Nonlinear Perturbation of Linear Evolution Equations in a Banach Space (*).

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Summary. – *Let X be a Banach space and $\{A(t)|t\in[0,T]\}$ a family of closed linear, densely defined m-accretive operators in X. This paper is concerned with the additive perturbation of $\{A(t)|t\in[0,T]\}$ by a continuous family of nonlinear accretive operators $\{B(t)|t\in[0,T]\}$. Namely solutions are provided for the integral equation $u(t,\tau,x)=W(t,\tau)x-\int_\tau^t W(t,s)B(s)\cdot$ $\cdot u(s,\tau,x)\,ds$, $u(\tau,\tau,x)=x$ where $W(t,s)$ is the linear evolution operator associated with the linear differential equation $v'(t,s,x)+A(t)v(t,s,x)=0$, $v(s,s,x)=x$.*

1. – Introduction.

Let X be a Banach space and $\{A(t)|t\in[0,T]\}$ be a family of linear accretive operators defined in X. We shall be concerned with the additive perturbation of $\{A(t)|t\in[0,T]\}$ by the continuous, nonlinear accretive family $\{B(t)|t\in[0,T]\}$. Namely we provide a solution to the integral equation:

$$u(t,\tau,x)=W(t,\tau)x-\int_\tau^t W(t,s)B(s)u(s,\tau,x)\,ds;\qquad u(\tau,\tau,x)=x\,, \tag{1.1}$$

where $W(t,s)$ is the linear evolution operator associated with the homogenous linear differential equation

$$v'(t,s,x)+A(t)v(t,s,x)=0;\qquad v(s,s,x)=x\,. \tag{1.2}$$

It is not difficult to see that solutions to,

$$u'(t,\tau,x)+A(t)u(t,\tau,x)+B(t)u(t,\tau,x)=0;\qquad u(\tau,\tau,x)=x\,, \tag{1.3}$$

satisfy (1.1). Thus using the terminology of F. BROWDER [1] one may consider solutions of (1.1) to provide « mild » solutions to (1.3).

In 1953 T. KATO [7] solved (1.2) via the method of product integration; for latter studies on the linear equation the reader is referred to [3], [4], [8], [9], [10],

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[13], [20]. In 1963 I. SEGAL [17] considered (1.1) for locally Lipschitz $B(t)$. More recently G. F. WEBB [18] solves (1.1) in case A and B are independent of t; D. L. LOVELADY [12] obtains solutions to (1.1) for the case of time independent A and time dependent $B(t)$. K. MAURO and N. YAMADA [16] and the author [4] have considered the case of time dependent $A(t)$ and $B(t)$.

2. – Definitions and Preliminaries.

Throughout our paper X shall denote a Banach space with norm $\|\cdot\|$ and dual space X^*; the pairing between $f \in X^*$ and $x \in X$ shall be denoted $\langle x, f\rangle$. The operator A is said to be *accretive* provided that

$$\|(x+\lambda Ax)-(y+\lambda Ay)\| \geqslant \|x-y\| , \qquad \text{for } x, y \in D(A) \text{ and } \lambda > 0 .$$

It is well known that this definition is equivalent to the statement that $\operatorname{Re}\langle Ax - Ay, f\rangle \geqslant 0$ for some $f \in F(x-y)$ where F is the duality map from X to X^*. An operator is said to be *strongly accretive* if $\langle Ax - Ay, f\rangle \geqslant 0$ for all $f \in F(x-y)$. An accretive operator is said to be m-accretive provided that $R(I+\lambda A) = X$ for all $\lambda > 0$.

It is well known (c.f. [20]) that a closed, densely defined, linear accretive operator is strongly accretive and R. H. MARTIN [14] shows that a continuous, nonlinear, accretive operator is strongly accretive. Thus, if A is a closed, densely defined, linear m-accretive operator and B is a continuous accretive operator the observation that

$$\langle (A+B)x-(A+B)y, f\rangle = \langle Ax - Ay, f\rangle + \langle Bx - By, f\rangle \geqslant 0 , \qquad \text{for all } f \in F(x, y)$$

insures the strong accretiveness of $A+B$. Martin also shows that if A is strongly accretive then

$$\lim_{h\to 0^+} 1h\left(\|x-y-h(Ax-Ay)\| - \|x-y\|\right) \leqslant 0 \qquad \text{for } x, y \in D(A) . \tag{2.1}$$

The foregoing property of strongly accretive operators will be used in conjunction with the following technique for computing the one sided derivative of the norm of a vector valued function.

LEMMA 2.1. – Let q be a function (a, b) to X and define $p(t) = \|q(t)\|$ for $t \in (a, b)$. If $q'^{+}(t)$ exists then $p'^{+}(t)$ exists and

$$p'^{+}(t) = \lim_{h\to 0^+} \frac{1}{h}\left(\|q(t) + hq'^{+}(t)\| - \|q(t)\|\right) . \tag{2.2}$$

A proof of the above lemma may be found in [3]. It is now convenient to place restrictions on our time dependent family of linear accretive operators $\{A(t)|t\in[0, T]\}$. These conditions are essentially mild well-posedness conditions for (1.2).

DEFINITION. – A family of closed densely defined m-accretive linear operators $\{A(t)|t\in[0, T]\}$ is said to satisfy *condition* E provided that the following are true

i) There exists a dense linear manifold $\mathfrak{D}$ such that $\mathfrak{D}\subseteq D(A(t))$ for all $t\in[0, T]$.

ii) There exists a unique family of linear operators $W(t, s)\colon X\to X$ for $0\leqslant s\leqslant t\leqslant T$ with the following properties

a) $W(t, s)$ is jointly continuous in s and t; $W(s, s)=I$ and $W(t, r)W(r, s)=W(t, s)$ for $0\leqslant s\leqslant r\leqslant t\leqslant T$.

b) If $x\in\mathfrak{D}$ then $D_t^+(W(t, s)x)=-A(t)W(t, s)x$ for $t\geqslant s$ and this derivative is weakly continuous in t and s.

