Oscillation, Nonoscillation, and Asymptotic Behavior for Third Order Nonlinear Differential Equations (*) (**).

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Summary. – *This paper discusses the behavior of real-valued solutions to the equation*

$$y''' + p(t)y'' + q(t)y' + r(t)y^{\mu} = 0\,, \tag{1.1}$$

where p, q, r are continuous and real-valued on some half-line $(t_0, +\infty)$ and μ is the quotient of odd positive integers. Criteria are obtained for the existence of nonoscillatory solutions, several stability theorems are proved, and the existence of oscillatory solutions is shown. Of primary concern are the two cases $q(t) \leqslant 0$, $r(t) > 0$, and $q(t) > 0$, $r(t) > 0$. Some of the main techniques used involve comparison theorems for linear equations and results in the theory of second order nonlinear oscillations.

1. – The purpose of this paper is to discuss the existence and asymptotic behavior of oscillatory and nonoscillatory solutions of the third order nonlinear differential equation

$$y''' + p(t)y'' + q(t)y' + r(t)y^{\mu} = 0 \tag{1.1}$$

where $\mu > 0$ is the quotient of odd positive integers, and $p(t)$, $q(t)$, $r(t)$ are (at least) continuous on some interval $(t_0, +\infty)$. A solution of (1.1) existing on $(t_1, +\infty)$ for some $t_1 \geqslant t_0$ is said to be oscillatory or nonoscillatory according as it does or does not have arbitrarily large zeros. Equation (1.1) with $p \equiv 0$ has been considered by HEIDEL [9], NELSON [15], SOLTES [16], ELIAS [2], WALTMAN [19], and for the case $\mu = 1$ we mention in particular the papers of LAZER [14], HANAN [7], BARRETT [1], and the book of SWANSON [18]. Although the coefficient of y'' can be removed by a simple change of variable, it is often useful to have oscillation and nonoscillation criteria with explicit relations holding among the various coefficients (cf. [13], [6]), and for that reason we prefer to consider the more general equation (1.1). In section 2 we establish the existence of nonoscillatory solutions of (1.1) with certain asymptotic properties under various assumptions on the coefficients (but always for the case $r(t) > 0$). In section 3 we consider the case $q \leqslant 0$, $r > 0$ and show that under certain additional assumptions, all nonoscillatory solutions of (1.1) tend to

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zero monotonically. Our results may be applied, in particular, to the case when $\int^{\infty} r(t)\,dt < \infty$, which is not treated in any of the references. These results improve and extend the results of [15], [16], [2], and [5], and for the case $\mu = 1$ results of [14]. Section 4 is devoted to the case $q \geqslant 0$, $r > 0$ where we establish stability theorems and oscillation criteria for (1.1) improving results of [9], [16] and [5]. Some of the results here are based on the theory of second order nonlinear oscillations along with appropriate changes of variable.

If $\mu \leqslant 1$, then solutions of (1.1) may be continued to all of $(t_0, +\infty)$. If $\mu > 1$, then under certain conditions a non-continuable solution of (1.1) has infinitely many zeros in a finite interval (see [9]). We shall be concerned here with the behavior of continuable solutions and shall assume that solutions of initial value problems for (1.1) are continuable to all of $(t_0, +\infty)$.

2. – In this section we shall establish some preliminary lemmas and theorems for the existence of nonoscillatory solutions of (1.1) which are interesting in their own right.

LEMMA 2.1. – *Let $q \leqslant 0$, $r > 0$. Then there exists a solution y of (1.1) with $y \not\equiv 0$ and*

$$y \geqslant 0\,, \quad y' \leqslant 0\,, \quad y'' \geqslant 0\,, \quad t \geqslant T \geqslant t_0\,. \tag{2.1}$$

PROOF. – The existence of a solution satisfying (2.1) for large t follows easily from a result of Hartman ([8], p. 510) by rewriting (1.1) as $(Py'')' + Qy' + Ry^{\mu} = 0$, with $P(t) = \exp\left(\int_T^t p\,ds\right)$, $Q(t) = P(t)q(t)$, and $R(t) = P(t)r(t)$. Setting $y_1 = y$, $y_2 = -y'$, $y_3 = y''$, then (1.1) may be written, with $Y = (y_1, y_2, Py_3)$, as

$$Y' + f(t, Y) = 0\,, \tag{2.2}$$

where $f(t, Y) = (y_2, y_3, Ry_1^{\mu} - Qy_2) \geqslant 0$ (in the usual component-wise sense) provided $Y \geqslant 0$. Therefore, by the result of Hartman, for any $k > 0$ there exists at least one solution $Y(t)$ with $\|Y(T)\| = k > 0$, and $Y(t) \geqslant 0$, $Y'(t) \leqslant 0$ for $t \geqslant T$.

REMARK. – If $\mu \geqslant 1$ so that initial value problems for (1.1) have unique solutions, then (2.1) may be replaced by

$$y > 0\,, \quad y' < 0\,, \quad y'' > 0\,, \quad t \geqslant T \geqslant t_0\,. \tag{2.1*}$$

To see this, note that if y satisfies (2.1) and $y''(t) = 0$ for some $t_1 \geqslant T$, then from (1.1) and our assumptions, $y'''(t_1) \leqslant 0$ and hence $y''' \equiv 0 \equiv y''$ for $t \geqslant t_1$. Therefore, $y' \equiv 0$, $t \geqslant t_1$, and since $r > 0$ we find $y \equiv 0$, $t \geqslant t_1$.

But then we must have, by uniqueness of solutions of initial value problems, that $y \equiv 0$ for $t \geqslant T$, contradicting Lemma 2.1.

The next result implies, among other things, the existence of a solution satisfying (2.1)* under different assumptions. Recall that an n-th order linear equation is said to be disconjugate on an interval I in case no nontrivial solution has more than $n-1$ zeros on I.

THEOREM 2.2. – Let $r > 0$ and assume the second order equation

$$y'' + (q - p'/2)y = 0 \tag{2.3}$$

is disconjugate on $(T, +\infty)$ for some $T \geqslant t_0$. Then there exists a solution of (1.1) satisfying for all large t

(i) $\qquad y > 0\,, \quad y' < 0\,, \quad \mu \geqslant 1\,,$

(ii) $\qquad y \geqslant 0\,, \quad y' \leqslant 0\,, \quad y \not\equiv 0\,, \quad 0 < \mu < 1\,.$

Furthermore, if $q - p' \leqslant 0$, $p \leqslant 0$, $p' \geqslant 0$ eventually, and if $y > 0$, $y' < 0$ holds for large t, then $y'' > 0$ for large t.

