Differentiability and Bifurcation Points for a Class of Monotone Nonlinear Operators (*).

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Sunto. – *Sia P l'operatore definito nello spazio $L^2(\Omega)$ ponendo $u = Pf$ in* (11″). *Si dimostra che questo operatore (differenziabile secondo Fréchet nell'origine (vedi teorema* 1) *e si utilizza questo risultato per caratterizzare i punti di biforcazione per il problema non lineare* (25) (*vedi teorema* 2).

Let Ω be a smooth open bounded set in the n-dimensional euclidean space $\boldsymbol{R}^n$ and let Γ be the boundary of Ω.

We shall assume that the $L^p(\Omega)$ spaces and the Sobolev spaces $W^{k,p}(\Omega)$, k positive integer, are familiar to the reader. We denote by $\| \,\|_p$ and $\| \,\|_{k,p}$ the usual norms in these spaces and we put $H = L^2(\Omega)$, $\| \,\| = \| \,\|_2$. If $1 \leqslant p < N$ we denote by p^* the Sobolev's embedding exponent $p^* = pn/(n-p)$. We shall also consider the spaces $L^p(\Gamma)$ and the Sobolev space $W^{\frac{1}{2},2}(\Gamma)$ with the usual norms $| \,|_p$ and $| \,|_{\frac{1}{2},2}$ respectively.

We shall assume that the reader is familiarized with the basic results on maximal monotone operators on Hilbert spaces.

Consider now a maximal monotone (m.m.) graph β on $\boldsymbol{R}\times\boldsymbol{R}$ such that $0 \in \beta(0)$ and define an operator $B\colon H \to 2^H$ as follows:

$$(1) \qquad \begin{cases} B = -\Delta \quad \text{with} \\ D(B) = \left\{u \in W^{2,2}(\Omega)\colon -\dfrac{\partial u}{\partial n} \in \beta(u) \ \text{ a.e. on } \Gamma\right\}, \end{cases}$$

where $D(B)$ denotes the set $\{u \in H\colon B(u) \neq \emptyset\}$.

On the other hand consider the convex, lower semicontinuous (l.s.c.) functional $\Phi\colon H \to]-\infty, +\infty]$ defined by

$$(2) \qquad \Phi(u) = \begin{cases} \frac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Gamma j(u)\, d\Gamma & \text{if } u \in W^{1,2}(\Omega) \text{ and } j(u) \in L^1(\Gamma), \\ +\infty & \text{otherwise}, \end{cases}$$

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where $j\colon \boldsymbol{R} \to]-\infty, +\infty]$ is convex and l.s.c., $\beta = \partial j$ and $j(0) = 0$. It is well known (cf. [5], theorem 12) that B is the subdifferential of Φ, i.e.,

$$B = \partial \Phi , \tag{3}$$

and consequently B is m.m. on H. Furthermore (cf. [4], theorem I.10)

$$\|u\|_{2,2} \leqslant c\| -\Delta u + u\| , \qquad \forall u \in D(B) . \tag{4}$$

On the other hand consider two measurable functions $a(x)$ and $b(x)$ defined on Ω with range in $[-\infty, +\infty]$ and verifying $a(x) < 0 < b(x)$ a.e. on Ω. Let $g(x, y)$ be a real function defined on

$$\Lambda = \{(x, y) \in \Omega \times \boldsymbol{R}\colon a(x) < y < b(x)\}$$

and suppose that for each $y \in \boldsymbol{R}$ the function $x \to g(x, y)$ is measurable in his domain $\{x \in \Omega\colon a(x) < y < b(x)\}$. Moreover we assume that $g(x, y)$ verifies the following conditions:

$$g(x, 0) = 0 , \qquad \text{a.e. in } \Omega , \tag{5}$$

(6) for almost all $x \in \Omega$ the function $y \to g(x, y)$, defined on $]a(x), b(x)[$, is continuous and nondecreasing. If $a(x) > -\infty$ then $\lim\limits_{y\to a(x)} g(x, y) = -\infty$; if $b(x) < +\infty$ then $\lim\limits_{y\to b(x)} g(x, y) = +\infty$.

Putting $g(x, y) = \emptyset$ if $y \notin]a(x), b(x)[$, the hypothesis (6) becomes equivalent to the maximal monotony of the graph $y \to g(x, y)$ in $\boldsymbol{R} \times \boldsymbol{R}$.