We remark that condition E is sufficient to guarentee that if $x\in\mathfrak{D}$ then

$$A(t)\int_\tau^t W(t, s)x\,ds=\int_\tau^t A(t)W(t, s)x\,ds\,. \tag{2.3}$$

3. – Local existence of solutions.

In this we present a result which will insure the local existence and uniqueness of solutions.

THEOREM 3.1. – Let $\{A(t)|t\in[0, T]\}$ be a family of closed densely, defined, linear m-accretive operators which satisfy condition E and $\{B(t)|t\in[0, T]\}$ a family of nonlinear accretive which are defined and continuous in a neighborhood of $(\tau, x_0)\in\in[0, T]\times X$. Then there exists a $T_{x_0}>\tau$ and unique function $u(\cdot, \tau, x_0)\colon[\tau, T_{x_0}]\to X$ so that $u(\cdot, \tau, x_0)=x_0$ and

$$u(t, \tau, x_0)=W(t, \tau)x_0-\int_\tau^t W(t, s)B(s)u(s, \tau, x_0)\,ds\,, \qquad \text{for } t\in[\tau, T_{x_0}]\,. \tag{3.1}$$

The proof of the foregoing theorem will be presented in two pieces which guarentee existence and uniqueness respectively. The proof is obtained via a modification of a technique employed by G. F. WEBB [18] for time independent A and B. The method has subsequently been employed by MAURO and YAMADA [16] and the author [4].

EXISTENCE PROOF. – For notational convenience we shall assume that $\tau=0$. Let V be a neighborhood of $(0, x_0)$ such that there exists an $M>0$ so that $\|B(s)v\|\leqslant M$ for all $(s, v)\in V$. Making use of the continuity properties of $W(t, s)$ and $B(t)$ and the denseness of $\mathfrak{D}$ we choose $T_{x_0}>0$ and sequence $\{x_n\}_{n=1}$ contained in the intersection of $\mathfrak{D}$ with the X-projection of V such that $x_n \to x_0$ and if $u = W(t, 0)x_n + w$ where $0 \leq t \leq T_{x_0}$ and $\|w\| \leq T_{x_0} M$ then $(t, u) \in V$ and so $\|B(t)u\| \leq M$. Let $n \in Z^+$ and define $t_0^n = 0$ and $u_n(t_0^n, x_0) = x_n$; define ∂_1^n to be the largest number so that

(3.2) 1) $0 \leq \partial_1^n$ and $t_0^n + \partial_1^n \leq T_{x_0}$.

2) If $\|z - u_n(t_0^n, x_0)\| \leq \partial_1^n M + \max\limits_{t^n \leq t \leq t_0^n + \partial_1^n} \|(W(t, t_0^n) - I) u_n(t_0^n, x_0)\|$, then

$$\|B(t)z - B(t_n^0) u_n(t_0^n, x_0)\| \leq 1/n , \qquad \text{for } t_0^n \leq t \leq t_1^n + \partial_1^n .$$

Set $t_1^n = t_0^n + \partial_1^n$ and if $t \in (t_0^n, t_1^n)$ let

$$u_n(t, x_0) = W(t, t_0^n) u_n(t_0^n, x_0) - \int_{t_0^n}^{t} W(t, s) B(t_0^n) u_n(t_0^n, x_0)\, ds .$$

Having defined $u_n(t_{k-1}^n, x_0)$ apply the foregoing procedure to define ∂_k^n, t_k^n and for $t \in (t_{k-1}^n, t_k^n]$ let

$$u_n(t, x_0) = W(t, t_{k-1}^n) u_n(t_{k-1}^n, x_0) - \int_{t_{k-1}^n}^{t} W(t, s) B(t_{k-1}^n) u_n(t_{k-1}^n, x_0)\, ds . \tag{3.3}$$

It is readily seen that for $t \in [t_{k-1}^n, t_k^n]$ we have,

$$\begin{aligned} u_n(t, x_0) = W(t, 0)x_n - \sum_{i=1}^{k-1} \int_{t_{i-1}^n}^{t_i^n} W(t, s) B(t_{i-1}^n) u_n(t_{i-1}^n, x_n)\, ds \\ - \int_{t_{k-1}^n}^{t} W(t, s) B(t_{k-1}^n) u_n(t_{k-1}^n, x_n)\, ds . \end{aligned} \tag{3.4}$$

From (3.4) we are able to show that $(t, u_n(t, x_0)) \in V$ and hence $\|B(t) u_n(t, x_0)\| \leq M$ By virtue of the fact that,

$$\begin{aligned} \Big\|(W(t, t_{i-1}^n) - I) u_n(t_{i-1}^n, x_0) - \int_{t_{k-1}^n}^{t} W(t, s) B(t_{i-1}^n) u_n(t_{i-1}^n, x)\, ds\Big\| \leq \\ \leq \|(W(t, t_{i-1}^n) - I) u_n(t_i^n, x_0)\| + \partial_i^n M , \qquad \text{for } t \in [t_{i-1}^n, t_i^n] \end{aligned}$$

we have

$$\|B(t) u_n(t, x_0) - B(t_{i-1}^n) u_n(t_{i-1}^n, x_0)\| \leq 1/n . \tag{3.5}$$

We now assert that there is an integer $N(n)$ such that $t_N^n = T_{x_0}$. This statement may be verified in the same manner as WEBB verifies it for of the case of time inde-

