The classes and representations of the groups of 27 lines and 28 bitangents.

Memoria di J. S. FRAME (a Princeton, New Jersey, U.S.A.)

Summary. - *For the simple group* H_0 *of the 1,451,520 automorphisms of the 28 bitangents to a plane quartic curve, and for its subgroup* G *of 51840 automorphisms of the 27 lines on the cubic surface, each related to an exceptional Lie algebra* (E_7 *and* E_8), *are determined all the classes and irreducible characters, some irreducible representations, certain basic invariants of* G, *and the large subgroups of* H_0.

1. Introduction.

Following the work of SEGRE [32], COXETER [10], TODD [34] and many others, this paper is concerned with the study of the classes and representations of two famous finite groups, related to the exceptional LIE groups E_6 and E_7, and each of geometric interest in its own right. One is the group G of automorphisms of the 27 lines on the general cubic surface, and the other is the group H_0 of automorphisms of the 28 bitangents to a plane quartic curve. As groups generated by reflections S_k whose products are of order 2 or 3, COXETER denotes the linear reflection groups associated with the adjoint groups of the exceptional simple LIE algebras E_6 and E_7 [8] by the symbols $[3^{2,2,1}]$ and $[3^{3,2,1}]$, and denotes their simple invariant rotation subgroups of index 2 by $[3^{2,2,1}]'$ and $[3^{3,2,1}]'$ respectively. We shall also denote these four groups by the letters G, H, G_0 and H_0.

The group G is the group of the 51840 automorphisms of the 27 lines on a general cubic surface, described in part in the books of HENDERSON [23] and SEGRE [32]. Its six dimensional irreducible orthogonal representation which we call F_2 (calling F_1 the identity representation) has been studied by COXETER as the group of symmetries of the six-dimensional polytope 2_{21}, and by CARTAN [8], WEYL [36], VAN DER WAERDEN [35], RACAH [28] and others in their studies of the exceptional LIE algebra E_6. Its invariant subgroup of index 2 is the simple group G_0 of order 25920. Hence the group G is not to be confused with another abstractly different group G' of order $51840 = (3^4-1)3^3(3^2-1)3$ studied by JORDAN [22], KLEIN [25], WITTING [37], MASCHKE [27], BURKHARDT [6] and others in connection with the trisection of the periods of the hyperelliptic modular functions, since the latter group G' has G_0 not as a subgroup but as a quotient group However, BURKHARDT' s five-dimensional representation of G_0 does induce a ten-dimensional irreducible representation (F_{15}) of our group G.

Closely related to the group G, and containing it as a subgroup of index 56, is the seven-dimensional reflection group $[3^{3,2,1}]$ associated with the exceptional LIE algebra E_7. This group H is the direct product of a group of order 2 (containing the identity and inversion in the origin) and a simple group $H_0 = [3^{3,2,1}]'$ of order $4 \times 9! = 1{,}451{,}520$, which is isomorphic with the group of automorphisms of the 28 bitangents to the general quartic curve. The geometrical relation between these 28 bitangents and the 27 lines of the general cubic surface has been described by GEISER [**20**], SEGRE [**31**], and others.

In § 2 we describe a class of 6×6 orthogonal matrices that represent in a simple way the 36 hyperplane reflections of which six generate the irreducible representation F_2 of the group G. In § 3 we describe the 25 conjugate classes of G, giving to each a cycle symbol $1^\alpha 2^\beta 3^\gamma 4^\delta \ldots$, in which α is the trace and α, β, γ, δ are integers. (The symbol $1\ 2\ 3^{-1}\ 6$ indicates that the six characteristic roots of a matrix in this class include a complete set of six 6th roots of unity from which the cube roots have been removed, and in addition the two square roots, 1, -1, and a first root 1). The classes of the various powers of an element are evident from this notation. We also prove that if T_k and U_k denote elements of order two that are respectively the products of two or three reflections S_i in mutually perpendicular hyperplanes, then each element of G_0 has the form $U_j U_k$ and each element of the odd coset $G_0 S$ has the form $U_j T_k$. Hence the elements of G may be written as products of at most six reflections. In § 4 a set of six basic invariants of degrees 2, 5, 6, 8, 9, 12 in the variables of F_2 are derived, whose Jacobian is factored into its 36 hyperplane factors In § 5 some other irreducible representations of G are studied, and the complete table of irreducible characters is formed. Finally in § 6 the 30 classes of conjugates for the group $H_0 = [3^{3,2,1}]'$ are obtained, and all the 30 ordinary irreducibile characters are found for this simple group of order 1,451,520. Several modular characters are also found, including the one for its representation as the simple Abelian linear group $A(6, 2)$ [**15**]. All large subgroups of H_0 (of index < 160) are found by a synthesis of permutation characters,

2. The 36 hyperplane reflections in 6-space.

The 27 lines on the general cubic surface are so related that each intersects 10 other lines and is skew to the remaining 16. An intersecting pair determines a third coplanar line, and the three lines form a triangle in one of the $27 \times 10/3! = 45$ tritangent planes. Each of the $27 \times 16/2 = 216$ pairs of skew lines determines one of the 36 double sixes. A double six consists of six pairs of skew lines forming two sextuplets, such that each line is skew to all the other five lines in the one sextuplet, but intersects each of their five partners that belong to the other sextuplet. Paired lines are in opposite sextuplets and are interchanged by the involutory transformation S_k associated with the double six D_k. The fifteen lines residual to the double six D_k are left invariant by S_k.

To describe these incidence relations succinctly we may use the 63 nonvanishing ordered triples that can be formed from the four marks 0, 1, w, w^2 (mod 2) of the finite field $GF(2^2)$. The 27 triples that contain just one 0 correspond to the lines of the cubic surface and may be called L-triples [17]. The other 36 triples having two or no 0's correspond to the double sixes, and may be called D-triples. Three L-triples represent coplanar lines if and only if their sum is 0. Two L-triples represent skew lines if and only if their difference is a D-triple, and this D-triple represents both the double six in which these lines are paired and the involutory transformation S_k that interchanges these lines. Two of the S_k are permutable if and only if the sum (or difference) of their D-triples is an L-triple. (The line of this L-triple intersects each of the four common lines of the two double sixes). Each S_k is permutable with 15 others but not with the remaining 20. Non-permutable S_k's fall into one of $36 \times 20/3\,! = 120$ sets of three associated S_k's that are related to each other as the three reflections in the altitudes of an equilateral triangle. Each two of a set of three associated S_k's have a product of order 3, and the three corresponding D-triples have sum zero modulo 2.

By introducing certain sign changes we may change from L-triples involving the mark w of the finite field to L-triples involving the complex cube root of unity ω. The L-triples and their differences the D-triples are then complex number triples whose real and imaginary parts are components of vectors in a real Euclidean 6-space. In fact, the L-triples determine the 27 vertices of the polytope 2_{21}, discovered in 1897 by T. GOSSET [21], whose relation to the 27 lines of the cubic surface was described in 1910 by P. SCHOUTE [29]. Each D-triple determines homogeneous coordinates for a symmetry hyperplane of 2_{21}, reflection in which is the corresponding transformation S_k.

COXETER [13] has shown that the 27 lines of the general cubic surface may be represented by the complex L-triples

$$(2.1) \qquad (0,\ \omega^\lambda,\ -\omega^\mu)(-\omega^\mu,\ 0,\ \omega^\lambda)(\omega^\lambda,\ -\omega^\mu,\ 0)\,;\ \lambda,\ \mu = 1,\ 2,\ 3\,;\ \omega = e^{2\pi i/3}.$$

Here again, three L-triples whose sum is zero represent coplanar lines. Differences of L-triples that correspond to skew lines determine the 36 distinct D- triples:

$$(2.2)\ (\omega^\varkappa, \omega^\lambda, \omega^\mu)(\omega^{\varkappa+1}-\omega^{\varkappa-1}, 0, 0)(0, \omega^{\lambda+1}-\omega^{\lambda-1}, 0)(0, 0, \omega^{\mu+1}-\omega^{\mu-1}),\ \varkappa, \lambda, \mu = 1, 2, 3.$$

Each of these 36 D-triples must be identified with its negative. For three D-triples of an associated set we have $D_i \pm D_j \pm D_k = 0$, where an appropriate choice of signs must be made to obtain equality.

Taking real and imaginary parts of the L-triples (2.1) as coordinates in a 6-space whose general point is

$$(x_1, y_1, x_2, y_2, x_3, y_3) \tag{2.3}$$

we obtain the following coordinates for the 27 vertices of the polytope 2_{21}:

$$\text{Vertices of } 2_{21}: \quad \begin{array}{l} (0, 0, c_\lambda, s_\lambda, -c_\mu, -s_\mu) \\ (-c_\mu, -s_\mu, 0, 0, c_\lambda, s_\lambda) \\ (c_\lambda, s_\lambda, -c_\mu, -s_\mu, 0, 0) \end{array} \qquad \lambda, \mu = 1, 2, 3; \tag{2.4}$$

where

$$c_\lambda = \cos 2\pi\lambda/3, \qquad s_\lambda = \sin 2\pi\lambda/3, \qquad \omega^\lambda = c_\lambda + i s_\lambda. \tag{2.5}$$

Pairs of non-adjacent vertices of this 2_{21} determine diagonals of length $6^{1/2}$ and represent intersecting lines of the cubic surface. Pairs of adjacent vertices determine edges of length $3^{1/2}$ and represent a pair of skew lines of the cubic surface. The difference of a pair of adjacent L-vectors (2.4) is a vector normal to one of the 36 hyperplanes of symmetry of 2_{21}. To obtain the unit column vectors D_k associated with the reflections S_k, we normalize each such difference by the factor $3^{-1/2}$. We denote the corresponding row vectors by D'_k or more explicitly by $D'_{\varkappa\lambda\mu}$, $D'_{\varkappa 00}$, $D'_{0\lambda 0}$, $D'_{00\mu}$, $(\varkappa, \lambda, \mu = 1, 2, 3)$ where

$$\begin{array}{ll} D'_{\varkappa\lambda\mu} = 3^{-1/2}(c_\varkappa, s_\varkappa, c_\lambda, s_\lambda, c_\mu, s_\mu), & \varkappa, \lambda, \mu = 1, 2, 3 \\ D'_{\varkappa 00} = \quad (-s_\varkappa, c_\varkappa, 0, 0, 0, 0) & \\ D'_{0\lambda 0} = \quad (0, 0, -s_\lambda, c_\lambda, 0, 0) & \\ D'_{00\mu} = \quad (0, 0, 0, 0, -s_\mu, c_\mu). & \end{array} \tag{2.6}$$

Vectors D_i and D_j having exactly one common subscript in the three-subscript notation of (2.6) represent permutable transformations S_i, S_j, and their scalar product is $D_i'D_j = 0$. Vectors D_i and D_j having two or no common subscripts in the three subscript notation of (2.6) represent non-permutable transformations S_i, S_j, and their scalar product is $D_i'D_j = \pm 1/2$. Thus they make angles of 60° or 120° with each other, and are parallel to two sides of an equilateral triangle. They determine a third vector D_k,

$$D_k = \pm (D_i - (2D_i'D_j)D_j), \tag{2.7}$$

which is a unit vector parallel to the third side of the equilateral triangle. The three vectors D_i, D_j, D_k are said to form an *associated set.* In the three subscript notation of (2.6), we may find the subscript triple for D_k by replacing 0, 1, 2, 3 in the subscript triples for D_i and D_j by the marks 0, w, w^2, w^3 $(= 1)$, then adding these triples mod 2, and converting back.

From the D-vectors we obtain an explicit representation for the 6×6 matrices that generate the real orthogonal representation F_2 of our group. Although it was derived independently, this representation is closely related to COXETER's representation by anti-collineations [**13**].

THEOREM 1. - If the 36 unit row vectors of (2.6) are denoted by D_k' and the corresponding column vectors by D_k, then the 36 real symmetric orthogonal matrices

$$S_k = I - 2D_kD_k' \tag{2.8}$$

form a complete set of 36 conjugate matrices, of which six suffice to generate the 6-dimensional irreducible linear group F_2 of order 51840.

PROOF: If X is any column vector perpendicular to D_k, then

$$S_kX = (I - 2D_kD_k')X = X - 2D_k(D_k'X) = X,$$
$$S_kD_k = (I - 2D_kD_k')D_k = D_k - 2D_k(1) = -D_k.$$

Since the matrix S_k reverses the sign of D_k, and leaves invariant all vectors perpendicular to D_k, it represents a reflection in the hyperplane through the origin perpendicular to D_k.

Using the triple subscript notation of (2.6), consider the six matrices S_{100}, S_{010}, S_{001}, S_{300}, S_{030}, S_{333} defined by (2.8). A pair of permutable S-matrices are recognized by having just one of their three subscripts in common. Each of a pair of non-permutable S-matrices transforms the other into the same third S-matrix, the new subscripts being obtained in the manner described above in (2.7) for the corresponding D-vectors in an associated set. Thus S_{100}, S_{010}, S_{001}, and their products transform S_{333} into each of the matrices $S_{\varkappa\lambda\mu}$ with $\varkappa$, λ, $\mu = 2$ or 3, and by transforming these by S_{300}, S_{030} and S_{300} S_{030} S_{333}, and by their products in turn, we may change one or all of the subscripts 2 in $S_{\varkappa\lambda\mu}$ to 1's. Finally each $S_{\varkappa00}$, $S_{0\lambda0}$, or $S_{00\mu}$ belongs to an associated set with two $S_{\varkappa\lambda\mu}$'s having two non-zero subscripts in common, and can be obtained by transforming one by the other. Thus each of the S-matrices is conjugate to S_{333}, and is expressible in terms of six generating S-matrices. Since the elements of a complete set of conjugates generate an invariant subgroup of G, this must include the simple subgroup G_{25920}. But since the individual S-matrices are not included in this even subgroup, the subgroup generated by the S_k must be the entire group G of order 51840.