Put now, for any $(x, y) \in \Omega \times \boldsymbol{R}$,

$$\psi(x,y) = \begin{cases} \displaystyle\int_0^y g(x,\eta)\, d\eta & \text{if } y \in [a(x), b(x)] , \\ +\infty & \text{otherwise}, \end{cases} \tag{7}$$

and define $\Psi\colon H \to [0, +\infty]$ by

$$\Psi(u) = \int_\Omega \psi(x, u(x))\, dx . \tag{8}$$

This functional is convex, l.s.c. and (see [3])

$$\partial \Psi = \bar{g} \tag{9}$$

where $\bar{g}$ is the operator

$$\bar{g}[u](x) = \begin{cases} g(x, u(x)) & \text{a.e. in } \Omega, \text{ if } g(x, u(x)) \in H , \\ \emptyset & \text{otherwise} . \end{cases} \tag{10}$$

Obviously $\bar{g}$ is m.m. (by (9)) and univalued on $D(\bar{g}) = \{u \in H: g(x, u(x)) \in H\}$. We suppose that the following condition holds:

(11) for any $f \in H$ the equation

$$Bu + \bar{g}[u] + u = f$$

has a solution u.

Since $B + \bar{g}$ is monotone on H the condition (11) is equivalent to

$$B + \bar{g} \quad \text{is m.m. in } H, \tag{11'}$$

since a monotone operator A is m.m. if and only if $R\,(I + A) = H$. Putting $u = Pf$ we have then

$$P = (I + B + \bar{g})^{-1} = \big(I + \partial(\Phi + \Psi)\big)^{-1}; \tag{12}$$

it is well known that such an operator P is a contraction on H.

Notice that (11) can be written more explicitly

(11″) for any $f \in H$ the problem

$$\begin{cases} -\Delta u + u + g(x, u(x)) = f & \text{a.e. in } \Omega, \\ -\dfrac{\partial u}{\partial n} \in \beta(u) & \text{a.e. in } \Gamma, \end{cases}$$

has a solution $u \in W^{2,2}(\Omega)$.

Finally we assume that

(13) The estimate $\|u\|_{2,2} \leq c\|f\|$ holds for any f in a neighbourhood of the origin of H.

Remark that from (11″) it follows trivially that

$$\|u - u_1\|_{1,2} \leq \|f - f_1\| \tag{14}$$

where $u = Pf$ and $u_1 = Pf_1$.

Sufficient conditions on $g(x, y)$ *in order that* (11) *holds were proved in* [3]. *In* [3] *we also give estimates on* $\|u\|_{2,2}$ *which can be used to prove* (13) (*see* [3] *theorem* 7.2 *and corollary* 7.3).

In this paper we seek conditions for Fréchet differentiability of the operator $P: H \to H$ at the origin. When g doesn't depend on x this problem was solved in [1] by assuming only that the function $g(y)$ is derivable for $y = 0$. The direct extension of this condition to the case when g depends also on x is to suppose that the functions $y \to g(x, y)$ are derivable at $y = 0$, uniformly in respect of almost all $x \in \Omega$. Under this last condition the results proved in [1] are easily extended to the case under consideration. The aim of this paper is to study the differentiability of P

at the origin when $y \to g(x, y)$ is not uniformly differentiable at $y = 0$ (see theorem 1). We apply then this result to the study of the bifurcation points for the problem (25) by using the Krasnosel'skii's theorem (see remark 4). The method used in this paper can also be adapted to the study of the differentiability at the infinity (see remark 5).

Suppose then that for almost all $x \in \Omega$ the function $y \to g(x, y)$ is differentiable at the point $y = 0$. Putting

$$\left[\frac{\partial g(x, y)}{\partial y}\right]_{y=0} = h(x) \tag{15}$$

we have

$$g(x, y) = h(x)y + \omega(x, y)y\,, \tag{16}$$

with $\lim\limits_{y\to 0} \omega(x, y) = 0$ for almost all $x \in \Omega$.

If $v(x)$ is a measurable function we put for commodity $\bar{\omega}[v](x) = \omega\big(x, v(x)\big)$. Remark that $v = v_1$ implies that $\bar{\omega}[v] = \bar{\omega}[v_1]$, with the usual convention $v = v_1$ if $v(x) = v_1(x)$ a.e. in Ω.