PROOF. – We show first that for each $t^* > T$ there exists a solution of (1.1), $y(t; t^*)$ satisfying the boundary conditions

$$y(T)^2 + y'(T)^2 + y''(T)^2 = 1\,, \qquad y(t^*) = y'(t^*) = 0 \tag{2.4}$$

and

$$y > 0\,, \qquad y' < 0 \qquad \text{for } T < t < t^*. \tag{2.5}$$

Clearly, if y is a solution of (1.1) satisfying $y(t^*) = y'(t^*) = 0 < y''(t^*)$, then the inequality in (2.5) holds, at least for awhile, in a left neighborhood of t^*. Suppose there is a point t_1, $T < t_1 < t^*$, with $y'(t_1) = 0$ and $y > 0$, $y' < 0$ on (t_1, t^*). If we multiply (1.1) by y' and integrate by parts from t_1 to t^* we obtain

$$\int_{t_1}^{t^*} r y^\mu y'\, dt = \int_{t_1}^{t^*} (y'')^2\, dt - \int_{t_1}^{t^*} (q - p'/2)(y')^2\, dt \tag{2.6}$$

Now the right hand side of (2.6) is easily seen to be positive by expanding

$$\int_{t_1}^{t^*} (y'' - y'v'/v)^2\, dt > 0\,,$$

where v is a solution of (2.3) with $v(t_1) = 0$, $v'(t_1) > 0$ on $(t_1, +\infty)$. But the left hand side of (2.6) is negative since $r y^\mu y' < 0$ on (t_1, t^*). This contradiction shows that (2.5) holds on (T, t^*). It remains to show that the first part of (2.4) holds. To this end, let S_k be the set of all numbers $y_k(T)^2 + y_k'(T)^2 + y_k''(T)^2$ where y_k is a solution of (1.1) with $y_k(t^*) = y_k'(t^*) = 0$, $y_k''(t^*) = k$. Then the set

$$S = \cup\{S_k : k \geqslant 0\}$$

is connected by the Kamke-Kneser Theorem ([8], [12]), and since $0 \in S$ it suffices to show that $\sup S < 1$ cannot hold. Suppose on the contrary that $\sup S < 1$. If we choose $M > 0$ so that $|p(t)| \leqslant M$, $|q(t)| \leqslant M$, and $|r(t)| \leqslant M$ on $[T, t^*]$, then an integration of (1.1) yields

$$|y_k''(t)| \leqslant C + M\int_T^t (|y_k''| + |y_k'|)\, ds \tag{2.7}$$

where C is a constant $(C = |y_k''(T)| + M(t^* - T))$. Since

$$|y_k'(t)| \leqslant |y_k'(T)| + \int_T^t |y_k''|\, ds\,,$$

upon adding this to (2.7) we get

$$|y_k''(t)| + |y_k'(t)| \leqslant C_1 + M_1\int_T^t (|y_k''| + |y_k'|)\, ds \tag{2.8}$$

for suitable constants C_1, M_1 (depending on C, M, $|y_k'(T)|$). Hence Gronwall's inequality shows that

$$|y_k''(t)| + |y_k'(t)| \leqslant M_2\,, \qquad T \leqslant t \leqslant t^*$$

for some constant M_2. This is a contradiction and therefore (2.4) and (2.5) hold for some $y = y_k$. Now letting $t_n \to \infty$, we obtain a sequence y_n of solutions of (1.1) satisfying (2.4), (2.5), and by a standard diagonalization argument, this yields a solution y of (1.1) on $[T, +\infty)$ with $y \geqslant 0$, $y' \leqslant 0$, and $y \not\equiv 0$. Now if $y > 0$ for $t \geqslant T$ and if $y'(t_1) = 0$ for some $t_1 > T$, then the first part of the argument above shows there cannot exist $t_2 > t_1$ with $y'(t_2) = 0$ and $y' < 0$ on $(t_1 \to t_2)$. Hence, either $y' < 0$, $t > t_1$, or $y' \equiv 0$, $t \geqslant t_1$. But the latter case clearly cannot hold since this leads to $ry^\mu \equiv 0$, a contradiction. Therefore, if $y > 0$ for $t \geqslant T$, then $y' < 0$ holds eventually. If $\mu \geqslant 1$, then as in the remark following Lemma 2.1, it must be the case that $y > 0$ for $t \geqslant T$, so that assertion (i) of the Theorem holds in this case. Finally, suppose $q - p' \leqslant 0$, $p \leqslant 0$, $p' \geqslant 0$ holds, $t \geqslant T_1$, and suppose $y > 0$, $y' < 0$, $t \geqslant T_1$. Since y is nonoscillatory, there must exist points arbitrarily far out for which $y'' > 0$. Suppose $y''(t_1) = y''(t_2) = 0$ and $y'' > 0$ on (t_1, t_2), $T_1 \leqslant t_1 < t_2$: Then an integration of (1.1) gives

$$py'\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2}(q-p')y'\,dt + \int_{t_1}^{t_1} ry^{\mu}\,dt = 0\,. \tag{2.9}$$

The sum of the two integrals in (2.9) is positive and since $y'(t_1) < y'(t_2) < 0$, (2.9) implies $p(t_2) < p(t_1)$, a contradiction. Therefore, we must have $y'' > 0$ eventually. This completes the proof of the Theorem.

Our next result yields the existence, but not the monotonicity, of nonoscillatory solutions to (1.1).

THEOREM 2.3. – Let $r > 0$, $p \leqslant 0$, and $p'' - q' \geqslant 0$ on $(t_0, +\infty)$. Then there exists a solution of (1.1) with $y \geqslant 0$ on $(t_0, +\infty)$, $y \not\equiv 0$. If $\mu \geqslant 1$, then $y > 0$.

PROOF. – For $t_0 < T < t^* < +\infty$, we show there exists a solution of (1.1) satisfying (2.4) with $y > 0$ on (t_0, t^*). With S and S_k as in Theorem 2.2 it suffices to show that $\sup S < 1$ cannot hold. Suppose y is a solution of (1.1) satisfying $y(t^*) = y'(t^*) = 0$, $y''(t^*) = k > 0$. We claim first that $y > 0$ on (t_0, t^*). If there exists $t_0 < t_1 < t^*$ with $y(t_1) = 0$ and $y > 0$ on (t_1, t^*), then multiplying (1.1) by y and integrating by parts gives

$$(pyy' - (y')^2/2)\Big]_{t_1}^{t^*} - \int_{t_1}^{t^*} p(y')^2\,dt + \int_{t_1}^{t^*}(q-p')yy'\,dt + \int_{t_1}^{t^*} ry^{\mu+1}\,dt = 0 \tag{2.10}$$

Now $\int_{t_1}^{t^*}(q-p')yy'\,dt = \int_{t_1}^{t^*}(p''-q')y^2/2\,dt$ so that from (2.10) we have

$$(y'(t_1))^2/2 - \int_{t_1}^{t^*} p(y')^2\,dt + \int_{t_1}^{t^*}(p''-q')y^2/2\,dt + \int_{t_1}^{t^*} ry^{\mu+1}\,dt = 0\,. \tag{2.11}$$

Since the left hand side of (2.11) is positive, this contradiction shows that $y > 0$ on (t_0, t^*).

