Parametrized $\bar{\partial}$ Operator on Pseudoconvex Sets (*) (**).

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Sunto. — *Si dimostra l'esistenza della soluzione per l'operatore di Cauchy-Riemann parametrizzato, nel caso continuo per varietà fortemente pseudoconvesse e nel caso differenziabile per varietà di Stein.*

§ 1. — Introduction.

For a given smoothly parametrized operator P_t of constant degree there is generally no continuously parametrized solution of the equation $P_t u_t = f_t \in RP_t$, even when f_t is regular with respect to the pair parameter-variable. A simple example is given by the equation $u_0 + t^2 \partial u_0 = (t^2 x - y) u_x + (t^2 y + x) u_y = t$ on the plane. In the present paper we show that the above fact does not happen if P_0 is the $\bar{\partial}$ operator on any strongly pseudoconvex set [prop. (4.1)] and we give a smoothly parametrized solution for the Stein case ⁽¹⁾. We work on a C^∞ family $\mathcal{W}$ of complex manifolds X_t ; the space of the parameter t is a ball B of $\mathbb{R}^m$ and could be reduced to a segment without any real loss of generality. Consider the equation

$$(1.1) \quad \bar{\partial} v = f$$

(it should be written $\bar{\partial}_t v_t = f_t$) on forms with values in a C^∞ vector bundle E such that $E_t = E|X_t$ is a holomorphic bundle (precise details about the algorithm are given in next section). In section 6 we shall prove the following

(1.2) THEOREM. — *Let $\mathcal{W} = \{X_t\}_{t \in B}$ be a C^∞ family of complex manifolds with X_0 Stein. Each compact set $K \subset X_0$ has a neighborhood Ω in $\mathcal{W}$ such that, for every smooth family $(f_t)_{t \in B}$ of C^∞ $(p, q+1)$ -forms with values in the fiber bundle E_t which satisfies $\bar{\partial}_t f_t = 0$, there exists on Ω a C^∞ family $(v_t)_{t \in B}$ of (p, q) -forms which resolves (1.1).*

Equation (1.1) is considered here with vector bundle valued forms out of necessity and not for excessive, apparent generality. Actually our proof goes through the existence of a continuous solution for the holomorphic tangent bundle which is, by the way, just of the type of E .

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⁽¹⁾ Indeed our manifold at time 0 need not to be Stein but only « limit of Stein » (i.e. every compact set has a Stein neighborhood), but since we do not know whether this hypothesis is really more general, we assume it to be Stein.

In section 7 we give two applications of this theorem:

pseudorigidity of Stein manifolds is proved for C^∞ deformations,

two proofs of the complex Frobenius theorem of L. NIRENBERG are given.

The first one is based on pseudorigidity and Newlander-Nirenberg theorem; the second one (only sketched) is a direct application of the present methods to KOHN proof ([7]) of the integrability theorem for almost complex structures.

§ 2. – *a)* Our total space is the product of a connected differential manifold X and an open ball B of $\mathbb{R}^m$. At each point (x, t) of $X \times B$ is defined a tensor $J: T_x X \rightarrow T_x X$ with $J^2 = -id$ which depends smoothly on (x, t) and satisfies the integrability conditions which are necessary (and sufficient) to give each $X \times \{t\}$ the structure of a complex manifold. We will note this complex manifold by X_t and the product $X \times B$ with this additional structure by $\mathcal{W}$. $\mathcal{W}$ is a C^∞ family of complex manifolds. It is useful to introduce the trivial map $\mathcal{G}: X \times B \rightarrow \mathcal{W}$ and the natural projection $\pi: \mathcal{W} \rightarrow B$. By the Frobenius-Nirenberg theorem [8] the structure of family of complex manifolds given by the tensor J can be described by an atlas (W_j, χ_j) of diffeomorphisms

$$\chi_j: W_j \rightarrow \text{open subset of } \mathbb{C}_{z_j}^n \times \mathbb{R}_t^m$$

in such a way that the t -component of $\chi_j(x, t)$ is t , the coordinates z_h are holomorphic functions of z_j in $W_j \cap W_h$ for fixed t and the complex structure on $X \times \{t\}$ is just the same that was induced by J , i.e. X_t .

Let $\varphi_0 \in C^\infty(X_0)$ be the function, plurisubharmonic at $\{\varphi_0 = 0\}$, defining $\Omega_0 = \{\varphi_0 < 0\}$ and φ be its trivial extension to $\mathcal{W}$ given by $\varphi[\mathcal{G}(x, t)] = \varphi_0(x)$. The restriction φ_t of φ to X_t defines a set $\Omega_t \subset X_t$ which is obviously still strongly pseudoconvex for $|t|$ small enough, say $|t| < \delta$. Thus we can suppose, up to an unessential restriction, that the basis ball B is just $|t| < \delta$. The open subset Ω of $\mathcal{W}$ given by $\varphi < 0$ is the neighborhood of Ω_0 mentioned in theorem (1.2).

If Ω_0 is Stein, a classical procedure shows that, up to a convenient restriction of B , by taking a sufficiently increasing and convex function $F: \mathbb{R} \rightarrow \mathbb{R}$, $\psi = F(\varphi)$ and the minimum eigenvalue λ_ψ of its Levi form can be assumed to be greater than any given continuous function on Ω , [9].

b) The complexified tangent bundle of $\mathcal{W}$ has a sub-bundle $\mathbf{T}$ of vectors which are linear combinations of $\partial/\partial z_1, \dots, \partial/\partial z_n$. The restriction $\mathbf{T}_t$ of $\mathbf{T}$ to X_t is the usual complex tangent bundle. Let Λ^{10} be the dual bundle of $\mathbf{T}$ and Λ^{01} its conjugate bundle. At each point $\zeta \in \mathcal{W}$, the fibers $\Lambda^{10}(\zeta)$ and $\Lambda^{01}(\zeta)$ generate an exterior algebra $\bigoplus_{p+q=n} \Lambda^{p,q}(\zeta)$. The vector space $\Lambda^{p,q}(\zeta)$ is the fibre of a bundle $\Lambda^{p,q}$ over $\mathcal{W}$ whose restriction $\Lambda_t^{p,q}$ to X_t is the usual bundle of (p, q) -forms on X_t . Note that $\Lambda^{p,q} = \Lambda^{p,0} \otimes \Lambda^{0,q}$. Let E be a complex C^∞ vector bundle on $\mathcal{W}$ whose restriction E_t to X_t is a holo-

morphic bundle. Two examples of such bundles are T and A^{p0} . In this paper we shall consider (p, q) -forms with values in an arbitrary bundle like E above, i.e. with cross sections of $A^{pq} \otimes E$. Since $A^{p0} \otimes E$ is still of the same type of E , it is no restriction to assume $p = 0$. For any subset A of $\mathcal{W}$, let $C_q^k(A)$ be the space of C^k sections of $A^{0q} \otimes E$ by respect to the z 's. Over any open neighborhood of A , $0 \leq k \leq \infty$. Then the operator $\bar{\partial}: C_q^k(\mathcal{W}) \rightarrow C_{q+1}^k(\mathcal{W})$ is well defined and its restriction $\bar{\partial}_i$ to X_i is the usual $\bar{\partial}$ operator on X_i for $(0, q)$ -forms with values in E_i . Thus the form f appearing in (1.1) belongs to $C_{q+1}(\mathcal{W}) \equiv C_{q+1}^\infty(\mathcal{W})$ and u belongs to $C_q(\mathcal{Q}) \equiv C_q^\infty(\mathcal{W})$.