Certain products of the matrices S_k play an important role in the study of the group G. Products of two, three, or four mutually permutable S factors will be denoted by T_k, U_k and V_k respectively. Thus if $D_i'D_j = 0$, etc.,

$$T_{ij} = S_iS_j = I - 2D_iD_i' - 2D_jD_j', \tag{2.9a}$$
$$U_{ijk} = S_iS_jS_k = I - 2D_iD_j' - 2D_jD_j' - 2D_kD_k', \tag{2.9b}$$
$$V_{ijkl} = S_iS_jS_kS_l = I - 2D_iD_i' - 2D_jD_j' - 2D_kD_k' - 2D_lD_l'. \tag{2.9c}$$

In particular we define U_0 to be the following diagonal matrix of trace 0:

$$U_0 = S_{300}\ S_{030}\ S_{003} = \text{diag}\,\{1,\ -1,\ 1,\ -1,\ 1,\ -1\}. \tag{2.10a}$$

If the four vectors D_i, D_j, D_k, D_l, are mutually perpendicular D-vectors of (2,6), they span a four-space perpendicular to the plane of an equilateral triangle formed by three vertices of 2_{21}, whose centroid is at the origin. Since there are 45 such triangles, there are 45 matrices V_i, representing reflections in a four space. However, each V_i can be represented in 3 distinct ways as a product of four factors S_k. Thus for example we may define the particular matrix V_0 in three ways as

$$V_0 = S_{300}S_{030}S_{003}S_{333} = S_{122}S_{212}S_{221}S_{111} = S_{211}S_{121}S_{112}S_{222}. \tag{2.10b}$$

Given one such factorization of a V matrix, the eight S factors in the other two factorizations are those not commutative with any of the factors of the first factorization. These eight fall uniquely into two sets of four mutually permutable factors forming the second and third factorization of V.

To prove this statement, let us define V_0 by its first factorization in (2.10b). Then we see that the second and third factorizations may be written as $S_{222}\ V_0\ S_{222}$ and $S_{111}\ V_0\ S_{111}$ respectively. On the other hand, it is easily verified that the product $V_0 = S_{300}S_{030}S_{003}S_{333}$ transforms each of the twelve S factors of (2.10b) into itself, so in particular V_0 is commutative with S_{111} and S_{222}. Hence the three products in (2.10b) are equal.

It is easily shown that if S_i, S_j, and S_k are mutually non-permutable members of an associated set, then S_iS_j is representable in the three ways:

$$S_iS_j = S_jS_k = S_kS_i, \quad \text{if} \quad (D_i'D_j)(D_j'D_k)(D_k'D_i) \neq 0, \tag{2.11}$$

Next consider an element of order 3 that can be represented as a product U_iU_j in which each of the three S-factors of U_i is non-permutable with just one of the S-factors of U_j, and conversely. Each such U_iU_j is an element of order 3 that has 108 distinct factorizations as a product of two such U factors. Clearly, from any given factorization U_iU_j, 27 arise (including this one) by applying (2.11) to each of the three pairs of non-permutable factors. But it is not quite trivial to obtain the four apparently unrelated factorizations of which the following is an example:

$$\begin{aligned}(S_{100}S_{010}S_{001})(S_{200}S_{020}S_{002}) &= (S_{233}S_{323}S_{332})(S_{311}S_{131}S_{113})\\ = (S_{231}S_{321}S_{222})(S_{312}S_{132}S_{333}) &= (S_{232}S_{322}S_{112})(S_{313}S_{133}S_{223}).\end{aligned} \tag{2.12}$$

To establish (2.12) however, it is only necessary to show that the left member transforms $S_{11\mu}\ S_{22\mu+1}$ into one of the equal products given by (2.11), whereas this product transforms the factors of the left member of (2.12) into those of the three other forms according as μ is 1, 2, or 3.

LEMMA 1. - Any matrix which is a product of factors S_i can be reduced to a product containing two less factors, under the following circumstances:

1. If it contains a repeated factor.
2. If it contains three factors from the same associated set.
3. If it contains four mutually permutable factors and a fifth that is permutable with none of them.

PROOF: To prove (1) we note that $S_i S_k S_i$ is a single S matrix and that

$$S_i(S_1 S_2 \dots S_n)S_i = (S_i S_1 S_i)(S_i S_2 S_i) \dots (S_i S_n S_i).$$

Thus the $n+2$ factors on the left are replaced by n factors on the right. To prove (2), let S_i, S_j, S_k be members of an associated set. Then we can replace S_j by the equal $S_i S_j S_k$, inserting two factors, and then eliminate two pairs of repeated factors $S_i\ S_i$ and $S_k\ S_k$ by (1).

To prove (3), we let the product of the four mutually permutable factors $S_i S_j S_k S_l$ be V, and note that the fifth factor S_n will appear in an alternate factorization $V = S_n S_p S_q S_r$. By introducing the identity as the product of these eight factors, we can then cancel five repeated pairs by (1), reducing the total number of factors by two.

DEFINITION. - A *completely reduced* product of S factors is defined to be one in which no further reductions are possible by Lemma 1, and in which any two non-permutable S factors are either adjacent or separated by a single S factor. Each pair of non-permutable factors shall be called a *link*.

THEOREM 2. - The trace of a completely reduced product of n S-factors with l links is given by the formula

$$\text{tr} \prod_{i=1}^{n} S_i = 6 - 2n + l. \tag{2.13}$$

PROOF: We expand the product in (2.13) by Theorem 1 as follows:

$$\prod_{i=1}^{n}(I - 2D_i D_i') = I - 2\sum_i D_i D_i' + 4\sum_{i<j} D_i(D_i'D_j)D_j' - 8\sum_{i<j<k} D_i(D_i'D_j)(D_j'D_k)D_k' + \dots \tag{2.14}$$

Taking traces we obtain

$$\text{tr}\prod_{i=1}^{n} S_i = 6 - 2n + \sum_{i<j}(2D_i'D_j)^2 - \sum_{i<j<k}(2D_i'D_j)(2D_j'D_k)(2D_k'D_i) + \dots . \tag{2.15}$$

The first summation in (2.15) reduces to the number of links, since $2D_i'D_j = \pm 1$ whenever S_i and S_j are non-permutable, and $2D_i'D_j = 0$ otherwise. The sums involving three products vanish, since we assumed that no three of the S_i belong to an associated set. Finally the sums involving four or more products vanish since we assumed that S_i and S_l are permutable, and hence $D_l'D_i = 0$. whenever S_i and S_l are separated by two or more factors S_j and S_k. Thus (2.13) is established, and the foundation is laid for a study of the classes of conjugate elements in G.

3. The classes of conjugate elements of G.

Since the even subgroup $[3^{2,2,1}]' = G_0$, of order 25920, is known to have ten rational characters and five pairs of complex characters [16], it follows that G will have ten pairs of associated characters belonging to the irreducible representations F_1, F_1^*, F_2, $F_2^* \dots F_{10}$, F_{10}^*, and five self associated characters belonging to the irreducible representatives $F_{11}, \dots F_{15}$. These characters are already known for elements of the even subgroup, and they will be determined for elements of the odd coset in § 5.

We also deduce that G will have $10+5$ « even » classes of conjugates $C_1, C_2, \dots C_{15}$ consisting of elements that are products of an even number of S factors, and 10 « odd » classes of conjugates $C_{16} \dots C_{25}$. The order of assigning subscripts to the class symbols C_λ is arbitrary, but we shall assign C_1 to the identity class, $C_2 \dots C_5$ and $C_{16} \dots C_{20}$ to those classes of elements whose orders are powers of 2, $C_6 \dots C_{14}$ and $C_{21} \dots C_{24}$ to those classes whose elements have an order divisible by 3, and C_{15} and C_{25} respectively to the classes of elements of orders 5 and 10.

Eighteen of the twenty-five classes may be described as r-chains or products of disjoint r-chains. By an r-chain we shall mean a product of $r-1$ S factors in which adjacent factors are linked (i. e. non-permutable) and non-adjacent factors are permutable. Such an element is illustrated by the product of $r-1$ transpositions $(12)(23)\dots(r-1\ r)$ in the symmetric group. By Theorem 2 its trace in the representation F_2 is the following:

$$\text{Trace of } r\text{-chain} = 6 - r. \tag{3.1}$$

Six classes of conjugates C_1, C_{16}, C_6, C_{18}, C_{10} and C_{23} have elements formed of r-chains for $r = 1, 2, 3, 4, 5, 6$, respectively, and we shall denote such a class by the symbol $1^{6-r}r$, which is analogous to the symbol for a permutation on 6 symbols, with trace $6-r$, that permutes r symbols in a single cycle.

Two or more r-chains are called *disjoint* if each S-factor in one is permutable with every S-factor in the other. Twelve additional classes of G have elements representable as products of disjoint chains. The order of such an element is the least common multiple of the orders of the permutable factors, and the trace (α) of its matrix in F_2 is equal to 6 minus the sum of the orders of its r-chains. For example, a product of a 2-chain (single S factor) and two 3-chains, all three mutually disjoint, will have order 6 and trace $6-2-3-3=-2$, and we shall denote it by $1^{-2}2\ 3^2$.

In general we shall characterize the classes of G by symbols of the form

$$1^\alpha 2^\beta 3^\gamma 4^\delta \dots = \prod_{k=1}^{n} k^{\alpha_k}, \qquad (\Sigma\, k\alpha_k = n), \tag{3.2}$$

where $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \alpha_3, \dots$ etc. are integers. Such a symbol in which α_i are non-negative integers is commonly used to characterize the class or classes in a permutation group consisting of permutations leaving α_1 symbols fixed (trace $= \alpha_1$), and containing α_k cycles of k letters. The mth power of such a permutation is obtained by simply replacing each factor k^{α_k} by $(k/d)^{d\alpha_k}$, where $d = (k, m)$, and combining exponents. The trace s_m of the mth power is given by the formula:

$$s_m = \sum_{k|m} k\alpha_k, \quad \text{or} \quad s_1 = \alpha,\ s_2 = \alpha + 2\beta,\ s_3 = \alpha + 3\gamma,\ s_4 = \alpha + 2\beta + 4\delta, \text{ etc.} \tag{3.3}$$

To find a symbol of the type (3.2) for a class of matrices, we may first compute s_m directly, and then solve equation (3.3) for the α_k. The solution is facilitated by expanding the Euler function $\Phi(k)$, and replacing the integers $\pm m$ in this expansion by the corresponding $\pm s_m$. Thus for example:

$$\Phi(12) = 12\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 12 - 6 - 4 + 2, \tag{3.4a}$$

$$12\alpha_{12} = \qquad s_{12} - s_6 - s_4 - s_2 . \tag{3.4b}$$

The numbers α_k so obtained will not in general be non-negative. But if the s_k are rational integers (as they are for the group G) and if the group element is conjugate to all these of its powers whose exponents are relatively prime to its order, then it follows that $s_{p^{\beta-1}n} \equiv s_{p^\beta n} \pmod{p^\beta}$, and hence that the α_k are integers [18].

The eighteen classes obtained as products of one or more disjoint r-chains are those in whose symbols all exponents except possibly α_1 (the trace) are non-negative. We find the numbers of elements g_λ in these 18 classes C_λ by building up the products from left to right (taking the chains in decreasing order of their length), multiplying together the numbers of ways each new factor can be obtained, and dividing by the number of distinct ways of representing a given group element as an ordered product of S-factors having a given pattern of links. For the classes $1^{-2}2^4$, $1^{-2}2^24$, and $1^{-2}26$ a non-trivial factor 3 enters the denominator because of the three-fold representation of V in (2.10b), and for the class $1^{-3}3^3$ a non-trivial factor 4 in the denominator arises from the identity (2.12). Any r-chain for $r > 2$ can be rewritten by transforming the first S-factor by the product of the remaining $r-2$ factors and placing this transformed S-factor at the end, so for $r > 2$ each r-chain introduces a factor r in the denominator of g_λ in (3.5). Also since α_r equal r-chains can be permuted among themselves in $\alpha_r!$ ways, the factor $\alpha_r!$ must be divided out when $r > 1$. The expressions for g_λ are as follows:

(3.5)

Class	Number of elements	$= g_\lambda$	Class	Number of elements	$= g_\lambda$
$1^6:$	$g_1 = 1$	$= 1$	$1^42:$	$g_{16} = 36$	$= 36$
$1^33:$	$g_6 = 36\cdot 20/3$	$= 240$	$1^24:$	$g_{18} = 36\cdot 20\cdot 9/4$	$= 1620$
$15:$	$g_{15} = 36\cdot 20\cdot 9\cdot 4/5$	$= 5184$	$6:$	$g_{23} = 36\cdot 20\cdot 9\cdot 4\cdot 1/6$	$= 4320$
$1^22^2:$	$g_2 = 36\cdot 15/2!$	$= 270$	$2^3:$	$g_{17} = 36\cdot 15\ 6/3!$	$= 540$
$123:$	$g_{21} = 36\cdot 20\cdot 6/3$	$= 1440$	$1^{-2}2^4:$	$g_3 = 35\cdot 15\cdot 6\cdot 1/4!(3)$	$= 45$
$3^2:$	$g_9 = 36\cdot 20\cdot 6\cdot 2/3^2 2!$	$= 480$	$1^{-1}2^23:$	$g_8 = 36\cdot 20\cdot 6\cdot 3/3\cdot 2!$	$= 2160$
$24:$	$g_5 = 36\cdot 20\cdot 9\cdot 2/4$	$= 3240$	$1^{-2}23^2:$	$g_{22} = 36\cdot 20\cdot 6\cdot 2\cdot 3/3^2 2!$	$= 1440$
$1^{-1}25:$	$g_{25} = 36\cdot 20\cdot 9\cdot 4/5$	$= 5184$	$1^{-3}3^3:$	$g_{11} = 36\cdot 20\cdot 6\cdot 2\cdot 3\cdot 2/3^3 3!(4)$	$= 80$
$1^{-2}26:$	$g_{10} = 36\cdot 20\cdot 9\cdot 4\cdot 1/6(3)$	$= 1440$	$1^{-2}2^24:$	$g_{19} = 36\cdot 20\cdot 9\cdot 2\cdot 1/4\cdot 2!(3)$	$= 540$