Next, since

$$\int_{T}^{t^*} py''\,dt = -y'(T)p(T) - p'(T)y(T) + \int_{T}^{t^*} p''y\,dt\,,$$

and

$$\int_{T}^{t^*} qy'\,dt = -q(T)y(T) - \int_{T}^{t^*} q'y\,dt\,,$$

if we integrate (1.1) from T to t^*, we get

$$k + \int_{T}^{t^*} ry^{\mu+1}\,dt = \int_{T}^{t^*}(q'-p'')y\,dt + y''(T) + (p'(T) + q(T))y(T) + y'(T)p(T) \tag{2.12}$$

and therefore

$$k \leqslant |y''(T)| + |(p'(T) + q(T))\,y(T)| + |y'(T)\,p(T)|\,. \tag{2.13}$$

Since the right hand side of (2.13) cannot remain bounded for all $k > 0$, it follows that $\sup S = +\infty$ and hence there exists a solution of (1.1) satisfying (2.4). Now a diagonalization argument gives a solution of (1.1) $y \not\equiv 0$, and $y \geqslant 0$ on $(t_0, +\infty)$. If $\mu \geqslant 1$, then the first part of the argument above shows that y can have at most one double zero so $y > 0$ eventually. (If $0 < \mu < 1$, then either $y > 0$ for large t or $y \equiv 0$ for large t).

Remark. – Some of the results of this section generalize results of [3], [14], and [17].

3. – In this section we shall investigate the case $q \leqslant 0$ and $r > 0$. For convenience, we state first the following Lemma, a proof of which may be found in [4], for example.

Lemma 3.1. – *The equation $y''' + a(t)y'' + b(t)y' + c(t)y = 0$ is disconjugate on the interval I iff there exists $\alpha(t)$, $\beta(t) \in C^2(I)$ with $\alpha < \beta$ and $\alpha'' + f(t, \alpha, \alpha') \geqslant 0$, $\beta'' + f(t, \beta, \beta') \leqslant 0$ on I, where $f(t, u, u') = 3uu' + a(t)u' + u^3 + a(t)u^2 + b(t)u + c(t)$.*

Remark. – Functions $\alpha(t)$, $\beta(t)$ as in Lemma 3.1 are said to be lower and upper solutions respectively, of the Riccati equation corresponding to $y''' + ay'' + by' + cy = 0$, obtained by the substitution $u = y'/y$ (see [11] for details).

Lemma 3.2. – *Let $q \leqslant 0$, $r > 0$, and let y be a solution of (1.1) with $y \geqslant 0$ and $y \not\equiv 0$ on any half-line $[t_1, +\infty)$. Then there exists $T > t_0$ such that either*

(i) $\quad y > 0\,, \quad y' < 0\,, \quad y'' > 0\,, \quad t \geqslant T\,,$ or

(ii) $\quad y > 0\,, \quad y' \geqslant 0\,, \quad t \geqslant T\,.$

Further, if (i) *holds, then $y > 0$, $y' < 0$, $y'' > 0$ on $(t_0, +\infty)$ and*

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0$$

Proof. – We show first there exists $T > t_0$ such that $y > 0$ for $t > T$. If not, then we may find consecutive double zeros $t_1 < t_2$ of y with $y > 0$ on (t_1, t_2), and we may choose $t_3 \in (t_1, t_2)$ such that $y''(t_3) = 0$, $y'' > 0$ $y' < 0$ on (t_3, t_2). Then the function

$$w(t) = y(t)\,y'(t)\,y''(t)\,P(t)$$

satisfies $w' > 0$ on (t_3, t_2), where $P(t) = \exp\left(\int_{t_3}^{t} p\,ds\right)$.

Since $w(t_3) = w(t_2) = 0$, we have a contradiction. Therefore, $y > 0$ for $t \geqslant T$. We show next that y' can change sign at most two times. If $y'(t_1) = y'(t_2) = 0$, and

$y' \neq 0$ on (t_1, t_2), then since the second order equation $z'' + pz' + qz = 0$ is disconjugate on $(t_0, +\infty)$ (i.e., $q \leqslant 0$; cf. [8]), it follows that $y' > 0$ on (t_1, t_2). Otherwise, with $v = -y' > 0$, then $v'' + pv' + qv \geqslant 0$ on (t_1, t_2) and this implies the existence of a solution $z(t)$ of $z'' + pz' + qz =$ with $z(t_1) = z(t_2) = 0$ and $0 < v(t) \leqslant z(t)$ on (t_1, t_2), (cf. [10], [11]), and this is a contradiction. Hence, either $y' \geqslant 0$ or $y' < 0$ for all large t. In the latter case, since $(Py'')' = -Pqy' - pry^\mu < 0$, we see that the function $w = yy'y''P$ satisfies $w' > 0$ on any interval on which $y'' > 0$. Therefore, since $y'' \leqslant 0$ cannot hold for all large t, it follows that $y'' > 0$ for all large, say $t \geqslant T$. Furthermore, in this case, one can show as in the first part of the proof, that $w(t) \neq 0$ for $t_0 < t < T$. That $y' \to 0$, $y'' \to 0$ as $t \to +\infty$ is clear. This completes the proof.

REMARK. – The previous lemma is similar to a result of LAZER [14] for the linear case.

The next result gives a sufficient condition that nonoscillatory solutions of (1.1) satisfy (2.1)* for all $t > t_0$.

LEMMA 3.3. – Let $q \leqslant 0$, $r > 0$, and let y be a solution of (1.1) with $y \geqslant 0$ and $y \not\equiv 0$ on any half-line $[t_1, +\infty)$. Further, assume for each $\lambda > 0$ the equation

$$y''' + py'' + qy' + \lambda ry = 0 \tag{3.1}$$

is not disconjugate on any half-line $[t_1, +\infty)$. Then:

(i) if $0 < \mu < 1$ and y is bounded, or

(ii) if $\mu \geqslant 1$,

it follows that $y > 0$, $y' < 0$, $y'' > 0$ for $t_0 < t < +\infty$ and $y' \to 0$, $y'' \to 0$ as $t \to +\infty$.
Further, if $\mu = 1$, then it is sufficient to assume (3.1) is not disconjugate for $\lambda = 1$.

PROOF. – We need only show that case (ii) of Lemma 3.2 cannot occur. So suppose $y > 0$, $y' \geqslant 0$, $t \geqslant T$. With $u = y'/y \geqslant 0$, $t \geqslant T$, we obtain from (1.1)

$$u'' + 3uu' + pu' + u^3 + pu^2 + qu + ry^{\mu-1} = 0\ . \tag{3.2}$$

In either case (i) or (ii) we may find $\lambda_0 > 0$ ($\lambda_0 = 1$, if $\mu = 1$) such that $y^{\mu-1}(t) > \lambda_0$ for all $t \geqslant T$. Since initial value problems for the Riccati equation have unique solutions, it follows that $u > 0$ for $t \geqslant T$, and by (3.2) that u is an upper solution of the Riccati equation corresponding to (3.1) with $\lambda = \lambda_0$. Since $\alpha \equiv 0$ is a lower solution, it follows by Lemma 3.1 that (3.1) with $\lambda = \lambda_0$ is disconjugate on $[T, +\infty)$, a contradiction. This proves the Lemma.