§ 3. — We recall briefly some classical results which can be found in [7] and [4], concerning the Laplace-Beltrami operator for a smooth, strongly pseudoconvex open subset A of a hermitian manifold X . Let be $A \equiv \{\varphi < 0\}$, where φ is a smooth function on X ; $d\varphi$ does not vanish at the boundary $\partial A \equiv \{\varphi = 0\}$. We shall consider $(0, q)$ -forms on X with values in a holomorphic hermitian vector bundle E ; the space of those among them which are C^∞ will be noted by C_q . $\mathcal{D}_q$ will denote the subspace of those with compact support. The hermitian product $\langle \cdot, \cdot \rangle$ on the fiber $E_x \otimes A_x^{0q}$ is defined by putting $\langle v \otimes \chi, u \otimes \theta \rangle = \langle v, u \rangle \langle \chi, \theta \rangle$, $(\cdot, \cdot) = \int_A \langle \cdot, \cdot \rangle \cdot e^{-\varphi} dV$ is the $\mathfrak{L}^2$ hermitian product, $\|u\| = (u, u)^{\frac{1}{2}}$ is the $\mathfrak{L}^2$ norm. Let be T and S the closed $\mathfrak{L}^2$ extension of the operators $\bar{\partial}: C_q(\bar{A}) \rightarrow C_{q+1}(\bar{A})$ and $\bar{\partial}: C_{q+1}(\bar{A}) \rightarrow C_{q+2}(\bar{A})$. The main result of the theory says that the strong pseudoconvexity of A implies that the operators S , T^* and $L = TT^* + S^*S$ have a closed range ([7]), hence we have

$$(3.1) \quad \|u\| \leq C \|Lu\|, \quad \text{for } u \in D_L.$$

D_L is the domain of L .

(3.2) **REMARK.** — The constant C is a continuous function of the metrics on T and E , of φ and their derivatives.

We fix now orthonormal frames $\omega^z(x)$ and $e_a(x)$ for T_x^* and E_x , depending smoothly on x in each coordinate neighborhood. Blocks of indices are always supposed to be strictly increasing, ω^A stands for $\omega^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_p}$, with $A = \alpha_1, \dots, \alpha_p$, $|A| = p$, and we put $\bar{\omega}_a^A = e_a \otimes \bar{\omega}^A$. Thus the punctual hermitian product between the vector valued forms $f = \sum_{aA} f_a^A \omega_a^A$ and $g = \sum_{aA} g_a^A \omega_a^A$ is given by $\langle f, g \rangle = \sum_{aA} f_a^A \bar{g}_a^A$. The operator ∇_α on functions is defined by $\bar{\partial}u = \sum_{aA} \bar{\nabla}_\alpha u \cdot \bar{\omega}^A$. By putting $\delta_\alpha u = e^\varphi \nabla_\alpha (u e^{-\varphi})$ we get the formal adjoint ϑ of $\bar{\partial}$ on $\mathcal{D}_q(X)$. ϑ is defined by $(g, \bar{\partial}f) = (\vartheta g, f)$ (f and g are smooth and one of them has compact support), and explicitly given by $\vartheta g = - \sum_{\alpha a A} \delta_\alpha g_a^A \omega_a^A +$ + linear part. The Green formula is

$$(3.3) \quad (f, \bar{\partial}g) = (\vartheta f, g) + \int_{\partial A} \langle f, g \wedge \bar{\partial}\varphi \rangle dS, \quad \text{with } f \in C_{q+1}(\bar{A}), g \in C_q(\bar{A}),$$

where dS is a suitable positive measure on ∂A . The restriction of L to $\mathcal{D}_q$ is the Laplace-Beltrami operator $\square = \bar{\partial}\partial + \partial\bar{\partial}$. By successive applications of Green formula (3.3) we get

$$(3.4) \quad (f, \square \varphi w) = (\square f, \varphi w) + \int_{\partial A} |\bar{\partial} \varphi|^2 \langle f, w \rangle dS, \quad \text{for } f, w \in C_q(\bar{A}).$$

(3.5) PROPOSITION. – *The form $f \in C_q(\bar{A})$ is in the domain D_L of L if and only if $f = \varphi^2 g$, with $g \in C_q(\bar{A})$.*

PROOF. – f is in D_L if and only if there exists a constant c such that

$$(3.6) \quad |(f, \square u)| \leq c \|u\|, \quad \text{for each } u \in C_q(\bar{A}).$$

Take any form $w \in C_q(\bar{A})$ and replace u in (3.6) by φw . Since we may have $\|\varphi w\| \leq 1$, with w arbitrarily large at the boundary, we conclude from (3.4) that f must vanish at ∂A i.e. $f = \varphi h$. Now we conjugate (3.4) with w replaced by h and f by u and we have

$$(3.7) \quad (f, \square u) = (\square f, u) - \int_{\partial A} |\bar{\partial} \varphi|^2 \langle h, u \rangle dS.$$

The same argument as before says that h must vanish at the boundary i.e. $h = \varphi g$, which means $f = \varphi^2 g$. Conversely (3.7) shows that, when $f = \varphi^2 g$, then $|(f, \square u)| \leq \|\square f\| \|u\|$. Q.E.D.

Our purpose is to prove the existence of a solution for the perturbed operator T_t which is regular with respect to the parameter. Since the parameter regularity is generally easier to prove when the solution is unique, it is more convenient to reduce the problem to one which has the unicity property. The above argument is the only reason we have to employ the Laplace-Beltrami operator $\square$.

All that we need of this theory can be resumed in the following.

(3.8) PROPOSITION. – *If u is the (unique) solution in D_L of the equation $Lu = f$, with $f \in C_q(\bar{A})$ and $\bar{\partial} f = 0$, then u is smooth because of the ellipticity of L , and $v = \partial u$ is the unique solution of the equation $\bar{\partial} v = f$ which is orthogonal to $\ker T$.*

PROOF. – It is enough to show that $f = \bar{\partial} \partial u$, i.e. that $w = \partial \bar{\partial} u$ vanishes. The ranges of T and T^* are closed, this implies ([4], p. 140) the inequality

$$(3.9) \quad \|g\| \leq \text{const} \|Tg\|, \quad \text{for each } g \in D_T \cap R_{T^*}.$$

Now the form $w \in C_q(\bar{A})$ belongs to $D_T \cap C_q(\bar{A})$. Moreover, since u vanishes at the boundary ∂A by Prop. (3.5), formula (3.3) gives that $\bar{\partial} u$ is in D_{T^*} . Thus $w \in R_{T^*}$. Finally the condition $\bar{\partial}(f = w + \bar{\partial} \partial u) = 0$ gives $Tw = \bar{\partial} w = 0$ and by (3.9) we have $w = 0$. Q.E.D.