The remaining seven classes have elements that are not expressible as products of S-factors with simple chain structure. The numbers of elements of five of the classes not listed above are found in the published character table of the group G_{25920} [16], together with the traces of their powers. These are elements of orders 4, 6, 6, 9 and 12 respectively, with traces 2, 1, 1, 0, -1 respectively in the six-dimensional representation F_2. The first two classes contain products of four S-factors with 4 and 3 links respectively, whereas the next three contain products of six S-factors with 7, 6, and 5 links respectively. Just 5/24 of the elements of the group are left for the two remaining classes of the odd coset, whereas only 1/12 of the order of the group is left as a total for the sum of the squares of the traces in these classes. Hence these classes contain 1/8 and 1/12 of the group elements respectively, and they include elements of order 8 and 12 respectively, each permutable only with its own powers. Each is the product of five S-factors, with four or five links respectively, so the traces are 0 and 1. From a study of the traces of powers we determine the symbols for these seven remaining classes as follows:

$$
\begin{aligned}
(3.5b)\qquad 1^2 2^{-2} 4^2 &: g_4 = 540 & 1 2 3^{-1} 6 &: g_7 = 1440 \text{ (four } S\text{-factors)} \\
1 2^{-2} 3^{-1} 6^2 &: g_{12} = 720 & 1^{-1} 2 3 4^{-1} 6^{-1} (12) &: g_{13} = 4320 \text{ (six } S\text{-factors)} \\
3^{-1} 9 &: g_{14} = 5760 & 1 2^{-1} 3^{-1} 4 6 &: g_{24} = 4320 \text{ (five } S\text{-factors)} \\
2 4^{-1} 8 &: g_{20} = 6480. & &
\end{aligned}
$$

It is interesting to note that the required link structure for every class is obtained by multiplying the fixed diagonal matrix U_0 of (2.10a) by the matrices U_j of C_{17} to obtain all even classes, or by the matrices T_j of C_2 to obtain all odd classes. We shall prove the following theorem:

THEOREM 3. – Every element of the simple subgroup G_{25920} is a product $U_i U_j$ of just two involutory elements U_i, U_j from the class C_{17} of G_{51840}, and every element of the odd coset is a product $U_i T_j$ where T_j lies in the class C_2. The elements T_j and U_i are defined to be products of two or three mutually permutable reflections S_k, respectively. Hence each element of G is a product of at most six factors S_k.

PROOF: We examine all the 540 products $U_0 U_j$ and the 270 products $U_0 T_k$, where U_0 is defined by (2.10a), and classify them according to the relationship of the factors of U_j or T_k to each of the three factors S_{300}, S_{030}, and S_{003} of U_0. To each S-factor of U_j or T_k we assign the value 2 if it is equal to one of the three factors of U_0, or the value $l-2$ if it is linked (non-permutable) with l of the three factors. To save printing space we denote -1 and -2 by $\bar{1}$ and $\bar{2}$, respectively. Furthermore, we indicate by 0^2 or $\bar{1}^2$ a pair of S-factors both related in the same way to each of the factors of U_0, but by 00 and $1\bar{1}$ a pair not both linked with the same factors of U_0. Thus each product U_0U_i is classified by three of the symbols 2, 1, 0, $\bar{1}$, $\bar{2}$, and each product U_0T_k by two of them. Furthermore, by Theorem 2, the trace of U_0U_j or U_0T_k is the sum of the numbers in its symbol.

Arranged in descending order, possible traces for the even and odd classes are:

$$(3.6)\quad \begin{array}{l} \text{Trace } (U_0U_j):\ 6,\ 3,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ -1,\ -1,\ -2,\ -2,\ -3 \\ \text{Trace } (U_0T_k):\ 4,\quad 2,\quad 1,\ 1,\quad 0,\ 0,\ 0,\ -1,\qquad -2,\ -2. \end{array}$$

Upon investigation it turns out that elements from different classes of conjugates never have the same symbol, and that all classes are represented. This fact proves Theorem 3. However, in nine cases two different symbols represent elements of the same class. To illustrate this, consider the product $U_0\ S_{111}\ S_{122}\ S_{133}$ which is represented by the symbol $11\bar{1}$, and the product $U_0\ S_{111}\ S_{123}\ S_{213}$ which is represented by the symbol 10^2. Both have trace 1. Their squares

$$U_0\ S_{200}\ S_{003}\ S_{030} = S_{300}\ S_{200} \quad \text{and} \quad U_0\ S_{121}\ S_{211}\ S_{112} = S_{333}\ S_{222}$$

are represented by the symbols $22\bar{1}$ and 111 respectively, and therefore have trace 3. Yet there is only one class of elements having trace 3, namely C_6, and only one class of elements of trace 1 whose squares have trace 3, namely the class C_7 of elements of order 6.

In Table I, we list for each class first its cycle symbol of type (3.2), which automatically gives the order of the element and its trace α in the six-dimensional representation, next its linkage symbol (or symbols) of the type just described, that enables one to write down elements U_iU_j or U_iT_j, in the class, next a particular element U_0U_j or U_0T_j in the class, then the number of elements g_λ, the order $n_\lambda = 51840/g_\lambda$ of the normalizer of an element of C_λ, and finally the traces of the elements in various permutation representations to be discussed later.

TABLE I.

Class structure of the group $[3^{2,2,1}]$.

Class	Cycle symbol	Link symbols	Representative element	g_λ	n_λ	Five Selected permutation characters of G.				
C_1	1^6	222	$1 = U_0 S_{300} S_{030} S_{003}$	1	51840	36	72	27	40	45
C_2	$1^2 2^2$	$22\bar{2}, 20^2$	$T = U_0 S_{300} S_{311} S_{322}$	270	192	8	12	7	0	5
C_3	$1^{-2} 2^4$	$0^2 \bar{2}$	$V = U_0 S_{311} S_{322} S_{333}$	45	1152	12	0	3	8	13
C_4	$1^2 2^{-2} 4^2$	110	$U_0 S_{121} S_{211} S_{223}$	540	96	0	0	3	4	1
C_5	24	$2\bar{1}^2, 20\bar{2}$	$U_0 S_{300} S_{311} S_{333}$	3240	16	2	2	1	0	1
C_6	$1^3 3$	$111, 22\bar{1}$	$U_0 S_{300} S_{030} S_{001}$	240	216	6	12	9	4	6
C_7	$123^{-1}6$	$11\bar{1}, 10^2$	$U_0 S_{111} S_{123} S_{213}$	1440	36	0	0	3	2	4
C_8	$1^{-1} 2^2 3$	$1\bar{1}^2$	$U_0 S_{111} S_{133} S_{100}$	2160	24	2	0	1	0	2
C_9	3^2	$2\bar{1}\bar{1}$	$U_0 S_{300} S_{010} S_{001}$	480	108	3	6	0	4	3
C_{10}	$1^{-2} 26$	$0\bar{1}^2, 00\bar{2}$	$U_0 S_{113} S_{311} S_{333}$	1440	36	3	0	0	2	1
C_{11}	$1^{-3} 3^3$	$\bar{1}\bar{1}\bar{1}$	$U_0 S_{100} S_{010} S_{001}$	80	648	0	0	0	13	9
C_{12}	$12^{-2} 3^{-1} 6^2$	100	$U_0 S_{111} S_{123} S_{321}$	720	72	0	0	0	5	1
C_{13}	$1^{-1} 234^{-1} 6^{-1} (12)$	$00\bar{1}, 1\bar{1}\bar{1}$	$U_0 S_{111} S_{133} S_{313}$	4320	12	0	0	0	1	1
C_{14}	$3^{-1} 9$	$10\bar{1}$	$U_0 S_{111} S_{213} S_{133}$	5760	9	0	0	0	1	0
C_{15}	15	$20\bar{1}$	$U_0 S_{300} S_{311} S_{010}$	5184	10	1	2	2	0	0
C_{16}	$1^4 2$	22	$S = U_0 S_{033} S_{003}$	36	1440	16	30	15	0	15
C_{17}	2^3	$2\bar{2}, 0^2$	$U = U_0 S_{311} S_{322}$	540	96	4	2	3	8	7
C_{18}	$1^2 4$	$20, 11$	$U_0 S_{111} S_{122}$	1620	32	2	4	5	0	3
C_{19}	$1^{-2} 2^2 4$	$\bar{1}^2, 0\bar{2}$	$U_0 S_{311} S_{333}$	540	96	6	0	1	0	3
C_{20}	$24^{-1} 8$	$1\bar{1}$	$U_0 S_{111} S_{133}$	6480	8	0	0	1	2	1
C_{21}	123	$2\bar{1}$	$U_0 S_{300} S_{010}$	1440	36	4	6	3	0	0
C_{22}	$1^{-2} 23^2$	$\bar{1}\bar{1}$	$U_0 S_{133} S_{313}$	1440	36	1	0	0	0	3
C_{23}	6	00	$U_0 S_{131} S_{113}$	4320	12	1	2	0	2	1
C_{24}	$12^{-1} 3^{-1} 46$	10	$U_0 S_{111} S_{123}$	4320	12	0	0	1	0	0
C_{25}	$1^{-1} 25$	$0\bar{1}$	$U_0 S_{113} S_{233}$	5184	10	1	0	0	0	0

4. Basic invariants of the 6-dimensional orthogonal representation of G.

It is clear that the 6-dimensional orthogonal group F_2 has the quadratic invariant

$$A = x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2. \tag{4.1}$$

G RACAH [28] has indicated that a set of six independent basic invariants of degrees 2, 5, 6, 8, 9, and 12 exist such that every rational invariant of the group is rationally expressible in terms of these. He also notes that since the product of these degrees is equal to the order of the group, these invariants form a complete set. His method of finding invariants is to form the power sums

$$\Phi_n = \sum_{\lambda,\mu=1}^{3} (\varphi_{2\lambda} - \varphi_{3\mu})^n + (\varphi_{3\lambda} - \varphi_{1\mu})^n + (\varphi_{1\lambda} - \varphi_{2\mu})^n \tag{4.2}$$

where

$$\varphi_{im} = x_i \cos 2\pi m/3 + y_i \sin 2\pi m/3, \qquad i,\ m = 1,\ 2,\ 3. \tag{4.3}$$

In general Φ_n represents the sum of the nth powers of the projections $\varphi_{i\lambda} - \varphi_{j\mu}$ of a vector of general position upon the lines joining the origin to the 27 vertices (2.4) of the polytope 2_{21}. Certain of the Φ_n, namely Φ_1 and Φ_3 are found to vanish identically. Φ_4 is proportional to Φ_2^2, Φ_7 to $\Phi_2\Phi_5$ and similarly all Φ_n except those for $n = 2, 5, 6, 8, 9, 12$ are found to be polynomial functions of those of lower degree. These six give a complete set of basic invariants, but the last three of them are quite complicated functions of x_i, y_i, involving some large coefficients.

We propose to obtain expressions for a different complete set of basic invariants of the degrees indicated, by introducing a new set of coordinates p_i, q_i defined by the equations

$$p_i = x_i^2 + y_i^2, \qquad q_i = \frac{1}{3} x_i^3 - x_i y_i^2, \tag{4.4}$$

and then operating on the invariant $A = p_1 + p_2 + p_3$ by applying successively the differential operators Θ and Δ, where

$$\Theta = \sum_{i=1}^{3} [3q_i(p_j - p_k) - 2p_i(q_j - q_k)] \frac{\partial}{\partial p_i} + \left[\frac{1}{2} p_i^2(p_j - p_k) - 3q_i(q_j - q_k)\right] \frac{\partial}{\partial q_i} \tag{4.5a}$$

$$(ijk) = (123),\ (231) \text{ or } (312).$$

$$\Delta = \sum_{i=1}^{3} 4 \frac{\partial}{\partial p_i} p_i \frac{\partial}{\partial p_i} + 12 q_i \frac{\partial^2}{\partial p_i \partial q_i} + p_i^2 \frac{\partial^2}{\partial q_i^2}. \tag{4.5b}$$

The operator Θ raises by 3 the degree in x_i, y_i, of a form to which it is applied, and the LAPLACE operator Δ lowers the degree by 2. We shall prove the following theorem:

THEOREM 4. - A set of six basic invariants of degrees 2, 5, 8, 6, 9, 12 for the orthogonal group F_2 are

$$A = p_1 + p_2 + p_3, \quad \Theta A, \quad \Theta^2 A, \quad \Delta\Theta^2 A, \quad \Theta\Delta\Theta^2 A, \quad \Theta^2\Delta\Theta^2 A. \tag{4.6}$$

where the operators Θ and Δ are defined by (4.5). The Jacobian of these six invariants, expressed as a function of degree 36 in x_i, y_i, factors into the 36 linear factors f_{im} and f_{kmn}:

$$\begin{aligned} f_{im} &= -x_i \sin 2\pi m/3 + y_i \cos 2\pi m/3, & i, m = 1, 2, 3, \\ f_{kmn} &= \varphi_{1k} + \varphi_{2m} + \varphi_{3n}, & k, m, n = 1, 2, 3, \end{aligned} \tag{4.7}$$

that vanish respectively on the 36 symmetry hyperplanes corresponding to the hyperplane reflections S_k of (2.7).