pendent A and B and the reader is referred to [18] for details. By use of the continuity of $B(\cdot)\colon V\subseteq[0,T]\times X\to X$ and $u_n(\cdot,x_0)\colon (0,T_{x_0}]\to X$ we insure the existence of a constant γ_n so if $v(\cdot)\colon [0,T_{x_0}]\to X$ and $\sup_{s\in[0,T_{x_0}]}\|u_n(s,x_0)-v(s)\|\leqslant\gamma_n$. Then $\sup_{s\in[0,T_{x_0}]}\|B(s)u_n(s,x_0)-B(s)v(s)\|\leqslant 1/n$.
We set $\gamma_n'=\min\{1/n,\gamma\}$. The continuity of $W(t,s)$ and the denseness of $\mathfrak{D}\subseteq X$ allow us to choose a sequence of points $\{B_{i-1}^n\}_{i=1}^{N(n)}\subseteq\mathfrak{D}$ such that if we define,

$$(3.6)\qquad v_n(t,x_0)=W(t,0)x_n-\sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t^n}W(t,s)B_{i-1}^n\,ds-\int_{t_{k-1}^n}^{t}W(t,s)B_{k-1}^n\,ds\,,\qquad \text{for } t\in[t_{k-1}^n,t_k^n],\ k=1,\dots,N(n)$$

we have

$$(3.7)\qquad \text{(i)}\ \sup_{s\in[0,T_{x_0}]}\|u_n(s,x_0)-v_n(s,x_0)\|<\gamma_n'.$$

$$\text{(ii)}\ \sup_{i=1\dots N(n)}\|B(t_{i-1}^n)u_n(t_{i-1}^n)-B_{i-1}^n\|<1/n.$$

From (3.5) and the subsequent remarks we have constructed a function $v_n(t,x_0)$ for each $n\in Z^+$ so that:

$$(3.6)\qquad \sup_{s\in[0,T_{x_0}]}\|v_n(s,x_0)-u_n(s,x_0)\|\leqslant 1/n\,,$$

and

$$\|B(t)v_n(t,x_0)-B(t_{i-1}^n)u_n(t_{i-1}^n,x_0)\|\leqslant 2/n\,,\qquad \text{for } t\in[t_{i-1}^n,t_i^n]\,.$$

Clearly if the sequence $\{v_n(t,x_0)\}$ converges uniformly to a function $u(t,x_0)$ on $[0,T_{x_0}]$ then $\{u_n(t,x_0)\}$ converges to $u(t,x_0)$ on $[0,T_{x_0}]$ and $u(t,x_0)$ is continuous on $[0,T_{x_0}]$.

Making use of condition E and (2.3) we are able to compute $v_n'^{+}(t,x_0)$ if $t\in(t_{k-1}^n,t_k^n)$ then,

$$(3.7)\qquad v_n'^{+}(t)=\lim_{h\to0^+}\Big\{\frac{W(t+h,0)x_n-W(t,0)x_n}{h}-1/h\sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_1^n}\big(W(t+h,s)B_{i-1}^n-W(t,s)B_{i-1}^n\big)\,ds-1/h\Big(\int_{t_{k-1}^u}^{t+h}W(t+h,s)B_{k-1}^n-W(t,s)B_{i-1}^n\Big)\,ds\Big\}=-A(t)v_n(t,x_0)-B_{k-1}^n.$$

We next show the sequence of functions $\{v_n(t, x_0)\}$ converges uniformly to a function $u(t, x_0)$ on $[0, T_{x_0}]$. Let m and n be positive integers and define $p_{mn}(t) = \|v_n(t, x_0) - v_m(t, x_0)\|$. Let $t \in [0, T_{x_0})$ and let $i, k \in Z^+$ be such that $t \in [t^m_{i-1}, t^m_i)$ and $t \in [t^n_{k-1}, t^n_k)$. From (2.1), (2.2) and (3.7) we may observe

$$(3.8)\qquad \begin{aligned} p^+_{mn}(t) &= \lim_{h \leftarrow 0^+} \Big(\|v_n(t, x_0) - v_m(t, x_0) - h\big(A(t)v_n(t, x_0) + B^n_{k-1} \\ &\quad - A(t)v_m(t, x_0) - B^m_{j-1}\big)\|\Big)/h \\ &\leqslant \|B^n_{k-1} - (t^n_{k-1})u_n(t^n_{k-1}, x_0)\| + \|B(t^n_{k-1})u_n(t^n_{k-1}, x_0) \\ &\quad - B(t)v_n(t, x_0)\| + \|B^m_{j-1} - B(t^m_{j-1})u_m(t^m_{j-1}, x_0)\| \\ &\quad + \|B(t^m_{j-1})u_m(t^m_{j-1}, x_0) - B(t)v_m(t, x_0)\| \\ &\leqslant 3(1/m + 1/n)\,. \end{aligned}$$

If we integrate (3.8) on $[0, t]$ we obtain $p_{mn}(t) = \|v_n(t) - v_m(t)\| \leqslant 2T_0(1/m + 1/n) + \|x_n - x_m\|$. We there by conclude that $\{v_n(t, x_0)\}$ converges uniformly to some $u(t, x_0)$ on $[0, T_{x_0}]$. Equation (3.4) can be applied to see that $u(t, x)$ satisfies (3.1). This argument may be readily modified to guarantee a function a local solution beginning at $\tau_0 \in [0, T]$.

The next lemma guarantees uniqueness of solutions by demonstrating the Lipschitz continuous dependence of solutions on initial data.