§ 4. – The continuous solution for the tangent bundle T .

We refer to the notation of § 2; consider again the open set $\Omega \subset \mathcal{W}$ of § 2a and suppose that E is the bundle T introduced in § 2b. A cross section $f \in C_{q+1}(\mathcal{W})$ is given; for every $t \in B$ consider the extended Laplace-Beltrami operator L_t relative to the pseudoconvex subset Ω_t of X_t as in § 3 and to the function $\varphi_t = \varphi|_{X_t}$. Suppose that $f_t \in RL_t$, for each t . Then as mentioned in (3.8), there exists a unique solution $u_t \in D_{L_t}$ of the equation $\square u_t = f_t$. The collection $(u_t)_{t \in B}$ gives obviously a cross section u over $\bar{\Omega}$ of the bundle $\Lambda^{0q} \otimes T$. The continuity of u is implicitly proved in [7], p. 144. We give here a more detailed proof which make use of some inequalities which will be useful later.

(4.1) PROPOSITION. – u is a continuous section of $\Lambda^{0q} \otimes T$.

Before starting with the proof we need to adapt to our case some classical inequalities for elliptic operators. For every subset $A_0 \subset\subset X_0$, put $A = \mathfrak{C}^{-1}(A_0 \times B)$ and $A_t = A \cap X_t = \mathfrak{C}^{-1}(A_0 \times \{t\})$. Since $\bar{\Omega}$ is compact it can be covered by a finite number $W_1, \dots, W_N$ of cards of the atlas mentioned in § 2a; in what follows the sets W_j will be fixed. A straightforward compactness argument shows that the ray δ of B can be chosen so small that $\pi W_j = B$ for each $j \leq N$. We denote by $\tilde{C}_q(\bar{A})$ the space of the families $\{f_j\}_{j \leq N}$ of C^∞ cross sections of $\Lambda^{0q} \otimes T$ over W_j and by $\tilde{C}_q(\bar{A}_t)$ the similarly defined space for $t \in B$. Although the elements of those spaces are not sections of $\Lambda^{0q} \otimes T$, nevertheless the Laplace-Beltrami operator $\square_t$ of X_t operates on $\tilde{C}_q(\bar{A}_\tau)$ and even on $\tilde{C}_q(\bar{A}_t)$ with $\tau \neq t$ as we shall see now. In the coordinate neighborhood W_j write the operator $\square$ in the form

$$(\square v)_A^a(z, t) = \sum_{bB\alpha} L_{AB\alpha}^{ab}(z, t) (D^\alpha v_B^b)_{(z,t)}; \quad b \leq N, |B| = q, |\alpha| \leq 2,$$

where $L_{AB\alpha}^{ab}$ are C^∞ functions on $\bar{W}_j$ and, for each $v_\tau \in \tilde{C}_q(\bar{A}_\tau)$, put

$$(4.2) \quad (\square_t v_\tau)_A^a(z, t) = \sum_{bB\alpha} L_{AB\alpha}^{ab}(z, t) (D_\alpha v_B^b)_{(z,\tau)}; \quad b \leq n, |B| = q, |\alpha| \leq 2.$$

We shall use in $\tilde{C}_q(\bar{\Omega}_t)$ and $\tilde{C}_q(\bar{A})$ the Sobolev norms given by

$$(\|v_t\|_s^A)^2 = \left\{ \sup (-2i)^n \int_{\bar{A} \cap W_j} |D^\alpha v_B^b|_{(z,t)}^2 dz \wedge d\bar{z}; \quad |\alpha| \leq s, j \leq N, b \leq n, |B| = q \right\},$$

where the derivatives D^α are made *only with respect to the variables $z, \bar{z}$* , and by

$$\|v\|_s^A = \sup_{t \in B} \|v_t\|_s^A, \quad (v_t = v|_{X_t}).$$

It will be also useful to introduce the position

$$(\|v_\tau - v_t\|_s^A)^2 = \sup_{A \cap \mathcal{W}_j} \left\{ (-2i)^n \int |D^\alpha [v_B^b(z, \tau) - v_B^b(z, t)]|^2 dz \wedge d\bar{z}; |\alpha| \leq s, j \leq N, b \leq n, |B| = q \right\}.$$

From the regularity of the coefficients $L_{AB\alpha}^{ab}$ appearing in (4.2) we get immediatly the following inequalities

$$(4.3) \quad \|(\square_\tau - \square_t)v_\tau\|_s^A \leq k_1 \|v_\tau\|_{s+2}^A |t - \tau|,$$

$$(4.4) \quad \|\square_\tau v_\tau\|_s^A \leq k_2 \|v_\tau\|_{s+2}^A$$

which are valid for every t and τ in B , $v \in \tilde{C}_q(\bar{A}_\tau)$ and the constants k_1, k_2 do not depend on t and τ .

$$(4.5) \quad \text{PROPOSITION.} - \text{If } u \text{ is the above cross section of } \Lambda^{0q} \otimes T \text{ then, for every } t \in B, \lim_{\tau \rightarrow t} \|u_t - u_\tau\|^\Omega = 0.$$

PROOF. - The fiber preserving diffeomorphism $\mathfrak{C}: \bar{\Omega} \rightarrow \bar{\Omega}_0 \times B$ introduced in section § 2a induces a diffeomorphism $\mathfrak{C}_\tau: \bar{\Omega}_t \rightarrow \bar{\Omega}_\tau$ which is the $\mathfrak{C}^{-1}$ -image of the trivial map $\bar{\Omega}_0 \times \{t\} \rightarrow \Omega_0 \times \{\tau\}$. $\mathfrak{C}_\tau \zeta$ depends smoothly on (ζ, t, τ) . Let $t \in B$ be fixed and let $\mathfrak{C}_\tau^q$ be the composed map

$$\mathfrak{C}_\tau^q: \Lambda^{0q} \otimes T_t \rightarrow \bigoplus_{r+m=q} [\Lambda_\tau^m \otimes (T_\tau \oplus \bar{T}_\tau)] \rightarrow \Lambda^{0q} \otimes T;$$

The first arrow above is induced by $\mathfrak{C}_\tau$ and the second is the natural projection. The map $\mathfrak{C}_\tau^q$ induces a map $C_q(\bar{\Omega}_t) \rightarrow C_q(\bar{\Omega}_\tau)$, is the identity for $t = \tau$, and depends smoothly on all its arguments i.e.: for each $w_t \in C_q(\bar{\Omega}_t)$ the cross section w of $\Lambda^{0q} \otimes T$ on $\bar{\Omega}$ given by $w_\tau = \mathfrak{C}_\tau^q w_t$ (τ varying in B) is a C^∞ extension of w_t . Since $u_t \in D_{L_t} \cap C_q(\bar{\Omega}_t)$ by prop. (3.5) it is $u_t = \varphi^2 w_t$, $w_t \in C_q(\bar{\Omega}_t)$. The above procedure gives a C^∞ extension w of w_t . Since w is regular up to the boundary of Ω , there exists a constant k_3 such that

$$(4.6) \quad \|u_t - \varphi^2 w_\tau\|_2^{\bar{\Omega}} \leq k_3 |t - \tau|, \quad \text{for each } \tau \in B.$$