Put

$$\begin{cases} s = n/3 & \text{if } n > 4\,, \\ s > 2n/(n+2) & \text{if } n = 4\,, \\ s = 2n/(n+2) & \text{if } n = 3\,, \\ s > 1 & \text{if } n = 2\,, \\ s = 1 & \text{if } n = 1\,. \end{cases} \tag{17}$$

Remark that $2n/(n+2) = (2^*)'$. On writting $s > (2^*)'$ or $s > 1$ we suppose, without loss of generality, that s is close to the indicated values. We assume that the remainder $\omega(x, y)$ verifies the following condition, which is weaker than uniform differentiability of $y \to g(x,y)$ at $y = 0$:

(18) for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$v \in D(\bar{g}) \quad \text{and} \quad \|v\|_\infty \leqslant \delta_\varepsilon \Rightarrow \|\bar{\omega}(v)\|_s < \varepsilon\,.$$

We can use « $v(x)$ measurable and $v(x) \in \,]a(x), b(x)[$ a.e. in Ω » instead of « $v \in D(\bar{g})$ ».

REMARK 1. – Condition (18) says that there exists in $L^\infty(\Omega)$ a neighbourhood U of the origin such that $\bar{\omega}\colon D(\bar{g}) \cap U \to L^s(\Omega)$ is continuous at the origin. If we put $s = +\infty$ this condition is related to the uniform differentiability at the origin of $y \to g(x, y)$ in respect of almost all $x \in \Omega$.

It is obvious that $h(x) \geqslant 0$ a.e. in Ω. In the following we suppose that

(19) $\quad h(x) \in L^m(\Omega) \quad$ with $m > n/2$ if $n \geqslant 4$ and $m = 2$ if $n < 4$.

REMARK 2. – We have assumed (for clearty) that $\lim_{y\to 0}\omega(x,y)=0$ a.e. in Ω, or equivalently, that $[\partial g(x,y)/\partial y]_{y=0}$ exists a.e. in Ω; however this condition is unnecessary. In fact it suffices that $g(x,y)=h(x)\,y+\omega(x,y)y$ for almost all $x\in\Omega$, with $\omega(x,y)$ verifying (18). Remark that $\omega(x,y)$ is well defined for $y\in\,]a(x),b(x)[$, $y\neq 0$ (we put by definition $\omega(x,0)=0$). One has $\omega(x,y)y\to 0$ when $y\to 0$ but not necessarily $\omega(x,y)\to 0$ when $y\to 0$.

One can prove that (5), (6) and (18) implies that $h(x)\geqslant 0$. The proof follows from the inequality $\omega(x,y)\geqslant -h(x)$ if $0\neq y\in\,]a(x),b(x)[$, and from the existence of a function $v(x)>0$, $v(x)\in D(\bar g)$ (1).

On the other hand notice that if $[\partial g(x,y)/\partial y]_{y=0}$ exists a.e. in Ω then it must coincide with $h(x)$ a.e. in Ω.

Let now β be the m.m. graph referred in (1). In the following we assume that β is differentiable at the origin in the following sense, introduced in [1]:

We say that β is differentiable at the origin with finite derivative β' if

$$\text{for any } \varepsilon>0 \text{ there exists } \delta_\varepsilon>0 \text{ such that}$$
$$|z-\beta' y|\leqslant\varepsilon|y|\,,\qquad \forall z\in\beta(y)\,,\tag{20}$$
$$\text{for all } y\in D(\beta)\cap\,]-\delta_\varepsilon,\ \delta_\varepsilon[\,.$$

We say that β is differentiable at the origin with $\beta'=+\infty$ if

$$\text{for any } \varepsilon>0 \text{ there exists } \delta_\varepsilon>0 \text{ such that}$$
$$|y|\leqslant\varepsilon|z|\,,\qquad \forall z\in\beta(y)\,,\tag{21}$$
$$\text{for all } y\in D(\beta)\cap\,]-\delta_\varepsilon,\ \delta_\varepsilon[\,.$$

On the other hand we consider the linear operator $A\colon H\to H$ defined by

$$\begin{cases} -\Delta Af+Af+h(x)Af=f & \text{a.e. in } \Omega\,,\\ \quad Af=0 & \text{a.e. in } \Gamma\,,\end{cases}\tag{22}$$

if $\beta'=+\infty$, or

$$\begin{cases} -\Delta Af+Af+h(x)Af=f & \text{a.e. in } \Omega\,,\\ -\dfrac{\partial Af}{\partial n}+\beta' Af=0 & \text{a.e. in } \Gamma\,,\end{cases}\tag{23}$$

if $\beta'<+\infty$.

