(\bar{x}^h)$ corresponding for ϱ_i , f_{ji} , $J_j{}^h$ and ξ^h respectively.

Finally, we shall show that M is the holomorphically subprojective Kählerian manifold of the first kind under the conditions (I), (II) and (III) stated in § 3. Now we set $z = \xi^r z_r$. Then we have after some computations

$$\begin{aligned}
 \frac{\partial z}{\partial x^i} &= \nabla_i z = \nabla_i \xi^r z_r + \xi^r \nabla_i z_r \\
 &= z_i + E \xi_i + F_i \\
 &= z_i
 \end{aligned}$$

by making use of (5.12), (6.1), (6.12), (6.13) and (6.15). Since we have $z^{(h)} = \xi^r z_r^{(h)}$, it follows that

$$\frac{\partial z^{(h)}}{\partial x^i} = z_i^{(h)} = \frac{\partial \bar{x}^h}{\partial x^i},$$

from which we get $z^{(h)} = \bar{x}^h$ and consequently $\bar{x}^h = \xi^r z_r^{(h)}$, or

$$(7.3) \qquad \bar{x}^h = \xi^h.$$

Taking account of (7.2) and (7.3), we obtain

$$\overline{\left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\}} = \bar{\varrho}_{(j} \delta_{i)}^h + \tilde{\varrho}_{(j} \bar{J}_{i)}^h + \bar{f}_{ji} \bar{x}^h - \bar{f}_{jr} \bar{J}_i{}^r \bar{x}^h$$

in a suitable local coordinate system $(\bar{x}^h)$. Therefore M is a holomorphically subprojective Kählerian manifold. At last, by Theorem A we can see that M is the holomorphically subprojective Kählerian manifold of the first kind.

Coversely, if M the holomorphically subprojective Kählerian manifold of the first kind, then M has the properties (I), (II) and (III) stated in § 3 by virtue of Theorem A.

Thus we have

MAIN THEOREM. — *In order that a $2n$ ($n \geq 3$) dimensional Kählerian manifold M is the holomorphically subprojective Kählerian manifold of the first kind, it is necessary and sufficient that the following properties (I), (II) and (III) are satisfied in M :*

I) M has a vanishing Bochner curvature tensor which has the Ricci tensor of the form

$$R_{ji} = \alpha g_{ji} + \beta(\sigma_j \sigma_i + \bar{\sigma}_j \bar{\sigma}_i),$$

where σ_i is a gradient vector, and α and β are some functions.

(I) The vector ξ^h given by $\xi^h = \kappa \sigma^h + \mu \bar{\sigma}^h$ for some functions κ and μ is contravariant analytic.

(III) There exist a gradient vector ϱ_i and a symmetric tensor f_{ji} satisfying $f_{r[i} J_{j]}^r = 0$ and

$$\nabla_k \varrho_j = -\varrho_k \varrho_j + \tilde{\varrho}_k \tilde{\varrho}_j + f_{kj} + \phi g_{kj} + \varphi(\xi_k \xi_j + \bar{\xi}_k \bar{\xi}_j),$$

where ϕ and φ are some functions.

By virtue of (7.2) and the Main Theorem, we have

THEOREM. — *The Christoffel symbols of the holomorphically subprojective Kählerian manifold of the first kind take the form*

$$\left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} = \varrho_{(j} \delta_{i)}^h + \tilde{\varrho}_{(j} J_{i)}^h + f_{ji} \xi^h - f_{jr} J_i^r \bar{\xi}^h$$

for suitable coordinate system (x^h) , where ξ^h is a contravariant analytic vector field such that

$$\nabla_j \xi^h = \alpha \delta_j^h + \beta J_j^h + \gamma_j \xi^h + \bar{\gamma}_j \bar{\xi}^h, \quad \gamma_j = \partial_j \gamma.$$

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