The choice of w in $C_q(\bar{\Omega})$ depends only on t which is fixed here. Thus we have

$$(4.7) \quad \|\varphi^2 w_\tau\|_2^\Omega \leq k, \quad \text{for each } \tau \in B.$$

From formula (3.1) and taking account of the remark (3.2) there exists a constant C such that we have

$$(4.8) \quad \|v_\tau\|^\Omega \leq C \|L_\tau v_\tau\|^\Omega, \quad \text{for each } v_\tau \in D_{L_\tau}, \text{ and each } \tau \in B.$$

Now u_τ belongs to D_{L_τ} by hypothesis and $\varphi^2 w_\tau$ is also in D_{L_τ} by prop. (3.5). Hence the inequality (4.8) can be applied to $u_\tau - \varphi^2 w_\tau$; and that gives

$$(4.9) \quad \|u_\tau - \varphi^2 w_\tau\|^2 \leq C \|\square_\tau(u_\tau - \varphi^2 w_\tau)\|^2, \quad \text{for each } \tau \in B.$$

We recall that the choice of w depends on t and shall prove that

$$(4.10) \quad \lim_{\tau \rightarrow t} \|\square_\tau(u_\tau - \varphi^2 w_\tau)\|^2 = 0.$$

Actually a direct application of the above inequalities (4.3, 4, 6, 7) gives

$$\begin{aligned} \|\square_\tau(u_\tau - \varphi^2 w_\tau)\|^2 &\leq \|\square_\tau u_\tau - \square_t u_t\|^2 + \|\square_t(u_t - \varphi^2 w_t)\|^2 + \\ &+ \|(\square_\tau - \square_t)\varphi^2 w_\tau\|^2 \leq \|f_\tau - f_t\|^2 + k_2 k_3 |t - \tau| + k_1 k |t - \tau|. \end{aligned}$$

The three terms of the last inequality are obtained as follows:

the first one from the fact that $\square u = f$, the second one from (4.4) and (4.6), the last one from (4.3) and (4.7). Since $f \in C_{q+1}(\bar{\Omega})$ the last inequality proves (4.10). Then the statement of prop. (4.5) can be derived from the fact that both terms at the left side of (4.6) and (4.9) tend to zero as τ tends to t . Q.E.D.

In order to prove the continuity of the solution u we remark that, for every pair A_0, A'_0 , with $A_0 \subset\subset A'_0 \subset\subset \Omega_0$ and $s \geq 0$, the constant appearing in the classical Friedrichs inequality

$$(4.11) \quad \|v_t\|_{s+2}^A \leq \text{const} \{ \|\square_t v_t\|_s^{A'} + \|v_t\|_s^{A'} \}, \quad \text{for each } v_t \in \tilde{C}_q(\bar{\Omega}_t),$$

can be chosen independent of t . Actually, by the regularity of the coefficients $L_{AB\alpha}^{ab}(z, t)$ appearing in (4.2), the punctual ellipticity constant

$$c(z, t) = \min \left\{ \sum_{aA} \left| \sum_{bB} \sum_{|\alpha|=2} L_{AB\alpha}^{ab}(z, t) \xi^\alpha \omega_b^B \right|, \xi \in \mathbb{C}^n, |\xi| = 1; \omega \in \mathbb{C}^{(n)_n}, |\omega| = 1 \right\}.$$

is bounded from above by a positive constant on each coordinate set $W_1, \dots, W_N$. Thus the standard Friedrichs procedure gives (4.11). First of all we deduce that, for each $A_0 \subset\subset \Omega_0$ the norms $\|u_\tau\|_s^A$ are bounded independently from τ . Actually by (4.11) we have

$$\|u_\tau\|_{s+2}^A \leq \text{const} \{ \|f_\tau\|_s^2 + \|u_\tau\|_s^2 \} \leq \text{const} \|f_\tau\|_s^2;$$

the last inequality being derived by (4.8). Thus we can write

$$(4.12) \quad \|u_\tau\|_s^A < c, \quad \text{for each } s \geq 0$$

where the constant c depends only on s and A_0 .

PROOF OF PROPOSITION (4.1). — Let (z, t) and (ζ, τ) be points in the same coordinate neighborhood W_j . We shall drop the subscript j (which is fixed) because our algorithm is already sufficiently ornate. A straightforward application of the Sobolev inequality gives, for s and s' large enough, $b \leq n$, $|B| = q$

$$(4.13) \quad |u_B^b(z, t) - u_B^b(\zeta, \tau)| \leq \text{const } \|u_t\|_{s'}^W |z - \zeta| + \text{const } \|u_t - u_\tau\|_{s+2}^W.$$

As, by (4.12), the first part tends to zero for $(\zeta, \tau) \rightarrow (z, t)$, hence we have only to prove that

$$(4.14) \quad \lim_{\tau \rightarrow t} \|u_t - u_\tau\|_{s+2}^W = 0.$$

Take some open subset A_0 of X_0 with $W \cap X_0 \subset A_0 \subset \Omega_0$ and apply (4.11) with A replaced by W and A' by A ; that gives

$$(4.15) \quad \|u_t - u_\tau\|_{s+2}^W \leq \text{const } \{ \|f_t - f_\tau\|_s^A + \|(\square_t - \square_\tau)u_\tau\|_s^A + \|u_t - u_\tau\|^4 \}.$$

We now show that the three terms of the right hand side of the last inequality tend to zero as $\tau \rightarrow t$, this will prove (4.14) and the proposition.

The first term tends to zero because f belongs to $C_\alpha(\bar{\Omega})$; by (4.3) the second one is smaller than $k_1 \|u_\tau\|_{s+2}^A |t - \tau| \leq k_1 c |t - \tau|$, see (4.12); the third one is smaller than $\|u_t - u_\tau\|^\Omega$ which tends to zero by prop. (4.5). Q.E.D.

(4.16) REMARK. — The inequality (4.13) is still valid if $D^\alpha u_B^b$ (derivatives only with respect to z and $\bar{z}$ variables) replaces u_B^b , provided s and s' are sufficiently large. This means that *all derivatives of u with respect to the variables z and $\bar{z}$ are also continuous functions of (z, t) .*

An immediate consequence of the above remark and of (3.8) is the following.

(4.17) PROPOSITION. — *If $\bar{\partial}f = 0$ and $f_t \in RL_t$, $\forall t$ (the last condition is empty when Ω is Stein), then the form $v_t = \partial_t u_t$ is continuous in $\bar{\Omega}$ together with all its z and $\bar{z}$ derivatives. It is the unique solution of the equation $\bar{\partial}v = f$ such that, for each $t \in B$, v_t is orthogonal to the kernel of T_t .*

For further applications we note that, for a fixed coordinate neighborhood $W \subset \Omega$ we have

$$(4.18) \quad |v_t|_1^W \leq \text{const } |f_t|_{n+1}^\Omega,$$

the constant depends only on Ω_0 and W_0 .