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(1) We put $w(x)=\min\{y\colon g(x,y)=1\}$ if this set is not empty and $w(x)=c>0$ otherwise, and we define $v(x)$ by $v(x)=\min\big(w(x),c\big)$.

We have $Af \in W^{2,2}(\Omega)$ and

$$\|Af\|_{2,2} \leq c\|f\| . \tag{24}$$

For the reader's convenience we verify in the appendix the validity of this statements under the condition (19).

Our aim is to prove the following result:

THEOREM 1. – *Assume that the conditions* (5), (6), (11″), (13) *and* (18) *hold and put* $u = Pf$ *in* (11″). *If* $\beta' = +\infty$ [*resp.* $\beta' < +\infty$] *the operator* P *is Fréchet differentiable at the origin and* $DP(0) = A$ *with* A *defined by* (22) [*resp.* (23)].

This theorem can be used to caracterize completely the bifurcation points λ for the problem

$$\begin{cases} -\Delta u + g(x, u(x)) + \lambda u = 0 & \text{a.e. in } \Omega , \\ -\dfrac{\partial u}{\partial n} \in \beta(u) & \text{a.e. in } \Gamma , \end{cases} \tag{25}$$

at it was done in [1] for the case in which g is independent of x. It is immediate that for $\lambda > 0$ the only solution of (25) is the null solution. Hence we may assume, without loss of generality, that $\lambda \leq 0$. If λ, u is a solution of (25) with $u \neq 0$ we say that λ is an eigenvalue. We say that λ_0 is a bifurcation point for (25) if for any $\varepsilon > 0$ there exists a solution λ, u with $0 < \|u\| < \varepsilon$ and $|\lambda - \lambda_0| < \varepsilon$.

The following result holds:

THEOREM 2. – *Assume that the hypothesis of theorem* 1 *hold. Then if* $\beta' = +\infty$ *the bifurcation points* λ *for problem* (25) *are the eigenvalues* λ *for the Dirichlet problem*

$$\begin{cases} -\Delta u + h(x)u + \lambda u = 0 & \textit{in } \Omega , \\ u = 0 & \textit{in } \Gamma . \end{cases} \tag{26}$$

If $\beta' < +\infty$ *then the bifurcation points* λ *for problem* (25) *are the eigenvalues* λ *for the linear problem*

$$\begin{cases} -\Delta u + h(x)u + \lambda u = 0 & \textit{in } \Omega , \\ \dfrac{\partial u}{\partial n} + \beta' u = 0 & \textit{in } \Gamma . \end{cases} \tag{27}$$

This theorem, which is essentially a consequence of theorem 1 and of the Krasnosel'skii theorem (cf. [8], § VI theorem 2.2, p. 332), can be proved in the following way:

It is easy to see that the solutions λ, u of (25) are transformed in the solutions of

$$v = \mu P v \tag{28}$$

by means of the change of variables

$$\mu = 1 - \lambda, \quad v = \mu u . \tag{29}$$

REMARK 3. – The change of variables (29) was introduced in [6].

In particular $\mu = 1 - \lambda$ transforms the bifurcation points λ for (25) onto the bifurcation points μ for (28). On the other hand $\mu = 1 - \lambda$ transforms the eigenvalues λ for (26) [resp. (27)] onto the characteristic values (2) for

$$v = \mu A v \tag{30}$$

where A is the linear operator defined by (22) [resp. (23)]. Therefore to prove theorem 2 it is enough to prove that the bifurcation points μ for (28) are the characteristic values μ for (30). But this follows from the Krasnosel'skii's theorem since by theorem 1 P is differentiable at the origin with $DP(0) = A$.

We recall that the Krasnosel'skii's theorem also requests the potentialness of P and some supplementary conditions, which are verified for our operator P, as was remarked in [6] for a formally analogous operator. Infact P is a potential operator since in a real Hilbert space every operator of the form $P = (I + \partial\theta)^{-1}$ with $\theta: H \to \boldsymbol{R} \cup \{+\infty\}$, θ convex and l.s.c. (and $\theta \not\equiv +\infty$) is a potential operator as proved by MOREAU in [9], proposition 7.d (3).