The above result implies that if y is a solution of (1.1) with $y(t_1)y'(t_1) = 0$ for some $t_1 > t_0$ and if (3.1) has on oscillatory solution for all $\lambda > 0$ (for $\lambda = 1$ if $\mu = 1$), then y is oscillatory. This extends a results of Lazer ([14], Lemma 1.2″) for the case $\mu = 1$ and gives an oscillation criterion for the case $\mu \neq 1$.

The next result is a stability result for (1.1) and implies that all nonoscillatory solutions tend monotonically to zero along with their derivatives.

THEOREM 3.4. – Let $q \leqslant 0$, $r > 0$ and let y be a nontrivial solution of (1.1) with $y \geqslant 0$ for large t. Assume further that equation (3.1) is not disconjugate for each $\lambda > 0$ (for $\lambda = 1$ if $\mu = 1$) and that either

(i) $\liminf_{t\to+\infty} \left(tp(t) - t^2 q(t)\right) > 2$, or

(ii) $\liminf_{t\to+\infty} t^3 r(t) > 0$.

Then $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0$.

PROOF. – We need only show that $\lim_{t\to\infty} y(t) = 0$. If not, suppose $y(t) \geqslant k > 0$ for all $t > t_0$, Let $\varepsilon > 0$ and choose $T > t_0$ so that $y(T) < (1+\varepsilon)k$. Since $y(t) - y(T) < y'(t)(t-T)$, $t > T$, from the Mean Value Theorem, we obtain from the above relations

$$y'(t)/y(t) > (t-T)^{-1}\left(1 - y(T)/y(t)\right) > -\varepsilon(t-T)^{-1} \tag{3.3}$$

for $t > T$. Now with $u = y'/y$ we see that u satisfies

$$u'' + 3uu' + pu' + u^3 + pu^2 + qu + \lambda_0 r \leqslant 0\,, \qquad t > T \tag{3.4}$$

where $\lambda_0 = 1$, if $\mu = 1$, $\lambda_0 = k^{\mu-1}$ if $\mu > 1$, and $\lambda_0 = (k(1+\varepsilon))^{\mu-1}$ if $\mu < 1$.

With $\alpha(t) = -\varepsilon(t-T)^{-1}$, a calculation shows that

$$\begin{aligned} &\alpha'' + 3\alpha\alpha' + p\alpha' + \alpha^3 + p\alpha^2 + q\alpha + \lambda_0 r = \\ &\quad -(t-T)^{-3}(2\varepsilon + 3\varepsilon^2 + \varepsilon^3) + (t-T)^{-2}p(t)(\varepsilon + \varepsilon^2) - (t-T)^{-1}q(t)\varepsilon + \lambda_0 r(t)\,. \end{aligned} \tag{3.5}$$

Using either (i) or (ii) of the hypothesis, one can show that the right hand side of (3.5) is nonegative for $\varepsilon > 0$ sufficiently small. Hence, by Lemma 3.1 equation (3.1) is disconjugate for $\lambda = \lambda_0$. This contradiction shows that $y(t) \to 0$ as $t \to +\infty$ and proves the Theorem.

REMARK. – It is shown in [14] (Theorem 1.5) that if $p \equiv 0$, $q \leqslant 0$, and $\int^{\infty} t^2 r(t)\,dt = +\infty$, and if (1.1) with $\mu = 1$ has an oscillatory solution, then any nonoscillatory solution tends to zero. The above result with $\mu = 1$ shows the conclusion of this theorem also holds when $\int^{\infty} t^2 r(t)\,dt < +\infty$, provided part (i) of the hypothesis of Theorem 3.4 holds.

THEOREM 3.5. – Let $\mu > 1$, $r > 0$, $q \leqslant 0$, $p \geqslant 0$, $p' - q \geqslant 0$, and $q' - p'' \geqslant 0$ for $t_0 < < t < +\infty$. Assume further that $\int^{\infty} r\,dt = +\infty$. Let y be a nontrivial nonoscillatory solution of (1.1). Then y satisfies for all large t, $\operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t)$, and $y \to 0$, $y' \to 0$, $y'' \to 0$, as $t \to +\infty$.

PROOF. – Assume that $y > 0$, $t > t_0$. Multiplying (1.1) by $y^{-\mu}$ yields

$$y'''y^{-\mu} + py''y^{-\mu} + qy'y^{-\mu} + r = 0\,. \tag{3.6}$$

For $t_0 < t_1 < t$, we integrate the first and second terms by parts twice and the third term once, from t_1 to t, and get after rearranging

$$\begin{aligned} &y''(t)\,y^{-\mu}(t) + \mu(y'(t))^2 y^{-\mu-1}(t)/2 + p(t)\,y'(t)\,y^{-\mu}(t) \\ &+ (p'(t) - q(t))\,y^{-\mu+1}(t)/(\mu - 1) + (\mu(\mu + 1))/2\int_{t_1}^{t} y'^3 y^{-\mu-2}\,ds \\ &+ (\mu - 1)^{-1}\int_{t_1}^{t} (q' - p'')\,y^{-\mu+1}\,ds + \mu\int_{t_1}^{t} y'^2 p y^{-\mu-1}\,ds + \int_{t_1}^{t} r\,ds = K\,. \end{aligned} \tag{3.7}$$

From (3.7) it is clear that if $y' \geqslant 0$ for all large t, then since $\int_{t_1}^{t} r\,ds \to +\infty$, we must have $y''(t)\,y^{-\mu}(t) \to -\infty$, and therefore, $y''(t) \to -\infty$, a contradiction. Therefore by Lemma 3.2 it follows that $y > 0$, $y' < 0$, $y'' > 0$, $t > t_0$, and $y' \to 0$, $y'' \to 0$, as $t \to +\infty$. If $y \to L > 0$, then from (3.6) we would have $y'''(t) + L^{\mu} r(t) < 0$, $t < t_0$, and integrating this inequality shows that $y''(t) \to -\infty$, again a contradiction. Therefore, $L = 0$ and the Theorem is proved.

From the proof of the previous theorem, we may obtain

COROLLARY 3.6. – Let $0 < \mu < 1$, $r > 0$, $q \leqslant 0$, $p \geqslant 0$, $p' - q \leqslant 0$, and $q' - p'' \leqslant 0$. Assume further that

$$\int^{t} r\,dt \to +\infty\,.$$

Then for any nonoscillatory solution y of (1.1), the conclusion of Theorem 3.5 holds.