LEMMA 3.1. – Let $\{A(t)|t \in [0, T]\}$ satisfy condition E and let $u(\cdot, \tau, x)$ and $v(\cdot, \tau, y)$ be solutions to (3.1) with initial conditions $u(\tau, \tau, x) = x$ and $v(\tau, \tau, y) = y$ on $[\tau, T_u]$ and $[\tau, T_v]$ respectively. If $B(\cdot)$ is continuous on neighborhoods containing the ranges of $u(\cdot, \tau, x)$ and $v(\cdot, \tau, y)$ we have $\|u(t, \tau, x) - v(t, \tau, y)\| \leqslant \|x - y\|$ for $t \in [\tau, T_1]$ where $T_1 = \min\{T_u, T_v\}$.

PROOF. – We again allow ourselves the convenience of considering $\tau = 0$ and denote our solutions $u(\cdot, x)$ and $v(\cdot, y)$; we proceed in a manner similar to the existence argument. For each $n \in Z_+$ let $\gamma_n > 0$ be such that

$$\sup_{s \in [0,T]} \|u(s, x) - f(s)\| < \gamma_n \qquad \text{and} \qquad \sup_{s \in [0,T]} \|v(s, x) - g(s)\| < \gamma_n$$

for $f(\cdot), g(\cdot)\colon [0, T_1] \to X$ implies that

$$\sup_{s \in [0,T_1]} \|B(s)u(s, x) - B(s)f(s)\| \leqslant 1/n \qquad \text{and} \qquad \sup_{s \in [0,T_1]} \|B(s)v(s) - B(s)g(s)\| \leqslant 1/n\,.$$

We set $\gamma'_n = \min\{1/n, \gamma_n\}$. We choose a partition $\{t^n_i\}^N_{i=1}$ of $[0, T_1]$ so that

$$\|B(t^n_i)u(t^n_i, x) - B(t)u(t, x)\| < \gamma'_n/3T_1 \qquad \text{and} \qquad \|B(t^n_i)v(t^n_i, y) - B(t)v(t, x)\| < \gamma'_n/3T_1$$

for $t_i^n < t < t_{i+1}^n$. We let $B\{_i^n(u)\}_{i=1}^{N(n)}$ and $\{B_i^n(v)\}_{i=1}^{N(n)}$ be sequences contained in $\mathfrak{D}$ chosen so that $\|B_i^n(u) - B(t_i^n)u_n(t_i^n, x)\| \leqslant \gamma_n'/3T_1$ and $\|B_i^n(v) - B(t_i^n)v_n(t_i^n, x)\| \leqslant \gamma_n'/3T_1$; sequences $\{x_n\}, \{y_n\} \subseteq \mathfrak{D}$ are chosen such that $\{x_n\}, \{y_n\} \subset \mathfrak{D}$, $\|x - x_n\| \leqslant \gamma_n'/3T_1$ and $\|y_n - y\| \leqslant \gamma_n'/3T_1$. If we define,

$$u_n(t, x) = W(t, 0)x_n - \sum_{i=1}^{k-1} \int_{t_{i-1}^n}^{t_i^n} W(t, s) B_{i-1}^n(u)\, ds - \int_{t_{k-1}^n}^{t} W(t, s) B_{k-1}^n(u)\, ds \tag{3.9}$$

and

$$v_n(t, x) = W(t, 0)y_n - \sum_{i=1}^{k-1} \int_{t_{i-1}^n}^{t_i^n} W(t, s) B_{i-1}^n(v)\, ds - \int_{t_{k-1}^n}^{t} W(t, s) B_{k-1}^n(v)\, ds$$

we have

$$\sup_{s\in[0,T_1]} \|u_n(s, x) - u(s, x)\| < \gamma_n \qquad \text{and} \qquad \sup_{s\in[0,T_1]} \|v_n(s, x) - v(s, x)\| < \gamma_n$$

and hence

$$\sup_{s\in[0,T_1]} \|B(s)u(s, x) - B(s)u_n(s, x)\| < 1/n$$

and

$$\sup_{s\in[0,T_1]} \|B(s)v(s, x) - B(s)v_n(s, x)\| < 1/n\,.$$

Arguing as before we see that if $t \in [t_{k-1}^n, t_k^n)$ we have $u_n'^{+}(t, x) = -A(t)u_n(t, x) - B_{k-1}^n(u)$ and $v_n'^{+}(t, y) = -A(t)v_n(t, y) - B_{k-1}^n(v)$. If $p_n(t) = \|u_n(t, x) - v_n(t, x)\|$ we can use (2.1) and (3.2) to differentiate $p_n(t)$ for $t \in [t_{k-1}^n, t)$ and obtain the estimate:

$$\begin{aligned} p_n'^{+}(t) &\leqslant \|B_{k-1}^n(u) - B(t)u(t, x)\| + \|B(t)u(t, x) \\ &\quad - B(t)u_n(t, x)\| + \|B_{k-1}^n(v) - B(t)v(t, x)\| \\ &\quad + \|B(t)v(t, x) - B(t)v_n(t, x)\| \\ &\leqslant 2(1/n + \gamma_n'/3T_1) \leqslant \frac{2}{n}(1 + 1/3T_1)\,. \end{aligned} \tag{3.10}$$

Thus $\{p_n'^{+}(t)\}$ converges uniformly to zero. Since $\{p_n(t)\}$ converges uniformly to $\|u(t, x) - v(t, x)\|$ we can integrate (3.10), to conclude that

$$\|u(t, x) - u(t, y)\| \leqslant \|u(0, x) - u(0, y)\| = \|x - y\|\,.$$

4. – Extension of solutions.

In this section we provide condition sufficient to guarantee the extension of local solutions of (3.1) from $[\tau, T_{x_0}]$ to $[\tau, T]$. We also indicate that we can associate a nonlinear evolution operator equation (3.1). Our first result in this direction in the following theorem.