PROOF: It is clear from (4.2) that the functions Φ_n are functions of φ_{im} completely symmetrized with respect to the second subscript m. Since for fixed i the elementary symmetric functions of φ_{im} are

$$\varphi_{i1} + \varphi_{i2} + \varphi_{i3} = 0, \quad \varphi_{i2}\varphi_{i3} + \varphi_{i3}\varphi_{i1} + \varphi_{i1}\varphi_{i2} = -3p_i/4, \quad \varphi_{i1}\varphi_{i2}\varphi_{i3} = 3q_i/4, \tag{4.8}$$

it follows that all rational invariants are rational functions of the six variables p_i, q_i, of (4.4). Cyclic permutations of the subscripts i in φ_{im} leave the functions Φ_n unchanged, whereas odd permutations change the sign of those whose degree in x_i, y_i, is odd, but leave unchanged those of even degree. This symmetry is introduced by using determinants of odd degree in q and permanents of even degree in q. For example we write

$$B = |\, q\, p\, 1\, | \equiv q_1(p_2 - p_3) + q_2(p_3 - p_1) + q_3(p_1 - p_2). \tag{4.9}$$

$$|\, qp^2\, q\, p\, |^+ \equiv q_1p_1^2(q_2p_3 + q_3p_2) + q_2p_2^2(q_3p_1 + q_1p_3) + q_3p_3^2(q_1p_2 + q_2p_1). \tag{4.10}$$

The first of these is a determinant of degree 5 in x, y which we shall prove to be the basic invariant $B = \Theta A/5$. The second is a permanent of degree 12, which resembles a determinant except for the fact that all six terms in its expansion are added without changes of sign. This particular permanent is not invariant under the group.

Every invariant of even degree is a linear combination of permanents similar to (4.10) and every invariant of odd degree is a linear combination of determinants similar to (4.9). A necessary and sufficient condition that such forms in p_i, q_i, be invariant under the group is that they be invariant under the single substitution S_{333}, whose explicit form is

$$S_{333}: x_i' = x_i - 2s, \qquad y_i' = y_i, \qquad \text{where } s = (x_1 + x_2 + x_3)/3. \tag{4.11}$$

This is true because the variables p_i and q_i are individually invariant under each of the nine reflections $S_{\varkappa 00}$, $S_{0\varkappa 0}$, $S_{00\varkappa}$, $\varkappa = 1, 2, 3$, and we know by Theorem 1 that these reflections together with S_{333} generate the group.

Since p_i and q_i are of degrees 2 and 3 in x_i, y_i, respectively, the only determinant of degree 3 is $|q\,11|$, which vanishes, and the only non-vanishing one of degree 5 is $|q\,p\,1|$. This must therefore be an invariant. We may verify directly the invariance under (4.11) of $A = \frac{1}{2}|p\,11|^{+}$ and $B = |q\,p\,1|$, as follows:

$$(4.12)\quad A' = \frac{1}{2}|p'\,11|^{+} = A - 2s\,|x\,11|^{+} + 2s^2\,|111|^{+} = A - 12s^2 + 12s^2 = A.$$

$$\begin{aligned}(4.12b)\quad B' &= |q'p'1 = |qp\,1| - 2s[\partial q/\partial x\,p\,1| + |q\,\partial p/\partial x\,1|] + 2s^2[|\partial q^2/\partial x^2 p 1| + 2|\partial q/\partial x \partial p/\partial x\,1|] \\ &\qquad + \text{vanishing terms,} \\ &= B - 2s[|x^2 - y^2\,2x^2 1| + |q\,2x\,1|] + 2s^2[|2x\,x^2 + y^2\,1| + |(2x^2 - 2y^2)\,2x\,1|] \\ &= B - 2s[|q/x\,2x^2\,1| + |q\,2x\,1| - 3s\,|q/x\,2x\,1|] = B.\end{aligned}$$

Under the orthogonal linear group F_2 the partial derivatives of an invariant B with respect to the original variables x_i, y_i, transform in the same manner as the coordinates themselves, and the scalar product of the gradients of two invariants B and C is a new invariant. Expressed in terms of the variables p_i, q_i, we have

$$\begin{aligned}\nabla B \cdot \nabla C &= \sum_{i=1}^{3} \frac{\partial B}{\partial x_i}\frac{\partial C}{\partial x_i} + \frac{\partial B}{\partial y_i}\frac{\partial C}{\partial y_i} \\ &= \sum_{i=1}^{3}\left(\frac{\partial B}{\partial x_i}\frac{\partial p_i}{\partial x_i} + \frac{\partial B}{\partial y_i}\frac{\partial p_i}{\partial y_i}\right)\frac{\partial C}{\partial p_i} + \left(\frac{\partial B}{\partial x_i}\frac{\partial q_i}{\partial x_i} + \frac{\partial B}{\partial y_i}\frac{\partial q_i}{\partial y_i}\right)\frac{\partial C}{\partial q_i} \\ &= \sum_{i=1}^{3}\left(4p_i\frac{\partial B}{\partial p_i} + 6q_i\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial p_i} + \left(6q_i\frac{\partial B}{\partial p_i} + p_i^2\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial q_i}.\end{aligned}$$

$$(4.13)\qquad \frac{1}{2}\nabla B \cdot \nabla C = \Theta C$$

where Θ is differential operator defined in (4.5a). Thus the operator Θ converts an invariant of degree n into a new one of degree $n+3$. Starting with A of degree 2, we find $\Theta A = 5B$ (of degree 5). Then $H = \Theta B$ is a new invariant of degree 8, and ΘH is an invariant of degree 11.

Using the LAPL CE operator Δ, which assumes the form (4.5b) in the variables p_i, q_i, we get nothing new from A or B, since $\Delta A = 12$, $\Delta B = 0$. However, ΔH is an invariant of degree 6 that involves q explicitly, and is therefore not proportional to A^3. Cancelling a numerical factor 16, we set $C = \Delta H/16$. Then we find that the invariants ΘC and $A^2 B$ of degree 9 are independent, and we take the simplest combination of these as $J = (\Theta C - 3A^2 B)/9$. Finally setting $K = 2\Theta J/3$, we obtain the following six linearly independent invariants of degree 12: A^6, $A^3 C$, $A^2 H$, AB^2, C^2, K.

Expressions for some of the invariants in terms of permanents and determinants are as follows:

(4.14)

2) $A = \frac{1}{2}|p11|^+$. 4) $A^2 = \frac{1}{2}|p^2 11|^+ + |p\,p\,1|^+$.

5) $B = \Theta A/5 = |q\,p1|$. 7) $AB = |qp\,p\,1| + |q\,p^2 1|$.

6) $C = \Delta H/16 = |p^2 p1|^+ - \frac{1}{2}|p\,p\,p|^+ + \frac{1}{2}|q^2 11|^+ - 5|q\,q1|^+$.

8) $H = \Theta B = \frac{1}{2}|p^2 p^2 1|^+ - \frac{1}{2}|p^2 p\,p|^+ + 2|q^2 p\,1|^+ - 6|q\,qp\,1|^+ + 4|q\,q\,p|^+$.

9) $J = (\Theta C - 3A^2 B)/9 = |qp\,p^2 1| + 2|q\,p\,p^2| + 4|q\,q^2 1|$.

12) $K = 2\Theta J/3 = \frac{1}{3}|p^3 p^3 1|^+ - |p^3 p^2 p|^+ + \frac{2}{3}|p^2 p^2 p^2|^+ + 4|q^2 p\,p^2|^+ + 2|q^2 p^3 1|^+ - 6|q^2 p^2 p|^+$
$- 10|qp^2 qp\,1|^+ + 4|qp^2 qp|^+ + 12|qp\,qp\,p|^+ - 10|qp\,q\,p^2|^+ + 4|q\,qp^3|^+$
$- 8|q^3 q\,1|^+ + 16|q^2 q^2 1|^+ - 8|q^2 q\,q|^+$.

Expressions for the power sum invariants Φ_n in terms of A, B, C, H, J, K, in addition to $\Phi_1 = \Phi_3 = 0$, are

(4.15)

$\Phi_2 \div 4 = A$ $\Phi_8 \div 4\cdot 3^2 = 3A^4 + 16AC + 20H$

$\Phi_4 \div 4\cdot 3 = A^2$ $\Phi_9 \div \binom{9}{3} 4\cdot 3^3 = A^2 B + J/2$

$\Phi_5 \div \binom{5}{3} 4\cdot 3 = B$ $\Phi_{10} \div 4\cdot 3^4 = A^5 + 10A^2 C + 30AH + 70B^2$

$\Phi_6 \div 4\cdot 3^2 = A^3 + 2C$ $\Phi_{11} \div \binom{11}{3} 4\cdot 3^2 = 9A^3 B + 11AJ + 4BC$

$\Phi_7 \div \binom{7}{3} 4\cdot 3^2 = AB$ $\Phi_{12} \div 4\cdot 3^4 = 3A^6 + 90K + 8C^2 + 48A^3 C + 250A^2 H + 1120AB^2$.

The particular invariants mentioned in Theorem 4 are also expressible in terms of A, B, C, H, J, K:

(4.16) $A,\quad \Theta A = 5B,\quad \Theta^2 A = 5H,\quad \Delta\Theta^2 A = 80C,\quad \Theta\Delta\Theta^2 A = 720J + 48A^2\Theta A$
$\Theta^2\Delta\Theta^2 A = 1080K + 72A^2\Theta^2 A + 144(\Theta A)^2$.

To complete the proof of Theorem 4 we must show that the Jacobian of the six functions A, B, C, H, J, K, with respect to the variables x_i, y_i, is not identically zero, and find its linear factors. A partial factorization is afforded by the relations:

$$(4.17)\qquad \frac{\partial(A\,B\,C\,H\,J\,K)}{\partial(x_1 y_1 x_2 y_2 x_3 y_3)} = \frac{\partial(A\,B\,C\,H\,J\,K)}{\partial(p_1 q_1 p_2 q_2 p_3 q_3)} \cdot \frac{\partial(p_1 q_1)}{\partial(x_1 y_1)} \cdot \frac{\partial(p_2 q_2)}{\partial(x_2 y_2)} \cdot \frac{\partial(p_3 q_3)}{\partial(x_3 y_3)}$$

$$(4.18)\qquad \frac{\partial(p_i q_i)}{\partial(x_i y_i)} = \begin{vmatrix} 2x_i & x_i - y_i^2 \\ 2y_i & -2x_i y_i \end{vmatrix} = 2y_i(-3x_i^2 + y_i^2) = (2f_{i1})(2f_{i2})(2f_{i3});$$

where f_{im} are defined by (4.7). Now we argue that since the nine functions f_{im} are linear factors of the Jacobian, and since the Jacobian is invariant under all the transformations of the group. it follows that all the 36 linear functions equivalent to f_{im} under the group must be factors of this Jacobian. Hence the first factor on the right of (4.17) is a constant multiple of the product of the 27 linear functions f_{kmn}. To find this constant multiple and show that it is not zero, we take a simple point (x, y) such as (1, 0, 0, 0, 0, 0) at which the product of f_{kmn} has the non vanishing value 2^{-18}, and compare it with the Jacobian of the six invariants with respect to the variables p_i, q_i, at the point $p_1 = 1$, $q_1 = 1/3$, $p_2 = q_2 = p_3 = q_3 = 0$.

The only terms from H, J, and K that have non vanishing partial derivatives at this point are $2\,|q^2p1|^+ - 6\,|q\,qp1|^+$ from H, $4\,|q\,q^21|$ from J, and $-8\,|q^3q1|^+$ from K.

Thus for $p_2 = p_3 = q_2 = q_3 = 0$ we obtain

$$\text{(4.18)}\qquad \frac{\partial(A\,B\,C\,H\,J\,K)}{\partial(p_1q_1p_2q_2p_3q_3)} = \frac{\partial(A\,B\,H\,C\,J\,K)}{\partial(p_1p_2p_3q_1q_2q_3)} =$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & q_1 & 2q_1^2 & p_1^2 & 0 & 0 \\ 1 & -q_1 & 2q_1^2 & p_1^2 & 0 & 0 \\ 0 & 0 & 0 & 2q_1 & 0 & 0 \\ 0 & -p_1 & -6q_1p_1 & -10q_1 & -4q_1^2 & -8q_1^3 \\ 0 & p_1 & -6q_1p_1 & -10q_1 & 4q_1^2 & -8q_1^3 \end{vmatrix} = (2q_1)^9.$$

Since $q_1 = 1/3$, the exact factorization of the Jacobian (4.17) is

$$\text{(4.20)}\qquad \frac{\partial(A\,B\,C\,H\,J\,K)}{\partial(x_1y_1x_2y_2x_3y_3)} = (16/3)^9\Big(\prod_{i,\,m=1}^{3} f_{im}\Big)\Big(\prod_{k,\,m,\,n=1}^{3} f_{kmn}\Big).$$

We shall find that some knowledge of the invariants of the representation F_2 is helpful in reducing its KRONECKER powers and finding precisely what variables undergo the different irreducible representations of G.

5. The irreducible characters and representations of $[3^{2,2,1}]$.

The following techniques which we denote by I, P, S, O, M, may be combined to compute the irreducible characters of a group G over the complex number field. (I) the analysis of characters *induced* by representations of a subgroup; (P) the analysis of KRONECKER *products* of two knowu characters; (S) the analysis of the KRONECKER mth power of a known character by first using *Schur*'s method of decomposing the mth power into components, one for each partition of m, that are irreducible for the full linear group, and then further decomposing these « partition characters » for the subgroup G; (O) the *orthogonality* relations among irreducible group characters χ_λ^x (expressed by the fact that $\chi_\lambda^x\, n_\lambda^{-1/2}$ is a unitary matrix); and (M) the theory of *modular* characters of the indecomposable and irreducible representations of G over a finite field whose characteristic p divides the order of G. This modular theory is extensively developed in papers of R. BRAUER [**4, 5**].

METHOD I: For the group G_{51840} we apply first the method I, using characters induced by the representations of the simple subgroup G_0 of order 25920 whose characters have been published [**16**]. From the ten real characters of G_0 we obtain ten induced characters of G, each of which splits into an associated pair $\chi^i + \chi^{i*}$ $(i = 1, \dots 0)$, such that $\chi_\lambda^i = \chi_\lambda^{i*}$ for even classes C_λ and $\chi_\lambda^i = -\chi_\lambda^{i*}$ for odd classes. Arbitrarily we designate by χ^i those characters having positive value in the class C_{16} $(= 1^4 2)$, and by χ^{i*} those with negative values in this class. The five remaining characters of G, $\chi^{(11)}, \dots, \chi^{(15)}$ are self associated, and are obtained for even classes by adding pairs of conjugate complex characters of the subgroup G_0. These self associated characters have value 0 for odd classes. In order to see at a glance the degrees of characters that are involved in the various equations it will also be helpful to denote a character by its degree (value for identity class) followed by a subscript, as follows.