On the other hand P is completely continuous in H (4). If not, it follows from (14) (notice that $W^{1,2}(\Omega) \hookrightarrow H$ is compact) that there exists a sequence $f_\nu \rightharpoonup f$ such that $Pf_\nu \to u \neq Pf$. But this is not possible since P is m.m. (cf. [5], (7)). Finally it is well known that from the compactness of the gradient P it follows the weak continuity of the potential θ (see for instance [10], theorem 8.2) and from the uniform continuity (in a ball) of P it follows the uniform diffferentiability (in the ball) of θ (see for instance the proof of theorem 4.2 in [10]).

REMARK 4. – Just like it has been done with Krasnosel'skii's theorem we can apply to problem (25) other known general results (5) which underly on the Fréchet differentiability of P at the origin (proved in theorem 1). We leave this exercise to the reader.

REMARK 5. – The basic ideas on which underly the study of the differentiability for P and for analogous operators were introduced in [1] for the study of the differentiability at the origin of P. Some of these ideas were later adapted in [7] for the study of the differentiability at infinity.

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(2) I.e. the values μ for which (30) has a non-null solution.

(3) There exists a potential θ, constructed in [9], such that $P = \nabla\theta$ in H, in the Fréchet sense.

(4) I.e. $f_n \rightharpoonup f$ in $H \Rightarrow Pf_n \to Pf$ in H. This property is also called «strong continuity» by some authors.

(5) As for example some results of Rabinowitz.

The device introduced in the present paper, concerning the case when g also depends on x, has a corresponding device for the study of the differentiability at infinity. Once this is done one can apply, as in [7], some known general results which underly on the differentiability of P at infinity.

REMARK 6. – By changing (16) in a suitable way the results extend immediately to the case in which $y \to g(x, y)$ is a graph in $\boldsymbol{R}\times\boldsymbol{R}$.

We prove now the theorem 1. First we recall two lemmas which are a particular case of some results of [1]:

LEMMA 1. – *Let α be a graph in $\boldsymbol{R}\times\boldsymbol{R}$ such that $0\in\alpha(0)$ and $\alpha' = +\infty$. Assume that $v\in W^{\frac12,2}(\Gamma)$, $w\in L^2(\Gamma)$, $v(x)\in D(\alpha)$ and $w(x)\in\alpha(v(x))$ a.e. in Γ. Then*

$$|v|_2 \leqslant \varepsilon|w|_2 + c\delta_\varepsilon^{(2-r)/2}|v|_{\frac12,2}^{r/2}, \qquad \forall\varepsilon>0, \tag{31}$$

with $r>2$ such that $W^{\frac12,2}(\Gamma)\hookrightarrow L^r(\Gamma)$.

LEMMA 2. – *Let α be a graph in $\boldsymbol{R}\times\boldsymbol{R}$ such that $0\in\alpha(0)$ and $\alpha' = 0$. Assume that $v, w\in W^{\frac12,2}(\Gamma)$, $v(x)\in D(\alpha)$ and $w(x)\in\alpha(v(x))$ a.e. in Γ. Then*

$$|w|_2 \leqslant \varepsilon|v|_2 + c\delta_\varepsilon^{(2-r)/2}\big(|v|_{\frac12,2}^{r/2} + |w|_{\frac12,2}^{r/2}\big), \qquad \forall\varepsilon>0, \tag{32}$$

with $r>2$ such that $W^{\frac12,2}(\Gamma)\hookrightarrow L^r(\Gamma)$.

LEMMA 3. – *Let $1\leqslant q\leqslant s\leqslant +\infty$, $q<t<+\infty$. Assume that*

$$w(x) = v(x)\,\omega\big(x, v(x)\big) \tag{33}$$

with $v(x)\in D(\bar g)$ [6] *and with ω verifying the condition* (18). *Then*

$$\|w\|_q \leqslant \varepsilon\|v\|_{sq/(s-q)} + c\delta_\varepsilon^{(q-t)/q}\big(\|v\|_t^{t/q} + \|w\|_t^{t/q}\big), \qquad \forall\varepsilon>0. \tag{34}$$

PROOF. – Put $\delta_\varepsilon = \delta$ and define $\Omega_{\delta,v}$ by

$$\Omega_{\delta,v} = \{x\in\Omega\colon |v(x)|>\delta\}.$$

Then

$$\|w\|_q^q = \int_{\Omega-\Omega_{\delta,v}} |w|^q\,dx + \int_{\Omega_{\delta,v}} |w|^q\,dx. \tag{35}$$

[6] If we assume (18) with « $v(x)\in\,]a(x), b(x)[$ a.e. in Ω » instead of « $v\in D(\bar g)$ » we do the same modification in lemma 3.