THEOREM 4.1. – Let $\{A(t)|t\in[0,T]\}$ be a family of linear m-accretive operators which satisfy condition E and $\{B(t)|t\in[0,T]\}$ be a family of nonlinear accretive operators such that $B(t)0=0$ for $t\in[0,T]$ and $B(\cdot)\colon [0,T]\times(S_R(0)\to X$ denotes the sphere of radius R about 0) is continuous and bounded. If $x\in S_R(0)$ and $u(t,\tau,x)$ exists a unique $u(\cdot,\tau,x)\colon[\tau,T]\to X$ such that, $u(\cdot,\tau,x)\colon[\tau,T]\to X$ $\tau\in[0,T)$ there satisfies (3.1) for $t\in[\tau,T]$.

PROOF. – The proof rests on the fact that if $u(\cdot,\tau,x)$ is a solution (3.1) on a subinterval of $[\tau,T]$ then $\|u(t,\tau,x)\|$ is nonincreasing in t. If the local solution guaranteed by Theorem 3.1 could not be continued to $[\tau,T]$ there exists a maximal interval of existence $[\tau,T')$ with $T'<T$. However, $\|u(t,\tau,x)\|\leqslant\|u(\tau,\tau',x)\|<R$. The boundedness of $B(\cdot)\colon[0,T]\times S\ (0)\to X$ insures the existence of N such that $\|W(t,s)B(s)u(s,\tau,x)\|\leqslant N$ for $t\geqslant s$ and $t,s\in[\tau,T]$ this implies

$$\lim_{t\to T'}\int_\tau^t\|W(t,s)B(s)u(s,\tau,x)\|\,ds$$

exists and hence we can define

$$u(T',\tau,x)=\lim_{t\to T'}\Big(W(t,\tau)x-\int_\tau^t W(t,s)B(s)u(s,\tau,x)\,ds\Big)\,.$$

Moreover $\|u(T,\tau,x)\|<R$ and we could extend the solution past T' via Theorem 3.1. This would contract the definition of T' as the least upper bound of the maximum interval of existence.

We now argue that $\|u(t,\tau,x)\|$ is nonincreasing. Let $\tau\leqslant T''<T$ and suppose a solution to (3.1) exists on $[\tau,T'']$. Let $\{x_n\}_{n=1}^\infty\subset\mathfrak{D}$ and $x_n\to x$; let $\{t_i^n\}_{i=1}^{N(n)}$ and $\{B_i^n\}_{i=1}^{N(n)}\subset\mathfrak{D}$ be a partition of $[\tau,T'')$ and sequence picked so that if $t\in[t_{k-1}^n,t_k^n)$ and

$$u_n(t,x)=W(t,\tau)x_n-\sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_i^n}W(t,s)B_{i-1}^n\,ds-\int_{t_{i-1}^n}^t W(t,s)\,B_{k-1}^n\,ds$$

then the following are true: $\|u_n(t,x)-u(t,\tau,x)\|\leqslant1/n$, $\|B_{k-1}^n-B(t)u(t,\tau,x)\|\leqslant1/n$ for $t\in[t_{i-1}^n,t_i^n]$ and $\|B(t)u(t,\tau,x)-B(t)u_n(t,x)\|\leqslant1/n$. If $p_n(t)=\|u_n(t,x\|$ and $t\leqslant[t_{k-1}^n,t)$ we observe that

$$\begin{aligned}(3.11)\quad p_n^{+\prime}(t)&\leqslant\lim_{h\leftarrow0}\Big(\|u_n(t,x)-h(A(t)u_n(t,x)+B_{k-1}^n)\|-\|u_n(t)\|\Big)/h\\&=\lim_{h\leftarrow0}\Big(\|u_n(t,x)-0+h(A(t)u_n(t,x)+B_{k-1}^n-A(t)0-B(t)0)\|-\|u_n(t,x)\|\Big)/h\\&\leqslant\|B(t)u_n(t,x)-B(t)u(t,\tau,x)\|+\|B(t)u_n(t,x)-B_{k-1}^n\|\\&\leqslant2/n\,.\end{aligned}$$

Integrating (3.11) on (τ,T), we can conclude that $\|u(t,\tau,x)\|$ is nonincreasing.

Our next theorem states that boundedness conditions on $B(\cdot)\colon [0, T]\times X \to X$ will guarantee the extendability of solutions and it will be used to define a nonlinear evolution operator on X.

THEOREM 4.2. – Let $\{A(t)|t\in[0, T]\}$ be a family of closed linear m-accretive operators which satisfy condition E and suppose that $\{B(t)\colon t\in[0, T]\}$ is a family of nonlinear accretive operators such that $B(\cdot)\colon [0, T]\times X\to X$ is continuous and maps bounded subsets of $[0, T]\times X$ to bounded subsets of X. If $x\in X$ then for each $t\in[0, T]$ there exists an unique $u(\cdot, \tau, x)\colon [0, T]\to X$ which satisfies equation (3.1).

PROOF. – Local existence and uniqueness are guaranteed by Theorem 3.1 and Lemma 3.1. The local solution can be extended in a manner similar to the preceeding theorem. We need only prove that $\|u(t, \tau, x)\|$ is bounded. Computations similar to those above indicate that if $u(\cdot, \tau, x)$ satisfies (3.1) on an interval $[\tau, T']$ then $\|u(t, \tau, x)\| \leqslant \|x\| + \sup\limits_{t\in[0,T]} \|B(t)0\|$.