That follows from the Sobolev inequality $|v_t|_1^W \leq \text{const } |u_t|_2^W \leq \text{const } \|u\|_{n+3}^W$, the Friedrichs inequality (4.11) $\|u_t\|_{n+3}^W \leq \text{const } \{ \|f_t\|_{n+1}^\Omega + \|u_t\|_0^\Omega \}$, from (4.8), which gives $\|u_t\|_0^\Omega \leq C \|f_t\|_0^\Omega$, and the fact that $\|f_t\|_{n+1}^\Omega$ is smaller than $\text{const } |f_t|_{n+1}^\Omega$.

§ 5. – Some lemmata for the product case.

The following lemmas give some statement for the parametrized C.R. equation

$$(5.1) \quad \bar{\partial}u = f$$

for C^k forms ($k = 0, \dots, \infty$) of type $(0, q)$ with values in a C^k family of holomorphic factor bundles on a *single* strongly pseudoconvex manifold Y (there exists on Y an exhaustive function which is plurisubharmonic outside a compact subset). The space of the parameter t is the open ball B of $\mathbb{R}^m$. The family mentioned above is a C^k complex vector bundle $F \xrightarrow{p} Y \times B$ whose restrictions $F_t \stackrel{\text{def}}{=} F|(Y \times \{t\})$ are holomorphic bundles. Let Λ_0^{0q} be the fiber bundle of $(0, q)$ -forms on Y and let Λ^{0q} be the pull-back of Λ_0^{0q} by natural projection $Y \times B \rightarrow Y$. Our forms are sections of $\Lambda^{0q} \otimes F$. The space of C^k forms of this type *whose restrictions to $Y \times \{t\}$ are C^∞ sections of F_t will be noted by C_q^k* . We note first that the problem (5.1) can be easily reduced to the equation

$$(5.2) \quad \bar{\partial}u_t = f_t, \quad t \in B$$

where u_t and f_t are C^k families of sections of a *single* bundle $F_0 \rightarrow Y$.

Actually, by well known results of Grauert on analytic bundles on Stein spaces ([2], see also [1]), since the bundles F_t are continuously homotopic, they are isomorphic as holomorphic bundles and the isomorphism $\phi_t: F_0 \rightarrow F_t$ can be chosen to be a C^k homotopy. More precisely there exists a C^k vector bundle isomorphism $\phi: F_0 \times B \rightarrow F$ such that $\phi(F_0 \times \{t\}) \subset F_t$ and $\phi_t: F_0 \times \{t\} \rightarrow F_t$ is a holomorphic vector bundle isomorphism. For sections $u, f, \dots$ etc. of $\Lambda^{0q} \otimes F$ we shall note by $\tilde{u}, \tilde{f}, \dots$ the ϕ^{-1} -corresponding sections of $\Lambda^{0q} \otimes (F_0 \times B)$. We have $\bar{\partial}\tilde{u} = (\bar{\partial}u)^\sim$. C_q^k has a natural Fréchet topology: Take a locally finite, relatively compact atlas (U_j, z_j) of Y such that $F_0|_{U_j}$ is trivial for each j and a basis $e_a^{(j)}$ of C^∞ sections of $F_0|_{U_j}$ ($a = 1, \dots, N = rk F$). The $e_a^{(j)}$ can be trivially extended as section $\tilde{e}_a^{(j)}$ of $(F_0 \times B)|(U_j \times B)$ and $e_a^{(j)} = \phi \circ \tilde{e}_a^{(j)}$ are a basis of $C_q^k(U_j \times B)$ (in particular $F|(U_j \times B)$ is trivial). Any form $u \in C_q^k$ can be written as follows

$$u(z, t) = \sum_{a_j}^N \sum_{|A_j|=q} u_{A_j}^{a_j}(z, t) d\bar{z}_j^{A_j} \otimes e_a^{(j)}.$$

Then the Fréchet topology is induced by the seminorms

$$|u|_s^L = \sup \{ |D_t^\alpha D_{z_j}^\beta u_{a_j}^{A_j}|_{(z,t)}, (z, t) \in L, (U_j \times B) \cap L \neq \emptyset, a_j \leq N, |A_j| = q, |\alpha| \leq k, |\beta| \leq s \}$$

where s is a non negative integer and L is a compact subset of $Y \times B$. We note by $C_q(Y, F_0)$ the Fréchet space of F_0 -valued $(0, q)$ -forms of class C^∞ on Y . ϕ induces a topological isomorphism between $C_q(Y, F_0) \hat{\otimes} C^k(B)$ and C_q^k which commutes with

the $\bar{\partial}$ operator. No confusion is possible for the topological tensor product because of the nuclearity of $C_q(Y, F_0)$. The first lemma we shall state occurs often in the literature and, roughly speaking, says that for a given differential operator $P: C^\infty \rightarrow C^\infty$, with closed range, the equation $Pu_i = f_i \in RP$ admits a parametrized solution with the same regularity of f_i . A proof based on Knneth formula for sheaves can be found in [9] prop. 10.1. We shall give here an other proof suggested by two Grothendieck's remarks.

(5.3) LEMMA. – *For every collection $(b_i)_{i \in B}$ of forms $b_i \in C_q(Y, F_0)$ such that $\bar{\partial}b_i$ defines a form g in C_{q+1}^k , there exists $v \in C_q^k$ such that $\bar{\partial}v_i = \bar{\partial}b_i$; $k = 0, \dots, \infty$.*

PROOF. – Since the image $B_{q+1} = \bar{\partial}C_q$ has finite codimension in $Z_{q+1} = \ker \{\bar{\partial}: C_{q+1} \rightarrow C_{q+2}\}$ which is closed, B_{q+1} is also closed in the C^∞ topology. By the Banach open map theorem, $\bar{\partial}$ induces a topological isomorphism $\bar{\partial}_*: C_q/Z_q \rightarrow B_{q+1}$. The C^k map $\bar{\partial}_*^{-1} \circ \bar{g}: B \rightarrow C_q/Z_q$ can be lifted to a C^k map $\tilde{v}: B \rightarrow C_q$. Actually, for $k = 0$ this is a lemma of Cartan ([1], p. 119). For $0 \leq k < +\infty$ $C^k(B)$ is a direct factor of a space of continuous functions on a locally compact space ([3] foot note p. 84) and so we are reduced to the case $k = 0$. Finally, for $k = \infty$ the statement follows directly by the nuclearity of $C^\infty(B)$, see [3], pp. 83-84. Hence if we note by p' the natural projection $C_q \rightarrow C_q/Z_q$, we have $\bar{\partial}\tilde{v} = \bar{\partial}_*p'\tilde{v} = \bar{g}$, then $\bar{\partial}v = g = \bar{\partial}b$. Q.E.D.

In what follows we shall note by Z_q^k the kernel of $\bar{\partial}: C_q^k \rightarrow C_{q+1}^k$.