(5.1) $1_p 1_n 6_p 6_n 15_p 15_n 20_p 20_n 30_p 30_n 64_p 64_n 81_p 81_n 15_q 15_m 24_p 24_n 60_p 60_n 20_s 90_s 80_s 60_s 10_s$.

Those with subscripts s are self associated, whereas those with subscripts p (or q) are positive for C_{16} and those with n (or m) are negative for C_{16}. The characters 15_p and 15_n have the value -1 in class C_2 whereas 15_q and 15_m are $+3$ in that class. In table II each character χ^i is shown as a column vector.

All characters are known for the even subgroup, and the self-associated characters vanish in the other classes. So it remains to compute the characters 15_p, 20_p, 30_p, 64_p, 81_p, 15_q, 24_p, 60_p, for the odd classes. Five of these can be computed by analyzing the characters induced by the 1-character of certain large subgroups of G. We decompose as follows the five transitive permutation characters given in Table I, using orthogonality relations to find irreducible components whose characters are already known.

(5.2)
$$\chi^{(27)} = 1_p + 6_p + 20_p \qquad \chi^{(45)} = 1_p + 20_p + 24_p$$
$$\chi^{(36)} = 1_p + 20_p + 15_q \qquad \chi^{(40)} = 1_p + 24_p + 15_p$$
$$\chi^{(72)} = \chi^{(36)} + 6_p + 30_p.$$

For a given class C_λ, $\chi_\lambda^{(27)}$ is the number of vertices of 2_{21} (or lines of the cubic surface) left fixed, $\chi_\lambda^{(45)}$ is the number of elements of C_3 (or tritangent planes of the cubic surface) left invariant, $\chi_\lambda^{(36)}$ the number of elements of C_{16} (or of double-sixes) left fixed, $\chi_\lambda^{(72)}$ the number of sextuples (half of a double-six) left fixed, and $\chi_\lambda^{(40)}$ the number of subgroups of order 3 from class C_{11} that are left fixed, by an element in the given class of conjugates. We note that each equation in (5.2) involves only one new character, since the character 6_p has already been found.

METHOD S: The « partition » characters which SCHUR [30] has shown to be the irreducible components of the KRONECKER m-th power of the full linear group, will be denoted by using the corresponding partition as a superscript. For the class whose cycle symbol is $1^\alpha 2^\beta 3^\gamma 4^\delta \ldots$ as described in (3.2) and (3.3) and whose character is $\chi^{[1]} = \alpha$, these characters have the following values:

$$(5.3)\quad \begin{aligned} \chi^{[1^2]} &= \binom{\alpha}{2} - \beta & \chi^{[1^4]} &= \binom{\alpha}{4} - \binom{\alpha}{2}\beta + \alpha\gamma - \delta + \binom{\beta}{2} \\ \chi^{[2]} &= \binom{\alpha+1}{2} + \beta & \chi^{[21^2]} &= 3\binom{\alpha+1}{4} - \binom{\alpha+1}{2}\beta + \delta - \binom{\beta}{2} \\ \chi^{[1^3]} &= \binom{\alpha}{3} - \alpha\beta + \gamma & \chi^{[2^2]} &= \binom{\alpha+1}{4} + \binom{\alpha+2}{4} - \alpha\gamma + \beta^2 \\ \chi^{[21]} &= 2\binom{\alpha+1}{3} - \gamma & \chi^{[31]} &= 3\binom{\alpha+2}{4} + \binom{\alpha}{2}\beta - \delta - \binom{\beta+1}{2} \\ \chi^{[3]} &= \binom{\alpha+2}{3} + \alpha\beta + \gamma & \chi^{[4]} &= \binom{\alpha+3}{4} + \binom{\alpha+1}{2}\beta + \alpha\gamma + \delta + \binom{\beta+1}{2}. \end{aligned}$$

Applying these formulas to the known character $\chi = 6_p$, we obtain decompostiions as follows:

$$(5.4)\quad \begin{aligned} 6_p^{[1^2]} &= \mathbf{15}_p & 6_p^{[1^4]} &= \mathbf{15}_n \\ 6_p^{[2]} &= 1_p + \mathbf{20}_p & 6_p^{[21^2]} &= 15_p + \mathbf{90}_s \\ 6_p^{[1^3]} &= \mathbf{20}_s & 6_p^{[2^2]} &= 1_p + 20_p + (\mathbf{24}_p + \mathbf{60}_p) \\ 6_p^{[21]} &= 6_p + \mathbf{64}_p & 6_p^{[31]} &= 15_p + 20_p + 30_p + 64_p + \mathbf{81}_p \\ 6_p^{[3]} &= 6_p + 20_p + \mathbf{30}_p & 6_p^{[4]} &= 1_p + 20_p + 6_p + 64_p + 20_p + \mathbf{15}_q. \end{aligned}$$

From these decompositions the characters 15_p, 20_p, 20_s, 64_p, 30_p, 15_n, 90_s, 81_p, and 15_q can be computed in succession without using the results of the other methods. This serves as a check on method I and gives the new characters 64_p and 81_p. By using the value of 24_p, obtained from the permutation character $\chi^{(45)}$, we see that 60_p can also be computed. This finishes the list of characters of $[3^{2,2,1}]$ given in Table II. We check the complete character table by the orthogonality relations which state that the matrix $\chi_\lambda^x n_\lambda^{-1/2}$ is a unitary matrix. Certain KRONECKER products may also be used as checks. For example we have:

$$(3.5)\qquad 6_p 15_q = 30_p + 60_p, \qquad 6_p 24_p = 64_p + 80_s, \qquad 6_p 10_s = 60_s.$$

METHOD *M*: The modular theory could have been used directly to obtain such characters as 81_p, 81_n, 64_p, 64_n and others. Since we shall use this theory later for the much larger group $[3^{3,2,1}]'$, we illustrate the modular theory here by two examples.

For the prime $p=3$, there are thirteen p-singular classes of G containing elements whose orders are divisible by p, and twelve p-regular classes containing elements whose orders are prime to 3 but divide $51840/3^4 = 2^7 5$. For only one regular class C_λ is the order n_λ of the normalizer of an element divisible by 3^4, so the theory of BRAUER [4] shows that there is but one block of characters of type 0 (mod 3). This block contains the 1-character and all characters with degrees not divisible by 3, besides several others. Within this block all the class multipliers $g_\lambda\chi^\varkappa_\lambda/\chi^\varkappa_1$ vanish (mod 3) except for class C_1. Actually all but two characters belong in this block, since the only characters for which this congruence does not hold (mod 3) are the two characters 81_p and 81_n, each of which forms a block of highest type. Characters of highest type must vanish for all p-singular classes. In the p-regular classes, however, these characters of degree 81 must in this case be odd for elements whose orders are a power of 2; and they have the value 1 in the class $1\cdot5$, and ±1 in the class $1^{-1}2\cdot5$. In each p-regular class therefore the character may be written $3^{e_\lambda}x_\lambda$ where 3^{e_λ} is the highest power of 3 dividing the order n_λ of the normalizer of an element; and is x_λ an odd integer. We do this for the odd coset, and use known values for the even subgroup. The three orthogonality relations involving the sum of squares of the characters and the scalar product with 1_p and 6_p now determine the characters 81_p and 81_n. We have, for the twelve 3-regular classes:

n_λ: $3^4(640)$, $3^2(128)$, $3(64)$, $3(32)$, 16, 10; $3^2(160)$, $3(32)$, (32), $3(32)$, (8), (10).

$$(5.6)\quad \sum_\lambda \frac{(\chi_\lambda)^2}{n_\lambda} = \frac{3^4}{640}+\frac{3^2}{128}+\frac{3}{64}+\frac{3}{32}+\frac{1}{16}+\frac{1}{10}+\frac{3^2x_1^2}{160}+\frac{3x_2^2+x_3^2+3x_4^2}{32}+\frac{x_5^2}{8}+\frac{x_6^2}{10}=1,$$

$$(5.7)\quad \sum_\lambda \frac{\chi_\lambda}{n_\lambda} = \frac{1}{640}+\frac{1}{128}-\frac{1}{64}-\frac{1}{32}-\frac{1}{16}+\frac{1}{10}+\frac{x_1}{160}+\frac{x_2+x_3+x_4}{32}+\frac{x_5}{8}+\frac{x_6}{10}=0,$$

$$(5.8)\qquad \text{(Scalar product with } 6_p-6_n\text{)}: \frac{4x_1}{160}+\frac{2x_3-2x_4}{32}-\frac{x_6}{10}=0.$$

Equation (5.6) is satisfied only if the positive integers x_i^2 are all 1, and from (5.7) and (5.8) we readily conclude that

$$(5.9)\qquad x_1=-x_2=-x_3=x_4=x_5=-x_6=\pm1.$$

The character values for 81_p in the 3-regular classes of the odd coset must be 9, -3, -1, 3, 1, -1 and in the 3-singular classes they are 0, whereas $81_n = -81_p$ in odd classes.

As a second example of the modular theory, we consider the prime 5, for which there are 23 regular classes, of which two have normalizers of an order n_λ divisible by 5 (the highest power of p dividing the group order). There must be two blocks of type 0 (and defect 1), and each block contains 5 ordinary characters with degrees alternately congruent to 1 and -1 (mod 5). Within each block the sum of the three characters of degrees $\equiv 1$ (mod 5) is equal to the sum of the other two characters, except for the two 5-singular classes. Thus we obtain the identities

$$(5.10)\qquad \begin{matrix} 1_p - 24_p + 81_n - 64_n + 6_n = 0 \\ 1_n - 24_n + 81_p - 64_p + 6_p = 0 \end{matrix} \quad \left(\begin{matrix}\text{Except in the two} \\ \text{classes } 1\cdot 5 \text{ and } 1^{-1}2\cdot 5.\end{matrix} \right.$$

Characters in the first block have equal values in the two classes $1\cdot 5$ and $1^{-1}25$, whereas those in the second block have values of opposite sign in the two classes.

Sums of consecutive characters in either row of (5.10) are indecomposable characters (mod 5). The remaining ordinary charecters whose degrees are divisible by 5 are both indecomposable and irreducible (mod 5). It is clear that 64_p is connected with 6_p and with 81_p in the indecomposable characters (mod 9) because the products

$$(5.11)\qquad \begin{aligned} 6_p 15_p &= 20_s + 6_p + 64_p \\ 6_p 30_p &= 20_p + 64_p + 81_p + 15_q \end{aligned}$$

must be sums of indecomposable characters (mod 5), since 15_p and 30_p are each indecomposable (mod 5).

The rules for combining characters, and the knowledge of the invariants shed some light on the irreducible representations themselves. Thus the SCHUR decompositions of the KRONECKER mth power [30] describe not only the characters but the homogeneous functions of the variables $x_1 \dots x_6$ in our basic six-dimensional representation 6_p that undergo these representations. To split these further for this particular finite group G we must make use of the basic invariants of the group. Thus the representation 15_p is a representation induced on the PLUECKER coordinates of lines in 6-space, or on such expressions as $x_i\partial I/\partial x_j - x_j\partial I/\partial x_i$, where I is any invariant function of $x_1 \dots x_6$. The 56 homogeneous symmetric products of degree 3 belonging to $6_p^{[3]}$ can be resolved into a set of six such as $x_i \Sigma_j x_j^2$ that undergo the representation 6_p, a set of 20 combinations of second partial derivatives of the fifth degree invariant B of (4.9) that undergo 20_p, and a residual set of 30 quantities belonging to 30_p.

Table II.

Character table of the group $[3^{2,2,1}]$ of order 51840.

Class	g_λ																n_λ
1^6 *Even classes*	1	1_p	6_p	15_p	20_p	30_p	64_p	81_p	15_q	24_p	60_p	20_s	90_s	80_s	60_s	10_s	51840
$1^2 2^2$	270	1	2	−1	4	2	0	−3	3	0	4	−4	−6	0	4	2	192
$1^{-2}2^4$	45	1	−2	−1	4	−10	0	9	7	8	−4	4	−6	−16	12	−6	1152
$1^2 2^{-2}4^2$	540	1	2	3	0	−2	0	−3	−1	0	0	4	2	0	4	2	96
24	3240	1	0	−1	0	0	0	−1	1	0	0	0	2	0	0	−2	16
$1^3 3$	240	1	3	3	5	3	4	0	0	0	−3	2	0	−4	−6	−2	216
$123^{-1}6$	1440	1	1	−1	1	−1	0	0	−2	2	−1	−2	0	2	0	0	36
$1^{-1}2^2 3$	2160	1	−1	−1	1	−1	0	0	0	0	1	2	0	0	−2	2	24
3^2	480	1	0	0	−1	3	−2	0	3	3	−3	2	0	2	0	4	108
$1^{-2}26$	1440	1	−2	2	1	−1	0	0	1	−1	−1	−2	0	2	0	0	36
$1^{-3}3^3$	80	1	−3	6	2	3	−8	0	−3	6	6	−7	9	−10	−3	1	648
$12^{-2}3^{-1}6^2$	720	1	1	2	−2	−1	0	0	1	2	2	1	−3	2	−3	−3	72
$1^{-1}234^{-1}6^{-1}(12)$	4320	1	−1	0	0	1	0	0	−1	0	0	1	−1	0	1	−1	12
$3^{-1}9$	5760	1	0	0	−1	0	1	0	0	0	0	−1	0	−1	0	1	9
15	5184	1	1	0	0	0	−1	1	0	−1	0	0	0	0	0	0	10
$1^4 2$ *Odd classes*	36	+1	4	5	10	10	16	9	5	4	10						1440
2^3	540	+1	0	−3	2	−2	0	−3	1	4	2						96
$1^2 4$	1620	+1	2	1	2	0	0	−1	−1	0	−2						32
$1^{-2}2^2 4$	540	+1	−2	1	2	−4	0	3	3	0	−2						96
$24^{-1}8$	6480	+1	0	−1	0	0	0	1	−1	0	0						8
123	1440	+1	1	−1	1	1	−2	0	2	−2	1						36
$1^{-2}23^2$	1440	+1	−2	2	1	1	−2	0	−1	1	1						36
6	4320	+1	0	0	−1	1	0	0	1	1	−1						12
$12^{-1}3^{-1}46$	4320	+1	1	1	−1	−1	0	0	0	0	1						12
$1^{-1}25$	5184	+1	−1	0	0	0	1	−1	0	−1	0						10

Self associated characters vanish for odd classes. Ten other characters are obtained from the first ten by multiplying these characters by −1 in odd classes.