We now turn our attention to the construction of a nonlinear evolution operator. Suppose that $\{A(t)|t\in[0, T]\}$ and $\{B(t)|t\in[0, T]\}$ satisfy the conditions of Theogem 4.2. If $x\in X$ and $\tau\in[0, T]$ and $u(t, \tau, x)$ denotes the solution to the integral equation. We define $U(t, \tau)\colon X\to X$ by the equation $U(t, \tau)x = u(t, \tau, x)$. Lemma (3.1) rives $\|U(t, \tau)x - U(t, \tau)y\| \leqslant \|x - y\|$. Since the function

$$u(t, r, u(r, s, x)) = U(r, s)u(r, s, x) = U(t, r)\,U(r, s)x$$

will provide a solution to the integral equation with initial condition $u(s, s, x) = x$ the uniqueness result yields the observation that $U(t, r)\,U(r, s)x = U(t, s)x$ for $0\leqslant s\leqslant t\leqslant T$ and $x\in X$. To see that $U(t, s)$ is continuous in t and s we examine the integral equation and make the observation that for $s > s_0$ we have $\|u(t, s)x - u(t, s_0)x\| \leqslant \|u(s, s_0)x - x\|$. We organize the foregoing remarks as the following theorem.

THEOREM 4.3. – Let $\{A(t)|t\in[0, T]\}$ and $\{B(t)|t\in[0, T]\}$ satisfy the conditions of Theorem 4.2. Then for each t, s such that $0\leqslant s\leqslant t\leqslant T$ there exists a nonlinear operator $U(t, s)\colon X\to X$ with the following properties:

1) $U(t, s)x = W(t, s)x - \int_s^t W(t, \tau)B(\tau)\,U(\tau, s)x\,d\tau$ for $x\in X$.

2) $\|U(t, s)x - U(t, s)y\| \leqslant \|x - y\|$ for $x, y\in X$.

3) $U(t, s)$ is strongly continuous in t and s and $U(t, t) = I$.

4) $U(t, r)\,U(r, s) = U(t, r)$ for $0\leqslant s\leqslant \tau\leqslant t\leqslant T$.

5. – The differentiability of $u(t, \tau, x)$.

If the Banach space X is not reflexive the function $u(t, \tau, x)$ need not be differentiable. WEBB [18] provides an example of a non-differentiable $u(t, \tau, x)$ in case A and B are independent of t. Thus equation (3.1) need not provide a solution to the quasi-linear Banach space differential equation,

$$u'(t, \tau, x) + A(t)u(t, \tau, x) + B(t)u(t, \tau, x) = 0\,; \qquad u(t, \tau, x) = x \in \mathfrak{D}\,, \tag{5.1}$$

even in case the linear equation (1.3) and the integral equation (3.1) are well-posed. In this section we show that solutions to the perturbed equation (5.1) may be guaranteed by either requiring the Banach X to be reflexive or by placing further restrictions on $A(t)$ or $B(t)$.

We now introduce our precise notice of a strong solution to a Banach space differential equation.

DEFINITION 5.1. – Let $\{C(t)|t \in [0, T]\}$ be a family of operators mapping a Banach space X to itself by a strong solution to the Cauchy initial value problem,

$$u'(t, \tau, x) + C(t)u(t, \tau, x) = 0\,; \qquad u(\tau, \tau, x) = x\,, \tag{5.2}$$

we mean a unique Lipschitz continuous function $u(\cdot, T, x)\colon [\tau, T] \to X$ such that $u(\tau, \tau, x) = x$ and $u'(t, \tau, x)$ exists and satisfies (5.2) for a.e. $t \in [\tau, T]$.

Our next theorem pertains to reflexive Banach spaces.

THEOREM 5.1. – Let X be a reflexive Banach space and suppose that $\{A(t)|t \in \in [0, T]\}$ and $\{B(t)|t \in [0, T]\}$ are families of linear accretive operators and nonlinear accretive operators respectively which satisfy the conditions of Theorem (4.2) then for each $x \in \mathfrak{D}$ and $\tau \in [0, \tau)$ there exists a unique strong solution to the Cauchy initial value problem:

$$u'(t, \tau, x) + A(t)u(t, \tau, x) + B(t)u(t, \tau, x) = 0\,; \qquad u(\tau, \tau, x) = x \in \mathfrak{D}\,.$$

PROOF. – Let $u(\cdot, \tau, x)$ denote the solution to the integral equation (3.1). Examination of the properties of the integral equation yields the observation that there exists $K > 0$ so that $\|u(t_1, \tau, x) - u(t_2, \tau, x)\| \leq |t_1 - t_2|K$ for $t, t_2 \in [\tau, T]$. Since X is reflexive $u'(t, \tau, x)$ is differentiable for a.e. $t \in [\tau, T]$. We now argue that $u(t, \tau, x)$ is a solution.

Making use of the continuity properties of $W(t, s)$ and $B(\cdot)$ we choose for $n \in Z^+$ a partition $\{t_i^n\}_{i=1}^{N(n)}$ and a sequence $\{B_i^n\}_{i=1}^{N(n)}$ so that if $t \in [t_{k-1}^n, t_k^n)$ and

$$u_n(t, \tau, x) = W(t, \tau)x - \sum_{i=1}^{k-1} \int_{t_i^n}^{t_i^n} W(t, s) B_{i-1}^n\, ds - \int_{t_{k-1}^n}^{t} W(t, s)\, B_{k-1}^n\, ds \qquad \text{for } t \in [t_{k-1}^n\, t_k] \tag{5.3}$$

we have

$$\|u_n(t, \tau, x) - u(t, \tau, x)\| < 1/n , \qquad \|B^n_{i-1} - B(t)u(t, \tau, x)\| < 1/n$$