(5.4) LEMMA. – *For $h \leq k$, C_q^k is dense in C_q^h and Z_q^k is dense in Z_q^h .*

PROOF. – Since $C_q(Y, F_0)$ and $Z_q(Y, F_0)$ are nuclear, by th. 5 of [3], $C_q(Y, F_0) \hat{\otimes} C^k(B)$ is dense in $C_q(Y, F_0) \hat{\otimes} C^h(B)$ and $Z_q(Y, F_0) \hat{\otimes} C^k(B)$ is dense in $Z_q(Y, F_0) \hat{\otimes} C^h(B)$. Then the statement follows directly by applying the isomorphism ϕ to all these spaces. Q.E.D.

(5.5) LEMMA. – *For every fixed compact $L \subset Y \times B$, every $b \in C_q^k$ such that $\bar{\partial}b \in C_{q+1}^{k+1}$ and $\varepsilon > 0$, there exists $u \in C_q^{k+1}$ such that $\bar{\partial}u = \bar{\partial}b$ and $\|b - u\|_k^L < \varepsilon$.*

PROOF. – By lemma (5.3) there exists $v \in C_q^{k+1}$ such that $\bar{\partial}v = \bar{\partial}b$. Thus $v - b \in Z_q^k$. By lemma (5.4) there exists $w \in Z_q^{k+1}$ such that $\|(b - v) - w\|_k^L < \varepsilon$. Then $u = v + w$ is the solution. Q.E.D.

§ 6. – a) In this first part of the present section we will refer implicitly to the special case that the vector bundle E is the tangent bundle T introduced in § 2. A real vector field w on $\mathcal{W}$ is said to be Killing if it is projectable in the parameter space B and its local one parameter group $\exp \lambda w$ induces holomorphic diffeomorphisms between open subsets of the fibers X_t . In particular if w is vertical (i.e. $\pi w = 0$), then for fixed λ $\exp \lambda w$ is a local holomorphic automorphism on each

fibre. Every vertical vector field w can be uniquely written in the form $w = u + \bar{u}$, where $u = \sum_{\alpha} u_{\alpha}(\partial/\partial z_{\alpha})$ is a section of $\mathbf{T}$. Then the condition for w to be Killing is simply $\bar{\partial}u = 0$. There are some particular vector fields on the patches W_j , defined in §2a, they are $D_j^v = (\partial/\partial t^v)_{x_j = \text{const}}$, $v \leq m$; their local groups are the inverse images by $\varphi_j: W_j \rightarrow \mathbb{C}^n \times \mathbb{R}^m$ of the translations in that t^v -axis direction. The projection πD_j^v is $\partial/\partial t^v$ and therefore, by the remark above, if we write $D_j^v - D_i^v = \theta_{ji}^v + \bar{\theta}_{ji}^v$, θ_{ji}^v becomes an holomorphic vector field on $X_t \cap W_i \cap W_j$ for fixed t . Clearly $(\theta_{ji}^v)_{ji}$ satisfies, by definition, the cocycle relation, thus, if one fix a partition of unity p_j relative to the covering (W_j) , formula $g^v|_{W_j} = \bar{\partial} \sum_i p_i \theta_{ij}^v$ defines a global $\bar{\partial}$ -closed form $g^v \in C_{01}^{\infty}(\mathcal{W}, \mathbf{T})$ which is not necessarily exact. g^v is nothing else than the Dolbeault correspondent of the cocycle θ_{ij}^v . In the classical theory of families of complex manifolds we have the following

(6.1) **SCHOLIUM** ([10]). — *Let U_0 be a relatively compact open subset of X_0 and assume that the equations*

$$(6.5) \quad \bar{\partial}v^v = g^v, \quad v = 1, \dots, m$$

admit a C^k solution ($k = 0, \dots, \infty$) in a neighborhood Δ of U_0 in $\mathcal{W}$. Integration of the vector fields $\sum_i p_i D_i^v - v^v - \bar{v}^v$ gives, for $\lambda > 0$ sufficiently small, a C^{k+1} diffeomorphism $\tau: A_0 \times B_{\lambda} \rightarrow \mathcal{W}$ such that $\pi \circ \tau = \text{pr}_B$, $\tau \circ \bar{\partial} = \bar{\partial} \circ \tau$, and $\tau(x, 0) = x$, $\forall x \in A_0$.

We shall also need a lemma of elementary analysis whose proof will be omitted:

(6.2) **LEMMA**. — *Let A' , A'' be open submanifolds of the riemannian manifold A , with $A' \subset \subset A'' \subset \subset A$. Take vector fields ξ^v ($v \leq m$) on $A \times B$ such that, for each $(a, t) \in A \times B$, $\xi^v(a, t)$ is tangent to $A \times \{t\}$, and suppose that $\xi^v(a, t)$ depends smoothly on a and is of class C^k as function of (a, t) . There exists a constant ϱ such that, if $|\xi^v| < \varrho$, there is a unique diffeomorphism $\phi: A' \times B \rightarrow A \times B$ with the following properties*

- (i) $\text{pr}_B \circ \phi = \text{pr}_B$;
- (ii) $\phi(a, 0) = a$, for each $a \in A''$;
- (iii) $\phi(\partial/\partial t^v) = \xi^v + \partial/\partial t^v$;
- (iv) $A' \times B \subset \phi(A'' \times B)$.

We shall denote by $C^{\infty k}$ the property of being C^k , and C^{∞} with respect to the first group of variables.

(6.3) **PROPOSITION**. — *Let U_0 be a relatively compact open subset of X_0 which admits a Stein neighborhood Δ_0 . There exists a neighborhood Ω of U_0 in $\mathcal{W}$, two se-*

quences (A_N) and (A'_N) of neighborhoods of U_0 in X_0 with

$$(6.4) \quad A'_N \subset\subset A'_{N+1} \subset\subset A_{N+1} \subset\subset A_N$$

and an injective $C^{\infty N+1}$ diffeomorphism $\tau_N: \bar{A}_N \times B \rightarrow \mathcal{W}$ with the following properties

- (i)_N $\text{pr}_B \circ \tau_N = \pi$;
- (ii)_N $\bar{\partial} \circ \tau_N = \tau_N \circ \bar{\partial}$ (i.e. τ_N sends $A_N \times \{t\}$ biholomorphically onto an open subset of X_t);
- (iii)_N $\bar{\Omega} \subset \tau_N(\bar{A}'_N \times B)$;
- (iv)_N The real vector field $\sum_i p_i D_i^v - \tau_N(\partial/\partial t^v) = v_N^v + \bar{v}_N^v$ (the equality defines v_N^v) is of class $C^{\infty N}$ and v_N^v satisfies the equation

$$(6.5)_N \quad \bar{\partial} v_N^v = g^v \quad \text{on } \tau_N(\bar{A}_N \times \bar{B});$$

- (v) there is $|v_N - v_{N+1}|_N^{\bar{\partial}} < 2^{-N}$ on Ω .