6. Characters of the group of automorphisms of the 28 bitangents to a plane quartic curve of genus three.

The group of automorphisms of the 28 bitangents to a plane quartic curve is a simple group H_0 of order 28×51840, called $[3^{3,2,1}]'$ by COXETER [**12**] and $A(6, 2)$ by DICKSON [**15**]. It contains as its subgroup leaving one bitangent fixed, the group $G = [3^{2,2,1}]$ of the 27 lines on the cubic surface. The tangent plane at a general point P of a cubic surface, and the 27 planes each passing through P and one of the lines of the cubic surface, meet an arbitrary section plane in the 28 bitangents. The plane quartic curve to which they are tangent is cut by the section plane from the quartic cone whose elements pass through P and are tangent to the cubic surface. The direct product of this group H_0 and a group of order 2 is a group H, of order 2,903,040, called $[3^{3,2,1}]$ by COXETER, which is the group of symmetries of the uniform 7-dimensional polytope 3_{21} of 56 vertices, and is also the group associated with the exceptional LIE group E_7.

A set of 56 vertices for the polytope 3_{21} [see **11**, p. 186] are

$$(6.1) \qquad \begin{array}{l} (\pm 1,\ 0,\ \pm 1,\ \pm 1,\ 0,\ 0,\ 0) \\ (0,\ \pm 1,\ 0,\ \pm 1,\ \pm 1,\ 0,\ 0), \text{ etc.} \end{array}$$

the rest being obtained by cyclical permutations on the seven coordinates. Pairs of opposite vertices correspond to the 28 bitangents. Each vertex is at distance 2 from its 27 nearest neighbors, and the $56 \times 27/2$ segments joining neighboring vertices are parallel in sets of twelve to the normals to 63 symmetry hyperplanes. There are 135 ways of choosing seven of these hyperplanes as mutually orthogonal coordinate hyperplanes, and each hyperplane lies in 15 of these sets of seven. Each hyperplane contains 16 pairs of opposite vertices that correspond to a KUMMER set [**20**] of 16 bitangents, and the corresponding hyperplane reflection S interchanges in pairs those vertices that correspond to lines of a double six on the cubic surface. From the coordinates (6.1) it is clear that 56 vertices fall into sets of eight that form the vertices of a cube. Each pair of orthogonal hyperplanes determines a unique third, forming a «cubic set», such that their three normals are parallel to the edges of one of the cubes. Exactly twelve more hyperplanes are orthogonal to each of the three in a cubic set, and contain the vertices of this cube; and these form three sets of four mutually orthogonal hyperplanes, whose reflections are related as the three distinct sets of S factors in (2.10*b*) for an element V of class C_3. We denote by $-V_k$ the product of the reflections in three hyperplanes of a cubic set. These involutory group elements $-V_k$ are *not* conjugate to the involutory elements U_j of classes C_{17}, that are products of reflections in three mutually orthogonal hyperplanes having four pairs of opposite vertices in common. Similarly, products of reflections in four mutually orthogonal hperplanes are of two types V and $-U$, that belong to different classes of conjugates in H_0. Both classes, however have the same cycle symbol $1^{-1}2^4$ in the 7-dimensional representation $\zeta^{(2)}$ of H_0. The fact that only odd powers of elements of the classes C_{16} to C_{25} can lie in these classes may be used to distinguish the classes C_{16} and C_3 in H_0. Similar considerations distinguish the classes C_{19} and C_5, both having the cycle symbol 124, and the classes C_{22} and C_{10}, both having the symbol $1^{-1}26$.

We wish to determine the table of characters of the group H_0. (From it the character table for H could easily be found by the direct product method). First we need to identify the classes of conjugates of H_0, of which there are 30. The first 15 classes are taken as those containing elements of the various even classes of G. To change the cycle symbol from 6 to 7 dimensions we simply add 1 to the exponent α. The classes $C_1 \ldots C_{15}$ are as follows.

(6.2*a*) 1^7, 1^32^2, $1^{-1}2^4$ (type V), $1^32^{-2}4^2$, 124, 1^43, $1^223^{-1}6$, 2^23, 13^2, $1^{-1}26$, $1^{-2}3^3$, $1^22^{-2}3^{-1}6^2$, $234^{-1}6^{-1}(12)$, $13^{-1}9$, 1^25.

Classes C_{16} to C_{25} of H_0 are obtained from the odd classes of G by multiplying each of their elements by -1, the inversion in the origin. Each product U_jT_k in G is replaced by $(-U_j)T_k$ in H_0, so *these elements are still products of at most six* S. *factors*. Multiplication by -1 takes a complete set of $(2k)$th roots of unity into itself, but changes a set of kth roots, when k is odd, into the $(2k)$th roots that are not kth roots. Hence to obtain the new cycle symbol for one of the classes C_{16} to C_{25}, we first increase α by 1, and then replace each power k^{α_k} of an odd order cycle by $k^{-\alpha_k}(2k)^{\alpha_k}$, and combine exponents. The new class symbols for C_{16} to C_{25} in H_0 are

(6.2*b*) $1^{-5}2^6$, $1^{-1}2^4$ (type $-U$), $1^{-3}2^34$, 124, $1^{-1}2^24^{-1}8$, $1^{-2}2^33^{-1}6$, $13^{-2}6^2$, $1^{-1}26$, $1^{-2}234$, $25^{-1}(10)$.

Denoting by g_λ and h_λ respectively the number of elements of G and H_0 in class C_λ of H_0, and denoting by n_λ and N_λ the orders of the corresponding normalizers of an element in G or in H_0, the equation

$$28g_\lambda/h_\lambda = 1 + \chi_\lambda^{(27)} = \zeta_\lambda^{(1)} + \zeta_\lambda^{(3)}. \tag{6.3}$$

determines the numbers h_λ, in those 25 classes of H_0 where $g_\lambda \neq 0$, in terms of the character of the permutation group P_{28} on the 28 cosets of G in H_0 (or the 28 bitangents). This character exceeds by one the subgroup character $\chi^{(27)}$ which was given in Table I. Since P_{28} is doubly transitive, $\chi^{(27)}$ is a simple irreducible character of H_0 which we shall call $\zeta^{(3)}$. It has the value -1 for the classes of H_0 that are not represented in G. By the orthogonality relations for group characters, we may compute the fractional contribution of these unknown classes to the 1-character $\zeta^{(1)}$ and to the 7-dimensional character $\zeta^{(2)}$, and its square. We divide the sum of these characters for the elements of known classes by the order of H_0 and subtract from 0 (or 1), to obtain the fractional sum (or sum of squares).

$$
\begin{aligned}
&\frac{1}{15}+\frac{1}{128}+\frac{5}{48}+\frac{1}{7}=\sum_\lambda 1/N_\lambda \quad \text{for unknown classes } C_{26} \dots \\
(6.4)\qquad &-\frac{1}{15}-\frac{1}{128}+\frac{1}{16} \quad =\sum_\lambda \zeta^{(2)}_\lambda/N_\lambda \qquad » \qquad » \qquad » \\
&\frac{1}{15}+\frac{1}{128}+\frac{1}{16} \quad =\sum_\lambda (\zeta^{(2)}_\lambda)^2/N_\lambda \qquad » \qquad » \qquad »
\end{aligned}
$$

Thus we are led to five additional classes, containing respectively 1/15, 1/24, 1/128. 1/16 and 1/7 of the elements of H_0, and having the traces -1, 0, -1, 1, 0 respectively in the 7-dimensional $\zeta^{(2)}$. A product of four S factors forming a 5-chain, and two other S-factors permutable with these and forming a 3-chain, can be constructed in the 7-space, giving a class of elements $1^{-1}35$ of order fifteen that are permutable only with their own powers. Six S factors can also be put together to form a 7-chain, which effects a cyclic permutation of the seven coordinates in (6.1), and represents a class of elements of order 7. An 8-chain of seven S factors can also be constructed, in which the first, third, fifth and seventh factors are permutable with each other. By moving these four factors to the left to form a product of type V, and then replacing them by a set of three factors equal to $-V$, we obtain an element of order eight and type $12^{-1}8$. Its normalizer is of order 16. Its square, of type $1^{-1}4^2$, is not represented in G, and belongs to a new class. Finally the product of a 3-chain and 4-chain, mutually disjoint, gives a new element of H of order 12 whose negative is of type $3^{-1}46$ in H_0. The traces in $\zeta^{(2)}$ check as they should, and we have the following classes:

(6.5) New classes in H_0: $1^{-1}35$, $3^{-1}46$, $1^{-1}4^2$, $12^{-1}8$, 7.
Order of normalizer N_λ: 15, 24, 128, 16, 7.

This completes the analysis of 30 classes of conjugates in H_0.

The thirty irreducible characters of H_0 (over the field of complex numbers) will be denoted by ζ^i, $(i=1,\dots,30)$, but in order to follow the computations more easily we shall also denote each character by attaching to the degree a subscript a, b or c, using b and c respectively for a second or third character of the same degree. As before we shall use a partition symbol in square brackets as a superscript to denote the (possibly reducible) character corresponding to SCHUR's irreducible components of the KRONECKER mth power of the full linear group. The derivation of the 30 irreducible characters may be carried out in three stages using different methods of attack that are available.

In the first stage, knowing the 1-character $\zeta^{(1)}=1_a$, and the 7-dimensional character $\zeta^{(2)}=7_a$, we use the method S, computing SCHUR's partition components of the KRONECKER m-th power by formulas (5.3) and then splitting off known components identified by use of the orthogonality relations. (The multiplicity of an irreducible ζ^i as a component of a reducible ζ is the mean value of $\zeta\bar{\zeta}^i$ averaged over all group elements). Six new characters may be isolated in this way as follows:

$$(6.6)\qquad \begin{array}{ll} 7_a^{[1^2]} = \mathbf{21}_a & 7_a^{[2]} = 1_a + \mathbf{27}_a \\ 7_a^{[1^3]} = \mathbf{35}_a & 7_a^{[21]} = 7_a + \mathbf{105}_a \\ 21_a^{[1^2]} = 21_a + \mathbf{189}_a & 21_a^{[2]} = 1_a + 27_a + 35_a + \mathbf{168}_a . \end{array}$$

In each of these decompositions the last character is the only new one, so its value is uniquely determined. The first stage ends when we can no longer obtain decompositions involving only one new character.

In the second stage we use the method I of induced representations. First we treat the eight known irreducible characters of H_0 as possibly reducible characters for the subgroup G, and reduce them by use of the character table of G (Table II). Then from certain of the irreducible characters χ^j of G which appear in these decompositions of ζ^i we form and reduce the induced character $I\chi^j$ of H_0 defined by the formula

$$(6.7)\qquad I\chi^j = (1_a + 27_a)\chi^j = \sum_i m_{ij}\zeta^i, \text{ for } H_0. \qquad \text{(See Table IV)}$$

The permutation character $1_a + 27_a$ of H_0 represents H_0 by permutations on the cosets of its subgroup G, and the non-negative integers m_{ij} indicate both the multiplicity with which ζ^i occurs as a component of the induced character $I\chi^j$, and the equal multiplicity with which χ^j occurs as a component of ζ^i when ζ^i is restricted to G. (This reciprocal formulation is not quite general, but holds in our case since the classes of H_0 de not split in G). Seven new characters are obtained as follows:

(6.8) Reduction of ζ^i in G — Reduction of induced $I\chi^j$ in H_0

Reduction of ζ^i in G		Reduction of induced $I\chi^j$ in H_0
$1_a = 1_p$	in G	$I = 1_a + \mathbf{27}_a$ (definition)
$7_a = 1_n + 6_n$	in G	$I1_n = 7_a + \mathbf{21}_b$
$27_a = 1_p + 6_p + 20_p$	in G	$I6_n = 7_a + 105_a + \mathbf{56}_a$
$21_a = 6_p + 15_p$	in G	$I6_p = 27_a + 21_a + \mathbf{120}_a$
$35_a = 15_n + 20_s$	in G	$I15_p = 21_a + 189_a + \mathbf{210}_a$
$105_a = 6_n + 15_n + 20_n + 64_n$	in G	$I15_n = 35_a + 105_a + \mathbf{280}_a$
$189_a = 15_p + 64_p + 20_s + 90_s$	in G	$I20_s = 35_a + 189_a + \mathbf{336}_a$
$168_a = 20_p + 64_p + 24_p + 60_p$	in G	$I20_p = 27_a + 168_a + 120_a + 210_a + \mathbf{35}_b$.

The method of forming and reducing KRONECKER products of 7_a with each of the known characters of small degree serves as a check at this stage, and yields one new character 189_b, bringing the total number of known characters up to 16.

$$
\begin{aligned}
(6.9)\quad & 7_a \times 21_a = 7_a + 35_a + 105_a & \quad 7_a \times 21_b &= 27_a + 120_a \\
& 7_a \times 27_a = 7_a + 105_a + 21_b + 56_a & \quad 7_a \times 56_a &= 27_a + 120_a + 210_a + 35_b \\
& 7_a \times 35_a = 21_a + 35_a + 189_a & \quad 7_a \times 35_b &= 56_a + 189_b
\end{aligned}
$$

A complete tabulation of products $\zeta^i \times \zeta^j$ that contain 7_a as a component, or of products $7_a\zeta^i$ that contain ζ^j as a component is a given in Table V at the end of this paper, based on the complete table of characters of H_0 (Table III). Note that if ζ^i is a component of the KRONECKER m-th power of 7_a, and if $\zeta^i\zeta^j$ contains 7_a, then ζ^j must be a component of either the $(m-1)$th power or the $(m+1)$th power of 7_a. Thus the characters fall into sets according to the least power of 7_a that contains them. Thus far we have obtained all components of the KRONECKER square, cube, and fourth power of 7_a, and some from the fifth power (280_a, 336_a and 189_b). But we have reached a point where products of known characters of small degree, either involve no new characters or at least two that must somehow be separated.