Moreover (cf. [1] (1.11) and (1.12)) we can prove that

$$(36)\qquad \int\limits_{\Omega_{\delta,v}} |w|^q\,dx \leqslant \int\limits_{\Omega_{\delta,v}} |v|^q\,dx + \int\limits_{\Omega_{\delta,w}} |w|^q\,dx \leqslant \frac{t}{t-q}\,\delta^{q-t}(\|v\|_t^t + \|w\|_t^t)\,.$$

On the other hand by using (33), Hölder's inequality and (18) it follows that (7)

$$(37)\qquad \int\limits_{\Omega-\Omega_{\delta,v}} |w|^q\,dx \leqslant \Big(\int\limits_{\Omega-\Omega_{\delta,v}} |\omega(x, v(x))|^s\,dx\Big)^{s/q}\Big(\int\limits_{\Omega-\Omega_{\delta,v}} |v|^{sq/(s-q)}\,dx\Big)^{(s-q)/s}$$
$$\leqslant \varepsilon^q \|v\|^q_{sq/(s-q)}$$

since the function defined by $\bar{v}(x) = v(x)$ if $x\in\Omega-\Omega_{\delta,v}$, $\bar{v}(x)=0$ if $x\in\Omega_{\delta,v}$ belongs to $D(\bar{g})$. From (35), (36) and (37) it follows (34).

From now on we fix s as indicated in (17) and q as follows

$$(38)\qquad \begin{cases} q = (2^*)' = 2n/(n+2) & \text{if } n>2\,,\\ q = s & \text{if } n\leqslant 2\,. \end{cases}$$

Finally we put $t=2$ in (34) and we define p by

$$(39)\qquad p = sq/(s-q)\,.$$

Remark that $p=(2^*)^*$ if $n>4$ and $p=+\infty$ if $n\leqslant 3$. From the Sobolev's embedding theorems it follows that $W^{1,2}(\Omega)\hookrightarrow L^{q'}(\Omega)$ and $W^{2,2}(\Omega)\hookrightarrow L^q(\Omega)$. Consequently the following result holds from lemma 3:

COROLLARY 1. – *Assume that ω verify the condition* (18) *and define $w(x)$ by* (33) *with $v(x)\in D(\bar{g})$. Then*

$$(40)\qquad \|w\|_q \leqslant \varepsilon\|v\|_p + c\delta_\varepsilon^{(q-2)/q}(\|v\|_2^{2/q} + \|w\|_2^{2/q})$$

where $q<2$, $W^{1,2}(\Omega)\hookrightarrow L^{q'}(\Omega)$ and $W^{2,2}(\Omega)\hookrightarrow L^p(\Omega)$.

PROOF OF THEOREM 1. – Suppose first that $\beta' = +\infty$. Putting $Rf = Pf - Af$ it follows from (22) and (11″) that

$$(41)\qquad \begin{cases} \Delta Rf - Rf - h(x)Rf = Pf\,\bar{\omega}(Pf) & \text{a.e. in } \Omega\,,\\ Rf = Pf & \text{a.e. in } \Gamma\,, \end{cases}$$

(7) With obvious changes if $s=q$ or $s=+\infty$.

Multiplying the first equation (41) by Rf, integrating in Ω and applying Green's formulae on has

$$\|Rf\|_{1,2}^2 \leqslant -\int_\Omega Pf\,\bar{\omega}(Pf)\,Rf\,dx + \int_\Gamma \frac{\partial Rf}{\partial n}\,Pf\,d\Gamma$$

since $h(x) \geqslant 0$ a.e. in Ω and $Rf = Pf$ in Γ.

Thus (⁸)

$$\|Rf\|_{1,2}^2 \leqslant c\|Rf\|_{q'}\|Pf\,\bar{\omega}(Pf)\|_q + \int_\Gamma \frac{\partial Rf}{\partial n}\,Pf\,d\Gamma \leqslant c\|f\|\,\|Pf\,\bar{\omega}(Pf)\|_q + c\|f\|\,|Pf|_2\,, \tag{42}$$

since $\|Rf\|_{q'} \leqslant c\|Rf\|_{1,2} \leqslant c\|f\|$ and $|\partial Rf/\partial n|_2 \leqslant c\|f\|$ by (13) and (24).