for $t \in [t^n_{i-1}, t^n_i]$ and

$$\|B(t)u_n(t, \tau, x) - B(t)u(t, \tau, x)\| < 1/n .$$

Clearly there exists a $K' > 0$ so that for all $t_1, t_2 \in [\tau, T]$ we have

$$\|u_n(t_1, \tau, x) - u_n(t_2, \tau, x)\| \leqslant |t_1 - t_2| K' .$$

Since X is reflexive $u_n'(t, \tau, x)$ exists for a.e. $t \in [\tau, T]$ and $\|u_n'(t, \tau, x)\| \leqslant K'$. Differentiating (5.3) we obtain

$$u_n'(t, \tau, x) = -A(t)u_n(t, \tau, x) - B^n_{k-1} \qquad \text{for a.e. } t \in [t^n_{k-1}, t^n_k) \tag{5.4}$$

and hence

$$u_n'(t, \tau, x) = -A(t)u_n(t, \tau, x) - B(t)u_n(t, \tau, x) + B(t)u_n(t, \tau, x) - B^n_{k-1} , \qquad \text{for a.e. } t \in [t^n_{k-1}, t^n_k] . \tag{5.5}$$

From the foregoing computation it is immediate

$$\|A(t)u_n(t, \tau, x)\| \leqslant K' + \sup \|B(t)u_n(t, \tau, x)\| + 1/n .$$

Thus the reflexiveness of X, the convergence of $u_n(t, \tau, x)$ to $u(t, \tau, x)$ and the closedness of $A(t)$ guarantee that $A(t)u_n(t, \tau, x) \rightharpoonup A(t)u(t, \tau, x)$. Applying an arbitrary $f \in X^*$ to (5.5) we obtain the integral equation

$$\begin{aligned} \langle u_n(t, \tau, x), f\rangle = \langle x, f\rangle - \int_0^t \langle A(s)u_n(s, \tau, x) + B(s)u_n(s, \tau, x), f\rangle\, ds \\ + \sum_{i=1}^{k-1} \int_{t^n_{i|1}}^{t^n_i} \langle B(s)u_n(s, \tau, x) - B^n_{i-1} f\rangle\, ds \\ + \int_{t^n_{k-1}}^{t} \langle B(s)u_n(s, \tau, x) - B^n_{k-1}, f\rangle\, ds . \end{aligned} \tag{5.6}$$

Examination of the two right most terms of (5.6) provides the estimate

$$\left\| \sum_{i=1}^{k-1} \int_{t^n_{i-1}}^{t^n_i} \langle B(s)u_n(s, \tau, x) - B^n_{i-1}, f\rangle\, ds - \int_{t^n_{k-1}}^{t} \langle B(s)u_n(s, \tau, x) - B^n_{k-1}, f\rangle\, ds \right\| \leqslant 2T\|f\|/n .$$

Consequently we take the limit of (5.6) as $n \to \infty$ to observe that

$$\langle u(t, \tau, x), f\rangle = \langle x, f\rangle - \int_{\tau}^{t} \langle A(s)u(s, \tau, x) + B(s)u(s, \tau, x), f\rangle\, ds$$

and hence

$$u(t, \tau, x) = x - \int_{0}^{t} A(s)u(s, \tau, x) + B(s)u(s, \tau, x)\rangle\, ds\,.$$

The fundamental theorem for vector valued functions insures that

$$u'(t, \tau, x) = -A(t)u(t, \tau, x) - B(t)u(t, \tau, x)\,, \qquad \text{for a.e. } t \in [\tau, T]\,.$$

Uniqueness follows by standard techniques.

In non reflexive spaces we must place further restrictions on $\{A(t)|t \in [0, T]\}$ to differentiate $u(t, \tau, x)$.

DEFINITION 5.2. – A family of closed linear densely defined linear m-accretive operators $\{A(t)|t \in [0, T]\}$ satisfying condition E is said to satisfy *condition* E' provided that the linear evolution operator $W(t, s)$ has the property that for each $x \in \mathfrak{D}$ and $s \in [0, T]$, $W'(t, s)x$ exists and satisfies $W'(t, s)x + A(t)W(t, s)x = 0$ for a.e. $t \in [s, T]$.

We have the following theorem for general Banach spaces.

THEOREM 5.2. – Let $\{A(t)|t \in [0, T]\}$ be a family of closed linear densely defined m-accretive operators defined in a Banach space which satisfy E' and suppose that $\{B(t)|t \in [0, T]\}$ is a family of nonlinear accretive operators such that $B(\cdot)$: $[0, T] \times \times X \to X$ is continuous and maps bounded subsets of $[0, T] \times X$ to bounded subsets of X. If either of the following two conditions hold:

i) $W(t, s)X \subseteq \mathfrak{D}$ for $t > s$ and $s, t \in [0, T]$.

or

ii) $B(t)\mathfrak{D} \subseteq \mathfrak{D}$ for $t \in [0, T]$.

then for each $x \in \mathfrak{D}$ and $\tau \in [0, T]$ there exists a unique strong solution $u(\cdot, \tau, x)$ to the Cauchy initial problem

$$u'(t, \tau, x) + A(t)u(t, \tau, x) + B(t)u(t, \tau, x) = 0\,, \qquad u(\tau, \tau, x) = x\,.$$

PROOF. – Theorem 4.2 guarantees the existence $u(\cdot, t, x)$ satisfying $u(\tau, \tau, x) = W(t, \tau)x - \int_{\tau}^{t} W(t, s)B(s)u(s, \tau, x)\, ds$. To establish our assertion we need to differentiate the right side of the equation. As in the preceeding theorem there exists K' so that

$$\|u(t_1, \tau, x) - u(t_2, \tau, x)\| \leqslant |t_1 - t_2| K' .$$

Since $d/dt W(t, \tau)x = -A(t) W(t, \tau)x$ for a.e. t and

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} W(t, s) B(s) u(s, \tau, x) = B(t) u(t, \tau, x)$$

we only need to compute

$$\lim_{h\to 0} \frac{1}{h} \int_\tau^{t+h} (W(t+h), x) - (W(t, s)) B(s) u(s, \tau, x)\, ds .$$