In the third stage we use the theory of modular characters combined with the induction method I. BRAUER [4] has shown that if p^a is the highest power of a prime p that divides the order $p^a g'$ of a finite group, then any ordinary irreducible character ζ^i whose degree z^i is divisible by p^a remains irreducible (mod p) and forms a block of the highest kind, which is also an indecomposable character (mod p). Next, an ordinary irreducible character ζ^i whose degree z^i is divisible by p^{a-1} but not by p^a belongs to a block of characters of defect 1 or type $a-1$, all of whose ordinary characters have this property and all of which have equal values (mod p) for the class multipliers $h_\lambda\zeta^i_\lambda/z^i$. For a group such as the group H_0 that we are studying, in which each family of p-conjugate classes consists of a single class, and therefore each family of p-conjugate irreducible representations has but one member, BRAUER's theorems show that a block of defect one consists of a chain of p ordinary irreducible representations ζ^i. The degree z^i of each is divisible by p^{a-1} but not by p^a, and the sum of characters of two consecutive members in the chain is an indecomposable character (mod p) which vanishes for p-singular classes (classes of elements whose orders are divisible by p) and has a value in a p-regular class C_λ that is divisible by the highest power of p dividing N_λ. The ordinary irreducible representations at the ends of the chain are irreducible (mod p), and the others contain two (mod p) irreducible components, one in common with each of its neighbors [5]. If we divide the ordinary characters of a block of defect 1 into two sets, so that adjacent members of the chain belong in different sets, then those characters in the same set have equal values in p-singular classes whereas two in opposite sets have values of sum zero in any p-singular class. Furthermore in any p-regular class the sum of the characters of one set is equal to the sum of those in the other set. If $a = 1$, the degrees in the one set are $\equiv 1 \pmod{p}$ and those in the other are $\equiv -1 \pmod{p}$, and the number of blocks of type 0 is equal to the number of p-regular classes for which $N^{\lambda} \equiv 0 \pmod{p}$.

When we apply this theory to the group H_0, we find that there are three blocks of type 0 (mod 5) in which the characters for the p-singular classes 1^25, $1^{-1}35$ and $25^{-1}10$ are respectively $\pm(1, 1, 1)$, $\pm(2, -1, 0)$ and $\pm(1, 1, -1)$. Each of these blocks contains 5 ordinary irreducible characters, and there remain fifteen characters of highest type with degrees divisible by 5. For the prime 3 we know that the characters 189_a and 189_b belong to a block of defect 1 which must contain a third ordinary character 378_a, and that the character 27_a belongs to a second block of defect 1 (mod 3). Thus we obtain five new characters 189_c, 378_a, 216_a, 512_a, and 84_a as follows:

		Relation in p-regular classes	Relation in p-singular classes
(6.10a)	$p=5$	$21_a - 189_a + 336_a - \mathbf{189}_c + 21_b = 0$	$(1,1,-1) = 21_a = -189_a = 336_a = -\mathbf{189}_c = 21_b$
(6.10b)	$p=3$	$189_a - \mathbf{378}_a + 189_b = 0$	$189_a = -\mathbf{378}_a = 189_b$
(6.10c)	$p=3$	$27_a - \mathbf{216}_a + 189_c = 0$	$27_a = -\mathbf{216}_a = 189_c$
(6.10d)	$p=5$	$27_a - 168_a + \mathbf{512}_a - 378_a + 7_a = 0$	$(2, -1, 0) = 27_a = -168_a = \mathbf{512}_a = -378_a = 7_a$
(6.10e)	$p=5$	$1_a - \mathbf{84}_a + 216_a - 189_b + 56_a = 0$	$(1,1,1) = 1_a = -\mathbf{84}_a = 216_a = -189_b = 56_a$.

The three characters of defect 1 (mod 3) in each of the blocks (6.10b) and (6.10c) are distributed in three different blocks of defect 1 (mod 5). That the ordinary characters in each of the (mod 5) blocks (6.10a), (6.10d) and (6.10e) are arranged so that the sum of consecutive characters is a modular indecomposable character (mod 5), is verified by analyzing into blocks the following characters of representations induced in H_0, by (mod 5) indecomposable representations of G.

$$
(6.11)\quad
\begin{array}{ll}
I15_p = (21_a + 189_a) & I20_s = (189_a + 336_a) \\
I15_m = (189_b + 216_a) + \mathbf{15}_a & I(1_p + 24_p) = (1_a + 84_a) + (27_a + 168_a) + \mathbf{420}_a \\
I30_n = (189_b + 56_a) + 280_a + \mathbf{315}_a & I(1_n + 24_n) = (21_b + 189_c) + (7_a + 378_a) + \mathbf{105}_c \\
I90_s = (189_a + 336_a) + (512_a + 378_a) + 280_a + 420_a + \mathbf{405}_a. &
\end{array}
$$

Not only do we conclude from (6.11) that the pairs in parentheses form indecomposable characters (mod 5) but we obtain from (6.11) the five new characters 15_a, 105_c, 315_a, 405_a and 420_a. We now have all seven of the characters that belong to the single block of defect 1 (mod 7). By examining the products $7_a \times 7_a$, $7_a \times 21_b$, (above 6.9), $15_a \times 56_a$ (below 6.13) and the induced representations $I15_m$ and $I90_s$ in (6.11) it is clear that

(6.12) $1_a - 27_a + 120_a - 405_a + 512_a - 216_a + 15_a = 0$, except in class 7

and that the sum of two adjacent characters in (6.12) is an indecomposable character (mod 7). The character 512_a is of highest type (mod 2), and 405_a is of highest type (mod 3).

The character 15_a remains irreducibile and is equal to 15_m for elements of the subgroup G. From it we readily obtain certain KRONECKER products from which all the remaining characters can be found.

$$
\begin{array}{llll}
 & 15_a \times 7_a = \mathbf{105}_b, & 15_a^{[1^2]} = 105_c, & 15_a^{[2]} = 1_a + 35_b + 84_a \\
(6.13) & 15_a \times 21_a = \mathbf{315}_a, & & 15_a \times 21_b = 35_b + \mathbf{280}_b \\
 & \multicolumn{2}{l}{15_a \times 56_a = 105_b + (120_a + 405_a) + \mathbf{210}_b,} & I10_s = 210_b + \mathbf{70}_a
\end{array}
$$

The list of the ordinary irreducible characters of the simple group H_0, also known as $[3^{3,2,1}]'$ or $A(6, 2)$, is now complete, and the values of the characters are collected in Table III. All the corresponding representations can be written with real coefficients, since the sum of the degrees is 5104, a number equal to the number of group elements that satisfy $S^2 = I$.

At this point we may indicate briefly the irreducible modular characters, excluding those of highest type (mod p), since they are the same as the ordinary irreducible characters whose degrees are divisible by the exact prime power p^a that divides the group order. From our previous discussion we derive the following values in p-regular classes for certain irreducible modular characters:

(6.14*a*) For $p = 7$: 1_a, $27_a - 1_a$, $120_a - 27_a + 1_a$, $512_a - 216_a + 15_a$, $216_a - 15_a$, 15_a.

(6.14*b*) For $p = 5$: 1_a, $84_a - 1_a$, $189_a - 56_a$, 56_a.
27_a, $168_a - 27_a$, $378_a - 7_a$, 7_a.
21_a, $189_a - 21_a$, $189_c - 21_b$, 21_b.

(6.14*c*) For $p = 3$: 189_a, 189_b; 27_a, 189_c.

There remains a single block of lowest type (mod 3) containing 23 ordinary characters and 10 modular characters, whose values for p-regular classes are

(6.14*d*) For $p = 3$: 1_a, 7_a, $15_a - 1_a$, 21_a, 35_a, 35_b, $56_a - 7_a$, $105_c - 15_a + 1_a$,
$105_b - 7_a$, $280_a - 35_a - 56_a + 7_a$.

The indecomposable character (mod 3) that contains the 1-character as one constituent is $1_a + 120_a + 512_a + 15_a$, and another indecomposable character whose degree is a small multiple of 81 is $35_b + 168_a + 84_a + 280_b$. The rest each contain more than four ordinary characters as constituents.

Finally for the prime 2, there remains a single block of lowest type containing 29 ordinary characters and 7 modular characters, whose values for p-regular classes are

(6.14e) For $p = 2$: $1_a,\ 7_a - 1_a,\ 15_a - 1_a,\ 21_a - 1_a,\ 70_a,\ 105_a - 7_a,\ 280_b - 105_a + 1_a$.

The indecomposable modular character that contains the 1-character is of degree $2^9 3^2 = 4608$, and it contains the sixteen ordinary characters of odd degree once each, and contains 56_a, 120_a, 210_a, 280_a, 336_a and 420_a twice each. All the (mod 2) indecomposable constituents of this block have a fairly complicated decomposition in a field of characteristic 0. Special mention should be made, however, of the 6-dimensional irreducible representation (mod 2) since this arises in DICKSON'S [15] definition of the group as the simple group $A(6, 2)$ of order $(2^6 - 1)2^5(2^4 - 1)2^3(2^2 - 1)2$.

We conclude this study of the characters of the group $[3^{3,2,1}]'$ by showing that this group has only eight classes of conjugate proper subgroups of order greater than 9000, of which five are classes of maximal subgroups. These subgroups include the reflection groups $[3^{2,2,1}]$ and $[3^{3,1,1}]$ and their even subgroups of index 2, the symmetric and alternating groups of degree 8, and two classes of groups of the orders $2^6 3^3 7 = 12096$ and $2^9 3 7 = 10752$ respectively. A subgroup of this last class leaves fixed one of the 135 possible products of seven S-factors that represent the inversion in the origin in the group $[3^{3,2,1}]$.

We show that no other large subgroups exist by forming all possible sums of irreducible characters of H_0 that satisfy the following necessary conditions for the character η of a transitive permutation representation $(\neq 1)$ of H_0.

1. - A transitive permutation character η contains the 1-character just once, and contains one or more other irreducible characters with non-negative multiplicity.

2. - The degree of η is a factor of the group order, and all the character values are non-negative integers.

3. - In no case may the value of η for an element of the group exceed its value for an integral power of the element.

Subject to those conditions the only possibilities for permutation characters of degree < 160 are the following eight, which actually are induced by subgroups.

(6.15) Permutation character	Degree	Order of subgroup	Type of subgroup
$1_a + 27_a$	28	51840	$[3^{2,2,1}]$
$(1_a + 27_a) + 7_a + 21_b$	56	25920	$[3^{2,2,1}]'$
$1_a + 35_b$	36	40320	$[3^6]$ (symmetric group)
$(1_a + 35_b) + 21_b + 15_a$	72	20160	$[3^6]'$ (alternating group)
$1_a + 27_a + 35_b$	63	23040	$[3^{3,1,1}]$
$(1_a + 27_a + 35_b) + 7_a + 56_a$	126	11520	$[3^{3,1,1}]'$
$1_a + 35_a + 84_a$	120	12096	Contains $HO(3, 9)$
$1_a + 35_b + 84_a + 15_a$	135	10752	

By working out its character table, we find that the subgroup of index 120 contains the simple group $HO(3, 9)$ of order 6048 as an invariant subgroup of index 2. To this simple subgroup of H_0 corresponds the permutation character $1_a + 35_a + 84_a + 15_a + 105_c$.

TABLE III.

Character table of the group $[3^{3,2,1}]'$ of order 1,451,520

Class symbol	Class size: h_λ	CHARACTERS OF ODD DEGREE															
1^7	1	1_a	7_a	27_a	21_a	35_a	105_a	189_a	21_b	35_b	189_b	189_c	15_a	105_b	105_c	315_a	405_a
$1^3 2^2$	$3^3 57$	1	3	7	1	−5	5	−11	5	7	13	1	3	9	−3	3	−3
$1^{-1}2^4(V)$	$3^2 57$	1	−1	3	−3	3	1	−3	5	11	−3	21	7	−7	17	−21	−27
$1^3 2^{-2} 4^2$	$2^2 3^3 57$	1	3	3	5	7	5	9	1	−1	−3	−3	−1	−3	−3	−5	−3
$1\ 2\ 4$	$2^4 3^4 57$	1	1	1	−1	−1	−1	1	1	1	1	−1	1	1	−1	−1	1
$1^4 3$	$2^5 37$	1	4	9	6	5	15	9	6	5	9	9	0	0	0	0	0
$1^2 2 3^{-1} 6$	$2^5 3^2 57$	1	2	3	0	−3	1	−3	2	−1	−3	3	−2	−4	2	0	0
$2^2 3$	$2^5 3^3 57$	1	0	1	−2	1	−1	1	2	1	1	1	0	0	0	0	0
$1 3^2$	$2^7 357$	1	1	0	0	2	−3	0	0	2	0	0	3	3	3	0	0
$1^{-1} 2 6$	$2^7 3^2 57$	1	−1	0	0	0	1	0	2	2	0	0	1	−1	−1	0	0
$1^{-2} 3^3$	$2^6 57$	1	−2	0	3	−1	−3	0	3	−1	0	0	−3	6	6	−9	0
$1^2 2^{-2} 3^{-1} 6^2$	$2^6 3^2 57$	1	2	0	3	3	1	0	−1	−1	0	0	1	2	2	3	0
$2 3 4^{-1} 6^{-1} (12)$	$2^7 3^3 57$	1	0	0	−1	1	−1	0	1	−1	0	0	−1	0	0	1	0
$1 3^{-1} 9$	$2^9 3^2 57$	1	1	0	0	−1	0	0	0	−1	0	0	0	0	0	0	0
$1^2 5$	$2^8 3^3 7$	1	2	2	1	0	0	−1	1	0	−1	−1	0	0	0	0	0
$1^{-5} 2^6$	$3^2 7$	1	−5	15	9	−5	−35	21	−11	15	−51	−39	−5	25	5	−45	45
$1^{-1} 2^4 (-U)$	$2^2 3^3 57$	1	−1	3	−3	3	1	−3	−3	3	−3	−3	−1	1	−7	3	−3
$1^{-3} 2^3 4$	$2^3 3^3 57$	1	−3	5	3	−1	−5	1	−3	1	1	−1	1	−3	−1	3	−3
$1\ 2\ 4$	$2^3 3^3 57$	1	1	1	−1	−1	−1	1	−3	5	1	−5	−3	−3	3	3	−3
$1^{-1} 2^2 4^{-1} 8$	$2^5 3^4 57$	1	−1	1	−1	1	1	−1	−1	−1	1	−1	1	−1	1	−1	1
$1^{-2} 2^3 3^{-1} 6$	$2^5 3^2 57$	1	−2	3	0	1	1	−3	−2	3	−3	3	−2	4	2	0	0
$1 3^{-2} 6^2$	$2^7 3^2 57$	1	1	0	0	−2	1	0	−2	0	0	0	1	1	−1	0	0
$1^{-1} 2 6$	$2^7 3^3 57$	1	−1	0	0	0	1	0	0	0	0	0	−1	1	−1	0	0
$1^{-2} 2 3 4$	$2^6 3^3 57$	1	−2	1	2	−1	−1	1	0	−1	1	1	0	0	0	0	0
$2 5^{-1} (10)$	$2^8 3^4 7$	1	0	0	−1	0	0	1	−1	0	−1	1	0	0	0	0	0
$1^{-1} 3 5$	$2^9 3^3 7$	1	−1	−1	1	0	0	−1	1	0	−1	−1	0	0	0	0	0
$3^{-1} 4 6$	$2^6 3^3 57$	1	0	−1	0	−1	1	1	0	1	1	−1	−2	0	2	0	0
$1^{-1} 4^2$	$2^2 3^4 57$	1	−1	−1	1	−1	1	1	1	3	−3	1	3	−3	1	3	5
$1 2^{-1} 8$	$2^5 3^4 57$	1	1	−1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	−1	1
7	$2^9 3^4 5$	1	0	−1	0	0	0	0	0	0	0	0	1	0	0	0	−1