PROOF. – Since Δ_0 is Stein, $\Delta_t = \mathfrak{T}^{-1}(\Delta_0 \times \{t\})$ is also Stein for $|t|$ small (see § 2a). Hence $L_{01}^2(\Delta_t, \mathbf{T}) \subset RL_t$, in particular $g_t^v \in RL_t$, thus, by prop. (4.17) there exists a $C^{\infty 0}$ solution v_0 of $(6.5)_0$ on Δ . By the above scholium there exist an open neighborhood A_0 of $\bar{U}_0$ in Δ_0 and, for $B = \{|t| < \delta\}$ small enough an injective $C^{\infty 1}$ diffeomorphism $\tau_0: A_0 \times B \rightarrow \mathcal{W}$ which satisfies (i)₀, (ii)₀. Condition (iv)₀ is also satisfied by construction of τ_0 . Let $A'_0 \subset\subset A_0$ be a neighborhood of $\bar{U}_0$ in X_0 , we put $\Omega = \tau_0(A'_0 \times B)$, thus (iii)₀ is trivially satisfied. We proceed now by induction on N . Choose open sets A_{N+1}, A'_{N+1} of X_0 such that (6.4) is satisfied and let ϱ_N be the smallest ϱ given by lemma (6.2) applied to the situations $A'_N \subset\subset A'_{N+1} \subset\subset A_{N+1}$ and $A'_{N+1} \subset\subset A_{N+1} \subset\subset A_N$. Since v_N^v satisfies $(6.5)_N$ on $\tau_N(\bar{A}_N \times \bar{B})$, by (ii)_N we have on $\bar{A}_N \times \bar{B}$ $\bar{\partial}(\tau_N^{-1} v_N^v) = \tau_N^{-1} g^v \in C^{\infty N+1}(\bar{A}_N \times \bar{B})$. Thus by lemma (5.5) there exists $w^v \in C^{\infty N+1}(\bar{A}_N \times \bar{B})$ such that $\bar{\partial} w^v = \tau_N^{-1} g^v$ and w^v is arbitrarily close to $\tau_N^{-1} v_N^v$ in the C^N -norm. Thus we may assume that

$$(6.6) \quad |w^v - \tau_N^{-1} v_N^v| < \frac{1}{2} \cdot \varrho_N \quad \text{on } \bar{A}_N \times \bar{B},$$

$$(6.7) \quad |\tau_N w^v - v_N^v|_N < 2^{-N} \quad \text{on } \tau_N(\bar{A}_N \times \bar{B}).$$

Put now $v_{N+1}^v = \tau_N w^v$, $u_{N+1}^v = \sum_i p_i D_i^v - v_{N+1}^v - \bar{v}_{N+1}^v$. Thus (v) follows immediately from (6.7) and (iii)_N. We note that, although τ_N is not of class C^{N+2} , vector fields of class C^{N+1} which are tangent to the fibers $A_N \times \{t\}$ are sent by τ_N into fields of the same class. The reason of this fact is that τ_N is of class $C^{\infty N+1}$ and that, by transporting such fields only a partial jacobian matrix of τ_N is involved, precisely the one where only derivatives with respect to the first group of variables appear and this matrix is of class C^{N+1} . This is the point where we gain one derivative. So v_{N+1}^v and u_{N+1}^v defined above are of class $C^{\infty N+1}$ on $\tau_N(\bar{A}_N \times \bar{B})$.

Now, from the identity $u_{N+1}^v - \tau_N(\partial/\partial t^v) = (v_N^v - v_{N+1}^v) + (\bar{v}_N^v - \bar{v}_{N+1}^v)$ on $\tau_N(\bar{A}_N \times \bar{B})$ we get $\tau_N^{-1}u_{N+1}^v - \partial/\partial t^v = (\tau_N^{-1}v_N^v - w^v) + (\bar{\tau}_N^{-1}\bar{v}_N^v - \bar{w}^v)$ on $\bar{A}_N \times \bar{B}$. Hence by (6.6) we have $|\tau_N^{-1}u_{N+1}^v - \partial/\partial t^v| < \varrho_N$ on $\bar{A}_N \times \bar{B}$. Let $\phi: \bar{A}_{N+1} \times \bar{B} \rightarrow A_N \times \bar{B}$ be the map defined in lemma (6.2) for the vectorfields $\tau_N^{-1}u_{N+1}^v - \partial/\partial t^v$ and the sets A_{N+1}' , A_{N+1} , A_N .

We have

$$(6.8) \quad A_N' \times B \subset \phi(A_{N+1}' B).$$

Put $\tau_{N+1} = \tau_N \circ \phi$. We have $\tau_{N+1}(\partial/\partial t^v) = u_{N+1}^v$, hence τ_{N+1} is of class $C^{\infty N+2}$.

Conditions (i)_{N+1} and (ii)_{N+1} follow directly by the same properties verified by τ_N and ϕ . Condition (iii)_{N+1} can be derived from (iii)_N by applying τ_N to both terms of (6.8). Finally the vector fields v_{N+1}^v are $C^{\infty N+1}$ in $\tau_{N+1}(A_{N+1} \times B) \subset A_N \times B$ as we have already shown and we have $\bar{\partial}v_{N+1}^v = \tau_N \bar{\partial}w^v = g^v$, thus (6.5)_{N+1} is also satisfied. Q.E.D.

PROOF OF THEOREM (1.2). — As we have seen there exists a neighborhood L of K in $\mathcal{W}$ which meets each X_t in a Stein open set L_t . Hence there exists, for each t , $u_t' \in C^\infty(L_t, E_t)$ such that $\bar{\partial}_t u_t' = f_t$. We do not know how u_t' depends on t . Now, by (iv) and (v) of prop. (6.3) $v^v = \lim_{N \rightarrow \infty} v_N^v$ is a solution of equation (6.5) in a neighborhood (in $\mathcal{W}$) of K . Hence, by scholium (6.1) there exists A_0 , with $K \subset A_0 \subset X_0$, $\lambda > 0$ and an injective diffeomorphism $\tau: A_0 \times B_\lambda \rightarrow \mathcal{W}$ such that $\pi \circ \tau = p_{B_\lambda}$, $\tau \circ \bar{\partial} = \bar{\partial} \circ \tau$ and $\tau(x, 0) = x$, $\forall x \in A_0$. We can assume that $\Omega \stackrel{\text{def}}{=} \tau(A_0 \times B_\lambda)$ is contained in L . Apply lemma (5.3) to the vector bundle $\tau^{-1}E \rightarrow A_0 \times B$ and the collection of forms $(\tau^{-1}u_t')$. Since $\bar{\partial}\tau^{-1}u' = \tau^{-1}f \in C_{0,q+1}^\infty(A_0 \times B_\lambda, \tau^{-1}E)$ there exists $v \in C_{0,q}^\infty(A_0 \times B_\lambda, \tau^{-1}E)$ such that $\bar{\partial}_t v_t = \bar{\partial}_t \tau^{-1}u_t' = \tau^{-1}f_t$. Hence $u = \tau v|_\Omega \in C_{0,q}^\infty(\Omega, E)$ is the required solution of (1.1). Q.E.D.

§ 7. — Applications.

a) Pseudorigidity of Stein Manifolds for C^∞ deformations.

DEFINITION. — A complex manifold X_0 is said to be pseudorigid for C^∞ deformations if, for every C^∞ family of complex manifolds in which X_0 is the manifold at the time $t = 0$ and every compact subset K of X_0 , there exist an open neighborhood A_0 of K in X_0 and a map τ as in scholium (6.1).