PROOF. – The proof follows from (3.7), as in Theorem 3.5.

EXAMPLE 3.7. – As remarked above, Theorem 3.4 applies to the case when

$$\int^{\infty} r(t)\,dt < +\infty\,,$$

in contrast to Theorem 3.5, Corollary 3.6, and the results in the references. As a simple illustration, for the case $p \equiv 0$, let $q(t) = -mt^{\gamma}$, and

$$k_1 t^{\delta_1} \leqslant r(t) \leqslant k_2 t^{\delta_2}$$

where $m, k_1, k_2 > 0$, $-3 < \delta_1 \leqslant \delta_2 < -1$, $\gamma < \delta_1$.

Then for any $\lambda > 0$, equation (3.1) has an oscillatory solution ([5]) and part (ii) of the hypothesis of Theorem 3.4 clearly holds. Hence, the conclusion of Theorem 3.4 holds.

Corollary 3.6 applies, for example, to any $r(t) > 0$ with

$$\int^{\infty} r\,dt = +\infty$$

and the choices $p(t) = k_1 t^{\delta}$, $q(t) = -k_2 t^{\gamma}$, where $k_1, k_2 > 0$, $\gamma + 1 \leqslant \delta < 0$. (If $\gamma + 1 = \delta$, then we also need $k_1\delta + k_2 \leqslant 0$). Such sublinear examples are not considered in the references.

We also wish to remark that Theorem 3.5 includes as special cases results of [16] and [15].

4. – In this section we shall consider the case $q \geqslant 0$, $r > 0$. We begin with a Lemma which is a generalization of a result of Heidel ([9], Theorem 3.6).

LEMMA 4.1. – *Let $q \geqslant 0$, $r > 0$ and assume the equation $z'' + (q - p'/2)z = 0$ is disconjugate on $(t_0, +\infty)$. Then for any nonoscillatory solution y of (1.1) there exists $T \geqslant t_0$ so that $y(t)y'(t) \geqslant 0$ or $y(t)y'(t) < 0$, for $t \geqslant T$.*

PROOF. – To be specific, assume $y > 0$, $t \geqslant T \geqslant t_0$. Suppose $y'(t_1) = y'(t_2) = 0$, $y' \neq 0$ on (t_1, t_2), for some $T \leqslant t_1 < t_2$: (Zeros of y' are isolated). Then as in Theorem 2.2 we multiply (1.1) by y' and integrate by parts to get relation (2.6), ($t^* = t_2$). Since the right hand side of (2.6) is positive, it follows that $y' > 0$ on (t_1, t_2). Hence, $y' \geqslant 0$ for all $t \geqslant T$, and this proves the Lemma.

LEMMA 4.2. – *Let $q \geqslant 0$, $r > 0$ and assume the equation $z'' + (q - p'/2)z = 0$ is disconjugate on $(t_0, +\infty)$. In addition, assume either*

(*a*) $\int_{t_0}^{\infty} P^{-1}\,dt = \int_{t_0}^{\infty} qP\,dt + \int_{t_0}^{\infty} t^{\mu} Pr\,dt = +\infty$, or

(*b*) *the equation $z'' + pz' + qz = 0$ is oscillatory, where in* (*a*) $P(t) = \exp\left(\int_{t_0}^{t} p\,ds\right)$.

Then any nonoscillatory solution y of (1.1) satisfies

$$y(t)y'(t) < 0 \quad \text{for all large } t.$$

Furthermore, if $p \leqslant 0$, $q - p' \leqslant 0$, $p' \geqslant 0$, $t \geqslant T$, and if $y > 0$, $y' < 0$ for all large t, then $y'' > 0$ eventually.

PROOF. – In view of Lemma 4.1, assume $y > 0$, $y' \geqslant 0$, $t \geqslant T$. If (b) holds, then $u = y'/y$ satisfies $u' + u^2 + py + q = -ry^\mu < 0$, $t \geqslant T$, so that $z'' + pz' + qz = 0$ is disconjugate ([8], p. 362), a contradiction. If (a) holds, then

$$(Py'')' = -qPy' - rPy^\mu < 0\,, \qquad t \geqslant T,$$

Hence, if $y''(t_1) < 0$ for some $t_1 > T$, then $P(t)y''(t) < P(t_1)y''(t_1) < 0$, $t > t_1$, and since $\int^\infty P^{-1}dt = +\infty$, this implies that $y'(t) \to -\infty$, a contradiction. Therefore, $y'' \geqslant 0$, $t \geqslant T$, so that $y'(t) \geqslant k > 0$, and $y(t) \geqslant kt$ for some $k > 0$ and all $t \geqslant t_2 \geqslant T$. But then from (1.1) we obtain

$$P(t)y''(t) \leqslant P(t_2)y''(t_2) - k\int_{t_2}^{t} Pq\,ds - k\int_{t_2}^{t} s^\mu Pr\,ds \tag{4.1}$$

and since the right hand side of (4.1) tends to $-\infty$, we have a contradiction. Hence, $y(t)y'(t) < 0$ for all large t.

Finally, if $p \leqslant 0$, $q - p' \leqslant 0$, $p' \geqslant 0$, $t \geqslant T$ and $y > 0$, $y' < 0$, for all large t, then $y'' > 0$ eventually, by Theorem 2.2. This completes the proof.

REMARK 4.3. – If $\int^\infty P^{-1}dt < +\infty$, so that (a) does not hold, then (b) holds, if for example,

$$\int^\infty qP\Big(\int_t^\infty P^{-1}ds\Big)^2 dt = +\infty\,,$$

(see [1], p. 421).

The next result gives several different criteria under which $y \to 0$, if y is a nonoscillatory solution of (1.1) with $yy' < 0$ for all large t.

THEOREM 4.3. – Let $q \geqslant 0$, $r > 0$, and let y be a nonoscillatory solution of (1.1) with $yy' < 0$, $t \geqslant T$. Then $y \to 0$ provided any one of the following three conditions holds:

(i) $pq + q' \leqslant 0$, $p \leqslant 0$, and $\int^\infty Pq\,dt = +\infty$,

(ii) $\int^\infty t^2 r\,dt = +\infty$, $tp(t) \leqslant 2$, and $t^2q(t) - (t^2p(t))' \leqslant M$ for some $M > 0$,

(iii) $p \leqslant 0$, $q - p' \leqslant 0$, $p' \geqslant 0$, there exists $k > 0$ such that $\liminf_{t\to\infty} t^3 r\big((t) - - k(t^{-1}q(t) - t^{-2}p(t))\big) > 0$ and for any $\lambda > 0$ ($\lambda = 1$, if $\mu = 1$) the equation $y''' + py'' + qy' + \lambda ry = 0$ is not disconjugate on any half-line $(t_1, +\infty)$.