On the other hand applying corollary 1 with $v = Pf$ and $w = Pf\,\bar{\omega}(Pf)$ we get

$$\|w\|_q \leqslant c\varepsilon\|v\|_{2,2} + c\delta_\varepsilon^{(q-2)/q}\left(\|v\|^{2/q} + \|w\|_{2/q}\right) \tag{43}$$

since $W^{2,2}(\Omega) \hookrightarrow L^p(\Omega)$. By using (13) and (24) it follows that (⁸)

$$\|w\| \leqslant c\|f\| \tag{44}$$

since $w = \Delta Rf - Rf - h(x)Rf$. From (43), (44) and (13) we get (⁸)

$$\|w\|_q \leqslant c\left(\varepsilon + \delta_\varepsilon^{(q-2)/q}\|f\|^{(2-q)/q}\right)\|f\|\,.$$

Therefore for any $\varepsilon > 0$ there exists $\delta'_\varepsilon > 0$ such that

$$\|f\| < \delta'_\varepsilon \Rightarrow \|w\|_q \leqslant \varepsilon\|f\|\,. \tag{45}$$

On the other hand by using the lemma 1 with β instead of α, $v = Pf$ and $w = -\,\partial Pf/\partial n$ one obtains as is [1] that for any $\varepsilon > 0$ there exists $\delta''_\varepsilon > 0$ such that

$$\|f\| < \delta''_\varepsilon \Rightarrow |Pf|_2 \leqslant \varepsilon\|f\|\,. \tag{46}$$

From (42), (45) and (46) we get

$$\lim_{\|f\|\to 0} \frac{\|Pf - Af\|_{1,2}}{\|f\|} = 0 \tag{47}$$

and thus P is Fréchet differentiable at the origin with $DP(0) = A$.

(⁸) For small values of $\|f\|$.

Assume now that $\beta' < +\infty$. Putting $Rf = Pf - Af$, and using (11′) and (23) it follows that

$$\|Rf\|_{1,2}^2 \leqslant -\int_\Omega Pf\,\bar{\omega}(Pf)\,Rf\,dx + \int_\Gamma \left(\frac{\partial Rf}{\partial n} + \beta' Rf\right) Rf\,d\Gamma,$$

since $\beta' \geqslant 0$. Then, as for (42),

$$\|Rf\|_{1,2}^2 \leqslant c\|f\|\,\|Pf_s\bar{\omega}(Pf)\|_q + c\|f\| \left|\frac{\partial Rf}{\partial n} + \beta' Rf\right|_2 . \tag{48}$$

Finally (47) follows from (48), (45) (9) and from the following result: for any $\varepsilon > 0$ there exists $\delta_\varepsilon'' > 0$ such that

$$\|f\| < \delta_\varepsilon'' \Rightarrow \left|\frac{\partial Rf}{\partial n} + \beta' Rf\right|_2 \leqslant \varepsilon\|f\| . \tag{49}$$

We prove this result exactly as is [1] by using the lemma 2 with $\alpha(y) = \beta(y) - \beta' y$, $v = Pf$ and $w = -\partial Rf/\partial n - \beta' Rf$. Remark that $\alpha' = 0$ and that $-\partial Rf/\partial n - \beta' Rf \in \in \beta(Pf) - \beta' Pf$ a.e. in Γ.

Remark by the way that P is also differentiable (at the origin) as an operator from $L^2(\Omega)$ into $W^{1,2}(\Omega)$, as follows from (47); this result can be generalized further (cf. [2]).

An Example. – The more usual example of functions converging to zero when $y \to 0$, not unifoımly in respect of a parameter, is the family $f_t(y) = |y|^t$, $t > 0$. From this family we can construct examples to which the theorem 1 (and consequently the theorem 2) applies. A very easy one is the following: put $\omega(x, y) = |y|^{r(x)}$ where $r(x)$ is the distance from x to the origin and Ω is a smooth open bounded set (more generally one can study the same problem if $r(x)$ is the distance from x to a line, a plan or a suitable set). For the sake of simplicity suppose that $n = 3$ and that $h(x) = 0$. Under this last assumption one has

$$g(x, y) = y|y|^{r(x)}, \qquad \forall x \in \Omega,\ \forall y \in \boldsymbol{R} .$$

For any fixed $x \neq 0$ one has $\lim_{y\to 0} \omega(x, y) = 0$; however, if $0 \in \bar{\Omega}$, this property is not verified uniformly in respect of $x \neq 0$. However one can verify that (18) holds for any $s \in [1, \infty[$. On the other hand the conditions (11) and (13) holds (10) and the remaining assumptions of theorem 1 are trivially verified (by assumption β is differentiable at the origin). Consequently the theorems 1 and 2 apply. In par-

(9) Which is proved as before.