Thus scholium (6.1) and theorem (1.2) say that every Stein manifold is pseudorigid for C^∞ deformations.

b) The complex Frobenius theorem.

The last statement on deformations permits to give a very simple proof of the following theorem of L. NIRENBERG ([8]).

THEOREM (L. NIRENBERG). — Let $\omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_p$ be C^∞ complex 1-forms on $\mathbb{R}^N$ independent at each point and let $\Omega(y)$ be the space generated over $\mathbb{C}$ by them at y . Suppose that the space $\Lambda = \Omega \cap \bar{\Omega}$ has constant dimension p . Then a (necessary and) sufficient condition for the existence of local coordinates $x_1, \dots, x_{2n}, t_1, \dots, t_p, s_1, \dots, s_q$ such that Ω is generated at each point by the differentials of $z_j = x_j + ix_{j+n}$ ($j \leq n$) and by $dt_1, \dots, dt_p$ is that

$$(7.1) \quad d\Omega \subset \mathfrak{I}\Omega, \quad d\Lambda \subset \mathfrak{I}\Lambda.$$

Ω stays for the space of C^∞ forms which are in Ω at each point, $d\Omega$ stays for the space of differentials of forms in Ω and $\mathfrak{I}\Omega$ stays for the ideal generated by Ω in the exterior algebra of all C^∞ complex forms.

We now give a proof which is based on Newlander-Nirenberg theorem (i.e. the above theorem in the case that $\Lambda = 0$ and that $\Omega + \bar{\Omega}$ is the space of complex 1-forms on $\mathbb{R}^N$) and on pseudorigidity of Stein manifolds. Put $\Gamma = \Omega + \bar{\Omega}$. Since $2 \dim \Omega = \dim \Gamma + \dim \Lambda$, we have $\dim \Gamma = 2n + p$. Put $q = N - 2n - p$. Since $\Lambda = \bar{\Lambda}$ and $\Gamma = \bar{\Gamma}$, the spaces Λ_R and Γ_R of forms of Λ and Γ with real coefficients generate them as follows: $\Lambda = \Lambda_R + i\Lambda_R$, $\Gamma = \Gamma_R + i\Gamma_R$. Λ can be supposed to have $\lambda_1, \dots, \lambda_p$ as basis and $\lambda_1, \dots, \lambda_p$ can be supposed to belong to Λ_R . One can choose $\gamma_1, \dots, \gamma_{2n}$ in Γ_R such that $\lambda_1, \dots, \lambda_p, \gamma_1, \dots, \gamma_{2n}$ is a basis of Γ_R over $\mathbb{R}$ and of Γ over $\mathbb{C}$. Conditions (7.1) may be rewritten as follows

$$(7.2) \quad d\omega_j = \sum_1^n \chi_j^h \wedge \omega_h + \sum_1^p \theta_j^m \wedge \lambda_m, \quad j \leq n; \quad d\lambda_m = \sum_1^p \varrho_m^k \wedge \lambda_k, \quad m \leq p;$$

where χ_j^h, θ_j^m and ϱ_m^k are suitable 1-forms. Since the γ 's are combinations of ω 's, (7.2) gives $d\gamma_j = \sum_1^{2n} \mu_j^h \wedge \gamma_h + \sum_1^p \nu_j^m \wedge \lambda_m$. Addition to this identity and to (7.2)₂ of their respective conjugates shows that the forms ϱ_m^k, μ_j^h and ν_j^m may be assumed to have real coefficients, hence $d\Lambda_R \subset \mathfrak{I}_R \Lambda_R$ and $d\Gamma_R \subset \mathfrak{I}_R \Gamma_R$, where $\mathfrak{I}_R$ stays for ideal generated over real forms. Thus, by the classical Frobenius theorem there exist coordinates $y_1, \dots, y_{2n}, t_1, \dots, t_p, s_1, \dots, s_q$ such that Λ_R is generated by $dt_1, \dots, dt_p$, say $\Lambda_R = \mathbb{R}dt$, and $\Gamma_R = \mathbb{R}dy \oplus \mathbb{R}dt$, hence $\Lambda = \mathbb{C}dt, \Gamma = \mathbb{C}dy \oplus \mathbb{C}dt$ and $\Omega = \mathbb{C}\omega \oplus \mathbb{C}dt$. The terms of the forms ω_j ($j \leq n$) which contain the differentials $dt_1, \dots, dt_p$ may obviously be suppressed and since all forms of Γ do not contain the differentials $ds_1, \dots, ds_q$, we can write $\omega_j = \omega_j(y, t, s) = \sum_1^n \omega_{jh}(y, t, s) dy^h$. Fix now t, s and take on the ball B of ray δ of $\mathbb{R}^{2n}$ the distribution $\Omega_{ts}(y) = \mathbb{C}\omega(y, t, s)$. Since the jacobian $(\omega_{jh}, \bar{\omega}_{jh})$ is non degenerate, all complex 1-form of $\mathbb{R}^{2n}$ are generated by $\Omega_{ts}, \bar{\Omega}_{ts}$ and $\Omega_{ts} \cap \bar{\Omega}_{ts} = \{0\}$. Thus we can apply Newlander-Nirenberg theorem and get a complex structure B_{ts} on B . Since the function $|y|^2$ is plurisubharmonic near the origin of B_{ts} , for δ small enough B_{00} is Stein. We apply now the pseudorigidity theorem (a), thus by successive shrinking of B_{00} if needed, we have a neighborhood U

of the origin of $\mathbb{R}^N$ and a biholomorphic map $\phi_{ts}: B_{ts} \cap U \rightarrow B_{00}$ which depends smoothly on (y, t, s) : thus $\phi_{ts*}\Omega_{00} = \Omega_{ts}$. Finally take complex coordinates $\zeta_j = \xi_j + i\xi_{j+n}$ ($j \leq n$) on B_{00} and put $x_j(y, t, s) = \xi_j[\phi_{ts}(y)]$, ($j \leq 2n$). We claim that x, t, s are the desired coordinates. Actually $dz = \phi_{ts*}d\zeta + \sum_{h=1}^p a_h dt^h$. Since $\phi_{ts*}d\zeta \in \Omega_{ts}$ we have $dz \in \mathbb{C}\omega + \mathbb{C}dt = \Omega$. The rest follows by straightforward dimensional arguments. Q.E.D.

c) A direct proof can be given by putting on $B \subset \mathbb{R}^{2n}$ the almost complex structure $\Omega_{isc}(y) = \mathbb{C}\omega(cy, t, s)$ and proceeding as in the classical proof: assume that $\omega_j(0, 0, 0) = dy_j + idy_{j+n}$ and apply (4.18) with f_t replaced by the $\bar{\Omega}_{isc}$ -component of $dy_j + idy_{j+n}$. For fixed c small enough, prop. (4.17) and theorem (1.2) give complex functions v_{isc}^j , $j \leq n$, such that the n functions $z_j(y, t, s) = (y_j + iy_{j+n})/c - v_{isc}^j(y/c)$, $j \leq n$, together with t and s give the desired coordinates.

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