PROOF. – We assume to be specific that $y > 0$, $y' < 0$, $t \geqslant T$. If (i) holds, then $(Pq)' \leqslant 0$, and since $p \leqslant 0$, $\int^\infty P^{-1}dt = +\infty$. Now from (1.1) we obtain through integration by parts, for $t > t_1 \geqslant T$

$$(4.2)\qquad P(t)(y''(t)+q(t)y(t))+\int_{t_1}^{t}(Pry^{\mu}-(Pq)'y)\,dt = P(t_1)(y''(t_1)+q(t_1)y(t_1))\,.$$

If there exists $t_1 > T$ with $y''(t_1)+q(t_1)y(t_1) \equiv -k < 0$, then from (4.2) we get $y''(t) < -q(t)y(t) - kP^{-1}(t)$, $t > t_1$, and therefore an integration of this inequality leads to $y'(t) \to -\infty$, a contradiction. Therefore, for some $k_1 > 0$, we must have

$$k_1 \geqslant P(t)(y''(t)+q(t)y(t)) \geqslant 0\,, \qquad t \geqslant T$$

and hence with $u = y'/y$ we have

$$u' + u^2 = y''/y \geqslant -q\,, \qquad t \geqslant T.$$

But then

$$(4.3)\qquad y(Pu' + Pu^2) \leqslant k_1 - qP\,, \qquad t \geqslant T$$

so that if $y \nrightarrow 0$, then an integration of (4.3) shows

$$(4.4)\qquad P(t)\,u(t) - \int_T^t pPu\,ds + \int_T^t Pu^2\,ds \to -\infty\,.$$

Since $p \leqslant 0$, this implies

$$(4.5)\qquad u(t) = y'(t)/y(t) < -P^{-1}(t)\,, \qquad t \geqslant t_2$$

and therefore $\log y(t) \to -\infty$, contradicting the assumption that $y(t) \nrightarrow 0$.

Suppose next that (ii) holds. Multiplying (1.1) by t^2 and integrating by parts from T to t we obtain

$$(4.6)\qquad t^2y''(t) - 2ty'(t) + t^2p(t)y'(t) + \int_T^t (s^2q(s) - (s^2p(s))' + 2)\,y'(s)\,ds + \int_T^t s^2r(s)\,y^{\mu}(s)\,ds = K$$

from which we get, using the condition in (ii),

$$(4.7)\qquad t^2y''(t) + (M+2)\,y(t) + \int_T^t s^2r(s)\,y^{\mu}(s)\,ds \leqslant K_1\,.$$

Thus, if $y(t) \nrightarrow 0$, then (4.7) implies that $t^2y''(t) \to -\infty$, which is a contradiction since $y'(t) < 0$, $t \geqslant T$.

Finally, suppose that (iii) holds. We may assume by Lemma 4.2 that $y > 0$, $y' < 0$, $y'' > 0$, $t \geqslant T$, and then the argument proceeds as in Theorem 3.4. Given

$\lambda > 0$, condition (iii) guarantees that $\alpha(t) = -\varepsilon(t-T)^{-1}$ is a lower solution of the Riccati equation corresponding to $y''' + py'' + qy' + \lambda ry = 0$ for sufficiently small $\varepsilon > 0$. Therefore, as in Theorem 3.4, we must have $y(t) \to 0$.

COROLLARY 4.4. – Let the hypotheses of Lemma 4.2 and Theorem 4.3 hold. Then for any nonoscillatory solution y of (1.1) $\lim y(t) = 0$.

RERMARK 4.5. – Part (ii) of Theorem 4.3 is a generalization of a result of HEIDEL [9]. It is easy to see that (i) and (ii) of Theorem 4.3 are independent. To see that (iii) may hold with neither (i) nor (ii) holding, let $q(t) \equiv 0$, $p(t) = -t^{-\delta}$, where $1 < 2\delta < 2$, and $r(t) = t^{-1}$. Clearly, neither (i) nor (ii) holds. To see that (iii) holds it is sufficient to show that

$$y''' - t^{-\delta}y'' + \lambda t^{-1}y = 0 \tag{4.8}$$

has an oscillatory solution for all $\lambda > 0$. With the change of variables

$$y = \exp\left(t^{1-\delta}/3(1-\delta)\right),$$

we obtain from (4.8)

$$w''' + B(t)\,w' + C(t)\,w = 0 \tag{4.9}$$

where $B(t) = -t^{-2\delta}(1 + 3\delta t^{\delta-1})/3$, and $C(t) = \lambda t^{-1} - 2t^{-3\delta}/27 + (\delta/3)(\delta+1)t^{-\delta-2}$. Since $C(t) > 0$, $B(t) < 0$, for large t and since

$$\int^{\infty}\left(C(t) - 2(-B(t))^{\frac{3}{2}}/3^{\frac{3}{2}}\right)dt = +\infty$$

for any $\lambda > 0$, it follows by a theorem of Lazer ([14], Theorem 1.3) that (4.9) has an oscillatory solution. Using results of [5] one can also give examples where $\int^{\infty} r(t)\,dt < +\infty$.

The next lemmas give criteria under which nonoscillatory solutions with certain initial conditions have asymptotic behavior much different from that in Lemma 2.1 and Theorem 2.2.

LEMMA 4.6. – Let $p \leqslant 0$, $q \geqslant 0$, $q - p' \geqslant 0$, $q' - p'' \leqslant 0$, and $\int^{\infty} p(t)\,dt > -\infty$. Assume further that for any $\lambda > 0$ there exists $T_\lambda > t_0$ such that

$$|\lambda p(t)| \leqslant q(t) + t^{\mu} r(t), \qquad t \geqslant T_\lambda.$$

Let y be a nontrivial nonoscillatory solution of (1.1) with $G(y(t_1)) \leqslant 0$, for some $t_1 > t_0$, where

$$G(y(t)) = 2y(t)y''(t) + 2y(t)y'(t)p(t) + \left(q(t) - p'(t)\right)y^2(t) - \left(y'(t)\right)^2.$$

Then there exists $T \geqslant t_0$ such that for $t \geqslant T$

$$\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'''(t) .$$

PROOF. – Assume $y \geqslant 0$, $y \not\equiv 0$, $t \geqslant t_1$. Then a calculation shows that

$$G'(y(t)) = 2(y')^2 p + (q' - p'') y^2 - 2ry^{1+\mu} \leqslant 0 , \qquad t \geqslant t_1$$

and $G'(y(t)) < 0$ if $y(t) \neq 0$. Therefore, if $G(y(t_1)) \leqslant 0$, then $G(y(t)) \leqslant 0$ for all $t \geqslant t_1$. If $y(t_2) = 0$ for some $t_2 > t_1$ then $G(y(t_2)) = 0$ so that $y \equiv 0$ for $t_1 \leqslant t \leqslant t_2$. Thus, we may assume that $y > 0$ and $G(y(t)) < 0$ for all $t > t_1$: If $y'(t_3) = 0$ for some $t_3 > t_1$, then y' cannot vanish again on $(t_3, +\infty)$. Now if $y' < 0$ for $t > t_3$, then clearly $y'' \leqslant 0$ cannot hold for all large t. If $y'' \geqslant 0$ for all large t, then

$$-y'(t)^2 \leqslant G(y(t)) \leqslant G(y(t_3)) < 0 , \qquad t > t_3$$

so that $y'(t) \to -k < 0$, a contradiction. Finally, if y'' changes sign for arbitrarily large t, then since $\limsup_{t \to +\infty} y'(t) = 0$, we may find a sequence $t_n \to +\infty$, with

$$\lim_{n \to \infty} y'(t_n) = 0 \qquad \text{and} \qquad y''(t_n) = 0 .$$

But then $-y'(t_n)^2 \leqslant G(y(t_n)) \leqslant G(y(t_3)) < 0$, for all n and this is a contradiction. Therefore, we may conclude that $y > 0$, $y' > 0$, $t > t_1$.