If either (i) or (ii) hold we see that

$$\lim_{h\to 0} \frac{1}{h} (W(t+h), s) - (W(t, s)) B(s) u(s, \tau, x) = -A(t) W(t, s) B(s) u(s, \tau, x) . \tag{5.7}$$

Thus the Lipschitz continuity of $u(\cdot, \tau, x)$ and the boundedness of $A(t) W(t, \tau)x$ and (2.3) allow us to apply the bounded convergence theorem and deduce that

$$\lim_{h\to 0} \frac{1}{h} \int_\tau^{t+h} (W(t+h), s) - (W(t, s)) B(s) u(s, \tau, x)\, ds = -\int_\tau^t A(t) W(t, s) B(s) u(s, \tau, x)\, ds$$

$$= -A(t) \int_\tau^t W(t, s) B(s) u(s, \tau, x)\, ds \qquad \text{for a.e. } t \in [\tau, T] .$$

Once again the uniqueness of solutions follows by standard techniques.

M. G. CRANDALL and A. PAZY [3] have recently studied nonlinear evolution equations and many of their results are applicable to perturbation questions. Their approach is different than ours and a comparison is difficult. Nevertheless, a few comments are in order. They consider Cauchy initial value problems of the form $0 \in u'(t, x) + A(t) u(t, x)$, $u(s, x) = x$ where $A(t)$ is a nonlinear (possibly multivalued) operator; (1.3) they provide conditions sufficient to guarantee the existence of a nonlinear evolution operator, $U(t, s)x = \lim_{n\to\infty} \Pi_1^n \big(I + t/nA(i(t-s)/n)\big)^{-1} x$. The product integral $U(t, s)x$ is shown to be a strong solution of the initial value problem in reflexive spaces and to represent solutions to the initial value problem in genera Banach spaces whenever such solutions exist. Interpreting equation (1.3) as

$u'(t, x) + A_1(t) u(t, x) = 0$, $u(s, x) = x$ where $A_1(t) = A(t) + B(t)$ with $A(t)$

linear and $B(t)$ nonlinear indicates the direct applicability of their work to perturbation problems. Their approach requires that t-continuity requirement be placed on the resolvent $\big(I+\lambda(A(t)+B(t))\big)^{-1}x$. Such a condition can be difficult to check and usually corresponds to perturbations of the form $A_1(t)=A+B(t)$ where A is linear and independent of t and $B(t)$ nonlinear with $B(t)x$ well behaved in x. Our methods do not produce the product integral representation. However, we do not place continuity conditions on the resolvents; we place well-posedness conditions on the linear equation and require that the nonlinear perturbation be continuous.

We conclude with an example which will hopefully illustrate the applicability of our material to quasi-linear partial differential equations. We consider Cauchy problems of the form

$$\frac{\partial u}{\partial t}=\sum_{ni,j=1}^{n} a_{ij}(t)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_1(t)\frac{\partial u}{\partial x_i}-c(t)u^3,\; u(0,x)=f(x) \tag{5.8}$$

where $x=(x_1,\dots,x_n)\in R^n$, $a_{ij}(\cdot)$, $b_1(\cdot)$ and $c(\cdot)$ $(1\leqslant i,j\leqslant n)$ are nonnegative and continuous from $[0,T]$ to R_+ and for each $t\in[0,T]$ the matrix $[a_{ij}(t)]$ is positive definite and symmetric. The analysis for the linear portion of (5.8) comes from J. A. GOLDSTEIN ([5], [6]). We set (5.8) in the Banach space X of bounded continuous functions on R^n that vanish at ∞ and are equipped with the supremum norm. We formally define operators $L(t)$ by the equation

$$\big(L(t)\big)u(x)=\sum_{i,j=1}^{n} a_{ij}(t)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x)+\sum_{i=1}^{n} b_i(t)\frac{\partial u}{\partial x_i}(x)$$

and we define a family of operators $\{A(t)|t\in[0,T]\}$ by the equation $A(t)f$ of $=L(t)f$ for f such that,

$$f\in C_0^2=\{g|g,\,\partial g/\partial x_i,\,\partial^2 g/\partial x_1\,\partial x_j\in X,\ 1\leqslant i,j\leqslant n \text{ and } g \text{ has compact support in } R^n\}.$$

If $f\in\mathfrak{D}=C_0^2$ we see that $\{A(t)|t\in[0,T]\}$ satisfies condition E', [5]. Moreover if we define $B(t)u=c(t)u^3$ we immediately see that $B(\cdot)\colon[0,T]\times X\to X$ is accretive, continuous, and maps bounded subsets to bounded subsets. Thus if $W(t,s)$ denotes the linear evolution operator generated by $\{A(t)|t\in[0,T]\}$, $f\in X$ and $\tau\in[0,T)$, there exists a unique $u(t,\tau,f)$ so that

$$u(t,\tau,f)=W(t,\tau)f-\int^{t} W(t,s)B(s)u(s,\tau,f)\,ds\,,\qquad u(\tau,\tau,f)=f\,.$$

We are consequently guaranteed the existence of a unique « mild solution » for 5.8 for any initial function $f\in X$. Further more it is readily apparent that the image of under $B(\cdot)$ is continued in $\mathfrak{D}$. Thus if the initial function $f\in\mathfrak{D}$ Theorem 5.2 guarantees the existence of a unique a.e. solution to (5.8).

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