TABLE III.

Character table of the group $3[^{3,\,2,\,1}]'$ (continued)

CHARACTERS OF EVEN DEGREE

														N_λ
168_a	56_a	120_a	210_a	280_a	336_a	216_a	512_a	378_a	84_a	420_a	280_b	210_b	70_a	1,451,520
8	8	8	2	—8	—16	8	0	2	4	—12	8	10	6	1,536
8	—8	—8	2	—8	16	24	0	—6	20	4	24	—14	—10	4,608
0	0	0	—2	0	0	0	0	6	4	—4	0	6	2	384
0	0	0	—2	0	0	0	0	2	0	0	0	—2	—2	32
6	11	15	15	10	6	—9	—16	—9	—6	0	—5	—15	—5	2,160
2	1	1	—1	—2	—2	—3	0	3	2	4	—3	1	—1	144
2	—1	—1	—1	—2	2	—1	0	—1	—2	0	—1	1	3	48
—3	2	0	0	1	0	0	—4	0	3	3	—2	3	1	108
—1	—2	—2	2	1	—2	0	0	0	—1	1	0	1	—1	36
6	2	—6	3	10	—6	0	8	0	3	—3	—8	—6	7	648
2	—2	—2	—1	—2	—2	0	0	0	—1	1	0	—2	—1	72
0	0	0	1	0	0	0	0	0	1	—1	0	0	—1	12
0	—1	0	0	1	0	0	—1	0	0	0	1	0	1	9
—2	1	0	0	0	1	1	2	—2	—1	0	0	0	0	30
40	—24	40	50	—40	—16	—24	0	—30	4	20	40	10	—10	23,040
8	0	0	—6	8	0	0	0	—6	4	4	0	2	—2	384
0	—4	4	2	0	0	4	0	2	0	0	—4	—2	2	192
0	4	—4	2	0	0	—4	0	2	0	0	4	—2	2	192
0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
—2	—3	1	—1	2	2	—3	0	3	—2	—4	1	1	—1	144
1	0	—2	2	—1	2	0	0	0	1	—1	—2	1	—1	36
—1	0	0	0	—1	0	0	0	0	1	1	0	—1	1	12
0	1	—1	—1	0	0	—1	0	—1	0	0	1	1	—1	24
0	1	0	0	0	—1	1	0	0	—1	0	0	0	0	10
1	1	0	0	0	1	1	—1	1	—1	0	0	0	0	15
0	—1	1	—1	0	0	1	0	—1	0	0	—1	1	—1	24
0	0	0	—2	0	0	0	0	—2	4	—4	0	—2	2	128
0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
0	0	1	0	0	0	—1	1	0	0	0	0	0	0	7

TABLE IV.

Irreducible components in the subgroup $G = [3^{2,2,1}]$ of the characters of $H_0 = [3^{3,2,1}]'$, and irreducible components in H_0 of the characters induced by those of G.

Characters of H_0 / of G	1_a	7_a	27_a	21_a	35_a	105_a	189_a	21_b	35_b	189_b	189_c	15_a	105_b	105_c	315_a	405_a	168_a	56_a	120_a	210_a	280_a	336_a	216_a	512_a	378_a	84_a	420_a	280_b	210_b	70_a
1_p	1		1																											
1_n		1						1																						
6_p			1	1															1											
6_n		1				1												1												
15_p				1			1													1										
15_n					1	1															1									
20_p			1						1								1		1	1										
20_n						1		1		1	1							1												
30_p													1			1			1	1										
30_n										1					1			1			1									
64_p							1									1	1		1	1							1	1		
64_n						1				1	1				1						1	1			1					
81_p														1		1				1		1		1			1	1		
81_n											1				1						1	1	1	1			1			
15_q									1				1															1		
15_m										1		1											1							
24_p																	1									1	1			
24_n											1			1											1					
60_p													1			1	1							1				1	1	
60_n										1					1								1	1	1					1
20_s					1		1															1								
90_s							1									1					1	1		1	1		1			
80_s															1	1								1	1		1		1	
60_s																							1	1	1	1		1	1	
10_s																													1	1

Each row indicates the decomposition of the character of H_0 induced by the character of G at the left. Each column indicates the decomposition in G of the irreducible character of H_0 at the top. (Note that no multiplicities exceed 1, in accordance with LITTLEWOOD' S conjecture, [26], for representations induced by a maximal subgroup G of H_0).

TABLE V.

Kronecker products that contain the character 7_a

1_a	x																	
	7_a	x	x															
	15_a	27_a		x	x	x												
		216_a	21_a			x	x											
			70_a	21_b					x									
				105_c	56_a			x	x	x								
	x	x			105_b	105_a			x	x	x	x						
		x				405_a	(35_a)					x						
		x				x	$((512_a))$	35_b					x					
		x	x				x	280_b	120_a				x	x	x			
		x	x				x		(210_b)	210_a			x	x	x	x	x	
				x		x	x			420_a	168_a		x	x		x		x
									x		84_a	189_a			x		x	x
					x	x		x					189_b					
						x				x				189_c				
						x	x	x		x					280_a			
				x	x	x	x	x		x						315_a		
						x	x			x							(336_a)	
						x	x	x	x	x	x							(378_a)

KRONECKER products that contain the character 7_a may be read from this table as follows. Two characters ζ^i and ζ^j (indicated by their degrees) are associated with each x in the table, one nearest to the x in the same row and the other in the same column. They are such that the product $\zeta^i\zeta^j$ contains 7_a as a component, and the product $7_a\zeta^i$ contains ζ^j as a component. The KRONECKER squares of the characters 35_a, 336_a, 378_a and 210_b each contain 7_a once, and $512_a \times 512_a$ contains 7_a twice. Decompositions such as (6.9) can all be read from the table. Characters above the diagonal steps are components of one of the first four KRONECKER powers of 7_a, those at the lower right are found in the KRONECKER fifth power (but in nol ower power) and those below the steps are first found in the sixth and seventh powers of 7_a.

REFERENCES

1. H. F. BAKER, *A locus with 25920 self transformations*, Cambridge University Press, 1946.
2. H. R. BRAHANA, *Generators of known simple groups*, « Annals of Math. », (2), 31, (1930), pp. 529-549, especially p. 533.
3. R. BRAUER and C. NESBITT, *On the modular representations of groups of finite order*, University of Toronto studies, Mathematical Series n. 4, 1937.
4. — — *On the modular characters of groups*, « Annals of Math. », vol. 42, n. 2, (April 1941), pp. 556-590.
5. R. BRAUER, *On groups whose order contains a prime number to the first power*, « Amer. Journal of Math. », vol. LXIV, n. 2, (1942), pp. 401-420.
6. H. BURKHARDT, *Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen*, II « Math. Annalen », 38, (1891), pp. 161-224.
7. — — III » Math. Annalen », 41, (1893). pp. 313-343, especially pp. 325-326.
8. E. CARTAN, *La géometrie des groupes simples*, « Annali di Matematica pura ed applicata », ser. 4, (1927), pp. 209-256, especially pp. 221-222, 230.
9. H. S. M. COXETER, *The pure archimedean polytopes in six and seven dimensions*, « Proc. Camb. Phil. Soc. », 24 (1928), pp. 1-9.
10. *The polytopes with regular-prismatic vertex figures*, Part. I, « Phil. Trans. Roy. Soc. London », (A), 229, (1930), pp. 329-425, esp. 388, 396, 418.
11. — — Part. II. « Proc. London Math. Soc. », (2), 34, (1932), pp. 126-189.
12 — — *Finite groups generated by reflections and their subgroups generated by reflections*, « Proc. Camb. Phil. Soc. », vol. 50, (1934), pp. 466-482.
13. — — *The polytope 2_{21}, whose 27 vertices correspond to the lines on the general cubic surface*, « Amer. Journ. of Math. », 62, (1940), pp. 457-486.
14. — — *Regular polytopes*, Methuen and Co., London, 1948.
15. L. E. DICKSON, *Linear groups, with an exposition of Galois field theory*, Leipzig 1901.
16. J. S. FRAME, *The simple group of order 25920*, « Duke Math. Journ. », 2, (1936), pp. 477-484.
17. — — *A symmetric representation of the twenty-seven lines on a cubic surface by lines in a finite geometry*, « Bull. Amer. Math. Soc. », 44, (1938), pp. 658-661.
18. — — *Congruence relations between the traces of matrix powers*, « Canadian Journal of Math. », vol. 1, n. 3, (1949), pp. 303-304.
19. G. FROBENIUS, *Ueber die Beziehungen zwischen den 28 Doppeltangenten einer ebenen Curve vierter Ordnung*, « Journal für Math. », 99, (1886), p. 276.
20. C. F. GEISER, *Ueber die Doppeltangenten einer ebenen Curve vierten Grades*, « Math. Annalen », 1, (1869), p. 129.
21. T. GOSSET, *On the regular and semi-regular figures in space of* n*-dimensions*, « Messenger of Mathematics », 24, (1900), pp. 43-48.
22. C. JORDAN, *Sur la trisection des fonctions abéliennes et sur les vingt-sept droites des surfaces du troisième ordre*, « Comptes Rendus », vol. LXVIII, (1869), pp. 865-869.
23. A. HENDERSON, *The twenty-seven lines upon a cubic surface*, Cambridge 1911.
24. O. HESSE, *Ueber die Doppeltangenten einer ebenen Curve vierter Ordnung*, « Journal für Math. », 49, (1855), p. 279.
25. F. KLEIN, *Sur la résolution, par des fonctions hyperelliptiques, de l'équation du vingt-septième degré, de laquelle depend la determination des vingt-sept droites d'une surface cubique*, « Journal de Math. », (Liouville), ser. 4, vol. 4, (1888), pp. 169-176.
26. D. E. LITTLEWOOD, *The theory of group characters*, Oxford, Clarendon Press, 1940.
27. H. MASCHKE, *Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen Substitutionen*. « Math. Annalen », 33, (1888), pp. 317-344.

28. G. RACAH, *Sulla caratterizzazione delle rappresentazioni irriducibili dei gruppi semi-semplici di Lie*, « Rendiconti dell'Acc. Naz. dei Lincei », ser. VIII, vol. VIII, (1950), pp. 108-112.
29. P. SCHOUTE, *On the relation between the vertices of a definite six-dimensional polytope and the lines of a cubic surface*, Kon. Akad. von Wetensch. te Amsterdam, « Proc. of Sec. of Sci. », vol. 13, (1910), pp. 375-383.
30. I. SCHUR, *Ueber die rationalen Darstellungen der allgemeinen linearen Gruppe*, Sitzungsber, Berlin, (1927), pp. 58-75.
31. B. SEGRE, *Quartiche piane e superficie cubiche*, « Boll. della Unione Matematica Italiana », ser. 1, vol. 8, (1929), pp. 203-210.
32. —— *The non singular cubic surfaces*, Oxford, Clarendon Press, 1942.
33. J. A. TODD, *On a quartic primal with forty-five nodes,in space of four dimensions*, « Quarterly Journal Math. », (Oxford series), 7, (1936), pp. 168-174.
34. —— *On the simple group of order 25920*, « Proc. Roy. Soc. London », (A), 189, (1947), pp. 326-358.
35. B. L. VAN DER WAERDEN, *Die Klassification der einfachen Lieschen Gruppen*, « Math. Zeitsch. », 37, (1933), pp. 446-462, especially p. 460.
36. H. WEYL, *Theorie der Darstellung kontinuerlicher halb-einfacher Gruppen durch lineare Transformation II*, « Math. Zeitsch. », 24, (1926), pp. 328-376; *III*, pp. 377-395.
37. A. WITTING, *Ueber eine Hesse'schen Configuration der ebenen Curve dritter Ordnung analoge Configurationen in Raume, auf welche die Transformationstheorie der hyperelliptischen Functionen* (p = 2) *führt*, Dresden 1887.