We show next that $y'' > 0$ for all $t > t_1$. If $y''(t^*) = 0$ for some $t^* > t_1$, then $y'''(t^*) = -q(t^*) y'(t^*) - r(t^*) y^\mu(t^*) < 0$, and therefore y'' cannot vanish again for $t > t^*$. Thus, $y'' < 0$ for $t > t^*$ and since $y''' = -py'' - qy' - ry^\mu < 0$, for $t > t^*$, it follows that $y' \to -\infty$, a contradiction. Therefore, $y'' > 0$, $t > t_1$. To see that $y'''(t) < 0$ for all large t, we need first the fact that y'' is bounded above. Multiply (1.1) by $(y'')^{-1}$ and integrate for $t \geqslant T > t_1$ to get

$$y''(t) \leqslant y''(T) \exp\Big(-\int_T^\infty p \, ds\Big) < +\infty , \qquad t \geqslant T.$$

Letting λ denote the right hand side of the above inequality and choosing $k > 0$ so that $y'(t) \geqslant k$, and $y(t) \geqslant kt$ for $t \geqslant T$, we obtain from (1.1)

$$y'''(t) + \lambda p(t) + kq(t) + k^\mu r(t) t^\mu \leqslant 0 , \qquad t \geqslant T. \tag{4.10}$$

But by the conditions of the Lemma, the last three terms of (4.10) are eventually positive which shows that $y'''(t) < 0$ eventually. This completes the proof of the Lemma.

LEMMA 4.7. – Let $p \geqslant 0$, $r > 0$, $pq + q' \leqslant 0$, $q \geqslant 0$. Let y be a nontrivial nonoscillatory solution of (1.1) with $H(y(t_1)) \leqslant 0$ for some $t_1 > t_0$ where

$$H(y(t)) = P(t)\big(2y''(t)y(t) - y'(t)^2 + qy(t)^2\big)\ ,$$

$P(t) = \exp\Big(\int\limits_{t_1}^{t} p(s)\,ds\Big)$. Then there exists $T \geqslant t_1$ so that $\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t) \neq \neq \operatorname{sgn} y'''(t)$, $t \geqslant T$.

PROOF. – A calculation shows that

$$H'(y(t)) = P(t)\big(y^2(pq + q') - 2ry^{1+\mu} - py'^2\big)\ . \tag{4.11}$$

Therefore, if $H(y(t_1)) \leqslant 0$, then $H(y(t)) \leqslant 0$ for $t \geqslant t_1$: The argument now is similar to, but easier, than in Lemma 4.6. If $y > 0$ for $t > t_1$, then $H'(y(t)) < 0$, $t > t_1$, and as in Lemma 4.6, it follows that $y' > 0$ and $y'' > 0$ for $t > t_1$. Hence, $y'''(t) = -p(t)y''(t) - q(t)y'(t) - r(t)y^\mu(t) < 0$, $t > t_1$, and this proves the Lemma.

Our next result includes an improvement of an oscillation criterion for (1.1) due to Heidel ([9], Corollary 3.4). We offer a different proof which is shorter and appeals to results in the theory of second order nonlinear oscillations.

THEOREM 4.8. – Let the hypotheses of Lemma 4.6 hold and, in addition, assume that the second order nonlinear equation

$$(P(t)u')' + f(t, u, \lambda) = 0 \tag{4.12}$$

is oscillatory for some $0 < \lambda < \frac{1}{2}$, (that is, all solutions of (4.12) are oscillatory), where $P(t)$ is as in Lemma 4.7 and

$$f(t, u, \lambda) = P(t)\big(q(t)u + \lambda^\mu t^\mu r(t)u^\mu\big)\ .$$

Then any nontrivial solution of (1.1) with $G(y(t_1)) \leqslant 0$ for some $t_1 \geqslant t_0$ is oscillatory.

PROOF. – If the theorem is not true, let y be a nonoscillatory solution with $G(y(t_1)) \leqslant 0$. In view of Lemma 4.6 we may then assume that $y > 0$, $y' > 0$, $y'' > 0$, and $y''' < 0$ for $t \geqslant T$. By a result of Lazer ([14], Lemma 3.2)

$$\liminf_{t \to +\infty} \frac{y(t)}{ty'(t)} \geqslant \frac{1}{2}$$

and hence we may also assume that $y(t)/ty'(t) \geqslant \lambda$, for $t \geqslant T$. Therefore, from (1.1) we obtain with $u = y'$

$$u'' + pu' + qu + \lambda^\mu t^\mu r(t)u^\mu \leqslant 0\ . \tag{4.13}$$

That is, $u = y'$ is an upper solution of the equation

$$z'' + pz' + qz + \lambda^\mu t^\mu r(t) z^\mu = 0 , \tag{4.14}$$

which is (4.12). Since any $k > 0$ is a lower solution of (4.13) (that is, satisfies the reverse differential inequality), it follows from well-known theorems in the theory of second order differential inequalities (see [10], [11]) that there exists a solution $v(t)$ of (4.14) with $0 < v(t) \leqslant u(t)$ on $(T, +\infty)$. Thus, (4.12) has a nontrivial non-oscillatory solution and this contradiction proves the Theorem.

THEOREM 4.9. – Let the hypotheses of Lemma 4.7 hold and assume further that (4.12) is oscillatory for some $0 < \lambda < \frac{1}{2}$. Then any nontrivial solution y of (1.1) with $H(y(t_1)) \leqslant 0$ is oscillatory, $t_1 \geqslant t_0$.

PROOF. – Follows as in Theorem 4.8.