(10) Apply by instance the corollary 7.3 of [3]; remark that

$$0 \leqslant -y^r \log y \leqslant 1/er \qquad \text{if} \ \ y \in [0, 1] \ \ \text{and} \ \ r > 0 .$$

ticular the bifurcation points λ for the problem $-\Delta u + u|u|^{r(x)} = -\lambda u$ in Ω, $-\partial u/\partial n \in \beta(u)$ in Γ, are the eigenvalues λ for the linear problem $-\Delta u = -\lambda u$ in Ω, $-\partial u/\partial n = \beta' u$ [resp. $u = 0$] in Γ, if $\beta' < +\infty$ [resp. $\beta' = +\infty$].

Appendix.

For the sake of convenience we verify in this appendix that the problem (23) ([11]), i.e. the problem

$$\begin{cases} -\Delta u + u + h(x)u = f & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n} + \beta' u = 0 & \text{in } \Gamma\,, \end{cases} \tag{50}$$

with $f \in L^2(\Omega)$, $\beta' \geqslant 0$, $h(x) \geqslant 0$ and $h(x)$ verifying (19) has a unique solution $u \in W^{2,2}(\Omega)$. Furthermore (24) holds, i.e. $\|u\|_{2,2} \leqslant c\|f\|$.

First, by using the variational method, we see easily that (50) has a unique weak solution $u \in W^{1,2}(\Omega)$, which verifies

$$\|u\|_{1,2} \leqslant c\|f\|\,, \tag{51}$$

since the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \big(1 + h(x)\big) uv \, dx + \beta' \int_\Gamma uv \, d\Gamma$$

is continuous and coercitive on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$.

Assume that $n \geqslant 4$ (if $n < 4$ the procedure is identical), put $q_1 = 2^* = 2n/(n-2)$ and define p_i and q_i by

$$\frac{1}{p_i} = \frac{1}{m} + \frac{1}{q_i}\,, \quad \frac{1}{q_{i+1}} = \frac{1}{p_i} - \frac{2}{n} \qquad (i \geqslant 1)\,. \tag{52}$$

Since $1/p_{i+1} = 1/p_i - \big((2/n) - (1/m)\big)$ the sequence p_i is increasing. If $p_1 < 2$ define i_0 by $1/p_{i_0+1} \leqslant \frac{1}{2} < 1/p_{i_0}$: Remark that $q_{i+1} = (p_i^*)^*$ if $1 \leqslant i \leqslant i_0$.

Since $u \in L^{q_1}(\Omega)$ and $h \in L^m(\Omega)$ one has $hu \in L^{p_1}(\Omega)$ with $\|hu\|_{p_1} \leqslant c\|u\|_{q_1} \leqslant c\|f\|$ by (51). On the other hand if $hu \in L^{p_i}(\Omega)$ with $1 \leqslant i \leqslant i_0$ it follows that $hu \in L^{p_{i+1}}(\Omega)$ with $\|hu\|_{p_{i+1}} \leqslant c\big(\|hu\|_{p_i} + \|f\|\big)$. For, it follows from (50), by regularisation, that $u \in W^{2,p_i}(\Omega) \hookrightarrow L^{q_{i+1}}(\Omega)$ and $\|hu\|_{p_{i+1}} \leqslant c\|u\|_{q_{i+1}} \leqslant c\|u\|_{2,p_i} \leqslant c\big(\|hu\|_{p_i} + \|f\|\big)$.

Hence, by induction, $hu \in L^{p_{i_0+1}}(\Omega)$ and $\|hu\|_2 \leqslant c\|hu\|_{p_{i_0+1}} \leqslant c\|f\|$. With a last regularisation one obtains that $u \in W^{2,2}(\Omega)$ with $\|u\|_{2,2} \leqslant c\|f\|$.

([11]) The procedure is identical for the Dirichlet problem (22).

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