REMARK 4.10. – Under the hypotheses of Lemma 4.6 or Lemma 4.7 it is clear that any criterion which guarantees that

$$u'' + pu' + qu = 0 , \tag{4.15}$$

or

$$u'' + pu' + \lambda^\mu t^\mu r(t) u^\mu = 0 , \quad \text{some } 0 < \lambda < \tfrac{1}{2} \tag{4.16}$$

is oscillatory also guarantees that (4.12) is oscillatory. As corollaries, therefore, we have

COROLLARY 4.11. – Let the hypotheses of Lemma 4.6 hold and in addition assume that

$$\text{(i)} \quad \int^\infty \big(t^\delta q(t) + t^{2\mu} r(t)\big)\, dt = +\infty , \quad 0 < \mu < 1, \quad \text{some } 0 < \delta < 1$$

$$\text{(ii)} \quad \int^\infty t^\delta \big(q(t) + t r(t)\big)\, dt = +\infty , \quad \mu = 1 , \quad \text{some } 0 < \delta < 1$$

$$\text{(iii)} \quad \int^\infty \big(t^\delta q(t) + t^{1+\mu} r(t)\big)\, dt = +\infty , \quad \mu > 1 , \quad \text{some } 0 < \delta < 1$$

Then any solution of (1.1) with $G(y(t_1)) \leqslant 0$ is oscillatory.

PROOF. – Since $P(t) = \exp\Big(\int_{t_1}^{t} p(s)\Big)\, ds$ is decreasing and bounded away from zero, a straightforward extension of a well-known oscillation criterion for $z'' + qz = 0$ (see [8], p. 368) shows that

$$\int^\infty t^\delta q(t)\, dt = +\infty$$

inplies that (4.15) is oscillatory: (i.e., $(Py')' + Pqy = 0$ is oscillatory). Likewise, (4.16) is oscillatory if:

$$(a)\qquad 0<\mu<1 \quad \text{and} \quad \int^{\infty} t^{2\mu} r(t)\,dt = +\infty\,,$$

$$(b)\qquad \mu=1 \quad \text{and} \quad \int^{\infty} t^{1+\delta} r(t)\,dt = +\infty\,, \quad \text{some } 0<\delta<1$$

$$(c)\qquad \mu>1 \quad \text{and} \quad \int^{\infty} t^{1+\mu} r(t)\,dt = +\infty\,,$$

(see [20] for additional details). This completes the proof.

Since either $\int^{\infty} P^{-1}(t)\,dt = +\infty$ or $\int^{\infty} P^{-1}(t)\,dt < +\infty$, we have as further corollaries for the case when the hypotheses of Lemma 4.7 hold:

COROLLARY 4.12. – Let the hypotheses of Lemma 4.7 hold with

$$\int^{\infty} P^{-1}(t)\,dt = +\infty$$

and assume one of the following conditions hold:

$$(a)\ \limsup_{t\to+\infty} \left(\int_{t_0}^{t} P^{-1}(s)\,ds\right)\left(\int_{t}^{\infty} P(s)q(s)\,ds\right) > 1;$$

$$(b)\ \int_{t_0}^{\infty} P(t)q(t)\,dt = +\infty;$$

$$(c)\ \int_{t_0}^{\infty}\left(\int_{t_0}^{t} P^{-1}(s)\,ds\right)P(t)\,t^{\mu} r(t)\,dt = +\infty,\ \mu>1;$$

$$(d)\ \limsup_{t\to\infty}\left(\int_{t_0}^{t} P^{-1}(s)\,ds\right)\int_{t}^{\infty} P(s)\big(q(s)+\lambda s r(s)\big)\,ds > 1,\ \mu=1,\ \text{some } 0<\lambda<\tfrac{1}{2};$$

$$(e)\ \int_{t_0}^{\infty}\left(\int_{t_0}^{t} P^{-1}(s)\,ds\right)^{\delta} P(t)\big(q(t)+tr(t)\big)\,dt = +\infty,\ \mu=1,\ \text{some } 0<\delta<1;$$

$$(f)\ \int_{t_0}^{\infty}\left(\int_{t_0}^{t} P^{-1}(s)\,ds\right)^{\mu} P(t)\,t^{\mu} r(t)\,dt = +\infty,\ 0<\mu<1.$$

Then any nontrivial solution of (1.1) with $H(y(t_0)) \leqslant 0$ is oscillatory.

PROOF. – Conditions (a) and (b) are sufficient for oscilllation of all solutions of (4.15) based on the Liouville transformation

$$s = \int_{t_0}^{t} P^{-1}(w)\,dw\,,$$

(see [1]). Likewise, conditions (d) and (e) imply (4.12) is oscillatory in the case $\mu = 1$. Finally, conditions (c) and (f) are sufficient for oscillation of all solutions of (4.16), based again on the Liouville transformation (see [20] for additional details).

Therefore, the Corollary follows by Remark 4.10.

COROLLARY 4.13. – Let the hypotheses of Lemma 4.7 hold with

$$\int_{t_0}^{\infty} P^{-1}(s)\,ds < +\infty\,,$$

and assume one of the following conditions hold:

$$(a)\qquad \int_{t_0}^{\infty} P(t)\,q(t)\Big(\int_{t}^{\infty} P^{-1}(s)\,ds\Big)^2 dt = +\infty\,,$$

$$(b)\qquad \int_{t_0}^{\infty} P(t)\,t^{\mu} r(t)\Big(\int_{t}^{\infty} P^{-1}(s)\,ds\Big)^{\mu} = +\infty\,,\qquad \mu > 1,$$

$$(c)\qquad \int_{t_0}^{\infty} P(t)\big(q(t) + \lambda t r(t)\big)\Big(\int_{t}^{\infty} P^{-1}(s)\,ds\Big)^2 dt = +\infty\,,$$

$$\mu = 1\,,\qquad 0 < \lambda < \tfrac{1}{2}$$

$$(d)\qquad \int_{t_0}^{\infty} P(t)\,t^{\mu} r(t)\Big(\int_{t}^{\infty} P^{-1}(s)\,ds\Big)\,dt = +\infty\,,\qquad 0 < \mu < 1.$$

Then any nontrivial solution of (1.1) with $H(y(t_0)) \leqslant 0$ is oscillatory.

PROOF. – Condition (a) is sufficient for oscillation of (4.15) (see [1], Corollary 1.4.1, p. 421), based on the change of independent variable

$$s = \Big(\int_{t}^{\infty} P^{-1}(w)\,dw\Big)^{-1},$$

and likewise condition (c) is sufficient for oscillation of (4.12) if $\mu = 1$. If we use the change of dependent and independent variables

$$s = \Big(\int_{t}^{\infty} P^{-1}(w)\,dw\Big)^{-1},\qquad v(s) = s y(t)\,,$$

then conditions (b) and (d) are straightforward extensions of well-known necessary and sufficient conditions for oscillation for second order superlinear and sub-linear equations, (again see [20], p. 227-28 for details). The proof is therefore complete.

REMARK 4.14. – Theorems 4.8, 4.9, and Corollaries 4.11, 4.12 may all be considered extensions and improvements (even in the case $p \equiv 0$) of oscillation criteria in [9], [3], [19], [16], and for the case $\mu = 1$ in [14], with however, different proofs. Corollary 4.13 gives new oscillation criteria in all cases.

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