Pseudo-Differential Operators (*) (1).

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Summary. – *We present here a number of results on some aspects of Kohn-Nirenberg's theory of pseudo-differential operators. We hope that some parts of Kohn-Nirenberg's paper* [1] *are presented here in a more detailed and explicit form; this could help a larger audience to understand their ideas and methods.*

1. – Preliminaries.

We assume basic knowledge of distribution theory; the spaces S, S', Hs; the Fourier transform in these spaces; we use the usual notations:

$$D_s = -i\frac{\partial}{\partial x_s}, \quad D = (D_1, \dots, D_n), \quad D^\alpha = D_1^{\alpha_1}, \dots, D_n^{\alpha_n}, \quad \xi^\alpha = \xi_1^{\alpha_1}, \dots, \xi_n^{\alpha_n}, \quad \partial_s = \frac{\partial}{\partial \xi_s},$$

$$\partial = (\partial_1, \dots, \partial_n), \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n, \quad \| \ \|_s = \| \ \|_{Hs},$$

$$\xi = (\xi_1, \dots, \xi_n), \quad |\xi|^2 = \xi_1^2 + \dots + \xi_n^2.$$

We say that the linear operator L, from S into S' is of order r, if $\|Lu\|_s \leqslant C\|u\|_{s+r}$, $\forall u \in \mathcal{S}$ and for any real s.

We define the Friedrichs operator $\varphi(D)$; $\varphi(D)u = \mathcal{F}^{-1}(\varphi(\xi)\tilde{u}(\xi))$.

We assume that $\varphi(\xi)$ applies S in S'; $\mathcal{F}u = \tilde{u}$ is the direct Fourier transform, $\mathcal{F}^{-1}$ the inverse Fourier transform.

EXAMPLE 1. – Let us consider a measurable function $\varphi(\xi)$ such that, $\forall \xi \in R^n$ $|\varphi(\xi)| \leqslant C(1+|\xi|^2)^\sigma$; it maps S into S'.

If $u \in \mathcal{S}$, $\Rightarrow \tilde{u} \in \mathcal{S}$ and $|\varphi(\xi)\tilde{u}(\xi)| \leqslant C_p(1+|\xi|^2)^{\sigma-p}$, $\forall p = 1, 2, \dots$. Hence

$$\mathcal{F}^{-1}(\varphi(\xi)\tilde{u}(\xi)) = (2\pi)^{-n/2}\int \exp(ix\cdot\xi)\varphi(\xi)\tilde{u}(\xi)\,d\xi$$

is an absolutely convergent integral, and $\varphi(D)u$ is continuous and bounded on $x \in R^n$.

We have estimates:

$$\|\varphi(D)u\|_s^2 = \int (1+|\xi|^2)^s|\varphi(\xi)|^2|\tilde{u}(\xi)|^2 d\xi \leqslant C\int (1+|\xi|^2)^{s+2\sigma}|\tilde{u}(\xi)|^2 d\xi = C\|u\|_{s+2\sigma}^2.$$

Hence, the operator $\varphi(D)$ is of order 2σ.

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EXAMPLE 2. – If $\psi(\xi)$ has compact support in R^n and is continuous, then, $\forall\xi\in R^n$ and $p=1,2,\ldots,\Rightarrow(1+|\xi|^2)^p|\psi(\xi)|\leqslant C_p$. If follows $\|\psi(D)u\|_s\leqslant C_{s,p}\|u\|_{s-p}$, $p=1,2,\ldots$. Hence, the inf. of the orders (named true order) is $-\infty$.

Another operator in S; if $a(x)\in\mathcal{S}$, then $a(x)u(x)\in\mathcal{S}$, $\forall u\in\mathcal{S}$. Moreover, we have the estimate $\|au\|_s\leqslant C\|u\|_s$, which shows that this multiplication operator is of order 0. In order to prove this estimate, we see first that:

$$\widetilde{au}(\xi)=(\tilde{a}*\tilde{u})(\xi)=(2\pi)^{-n/2}\int\tilde{a}(\xi-\eta)\tilde{u}(\eta)\,d\eta\,.$$

Therefore:

$$\|au\|_s=\|(1+|\xi|^2)^{s/2}(2\pi)^{-n/2}\int\tilde{a}(\xi-\eta)\tilde{u}(\eta)\,d\eta\|_0=$$
$$=\|(2\pi)^{-n/2}\int(1+|\xi|^2)^{s/2}(1+|\eta|^2)^{-s/2}\tilde{a}(\xi-\eta)(1+|\eta|^2)^{s/2}\tilde{u}(\eta)\,d\eta\|_0\,.$$

We know the inequality:

$$\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^{s/2}\leqslant 2^{|s|/2}(1+|\xi-\eta|^2)^{|s|/2};$$

furthermore, if $|f(\xi)|\leqslant|g(\xi)|\Rightarrow\|f\|_0\leqslant\|g\|_0$. Consequently, as

$$\left|\int(1+|\xi|^2)^{s/2}(1+|\eta|^2)^{-s/2}\tilde{a}(\xi-\eta)(1+|\eta|^2)^{s/2}\tilde{u}(\eta)\,d\eta\right|=f(\xi)\leqslant$$
$$\leqslant 2^{|s|/2}\int(1+|\xi-\eta|^2)^{|s|/2}|\tilde{a}(\xi-\eta)|(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta=g(\xi)\,,$$

it follows that

$$\|au\|_s\leqslant(2\pi)^{-n/2}2^{|s|/2}\left\|\int(1+|\xi-\eta|^2)^{|s|/2}|\tilde{a}(\xi-\eta)|(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta\right\|_0\,. \tag{1.1}$$

Let us remember Minkowski's inequality for integrals

$$\left(\int\left(\int|f(\xi,\eta)|\,d\eta\right)^2d\xi\right)^{\frac12}\leqslant\int\left(\int|f(\xi,\eta)|^2d\xi\right)^{\frac12}d\eta\,. \tag{1.2}$$

Changing the variables: $\xi-\eta=\eta'$ in (1.1), we have obviously

$$\|au\|_s\leqslant(2\pi)^{-n/2}2^{|s|/2}\left\|\int(1+|\eta'|^2)^{|s|/2}|\tilde{a}(\eta')|(1+|\xi-\eta'|^2)^{s/2}|\tilde{u}(\xi-\eta')|\,d\eta'\right\|_0\,. \tag{1.3}$$

Let be $f(\xi,\eta)=(1+|\eta|^2)^{|s|/2}|\tilde{a}(\eta)|(1+|\xi-\eta|^2)^{s/2}|\tilde{u}(\xi-\eta)|$; we have then, by (1.2)-(1.3)

$$\|au\|_s\leqslant C_s\left(\int\left(\int f(\xi,\eta)\,d\eta\right)^2d\xi\right)^{\frac12}\leqslant C_s\int\left(\int f^2(\xi,\eta)\,d\xi\right)^{\frac12}d\eta= \tag{1.4}$$
$$=C_s\int\left(\int(1+|\eta|^2)^{|s|}|\tilde{a}(\eta)|^2(1+|\xi-\eta|^2)^s|\tilde{u}(\xi-\eta)|^2d\xi\right)^{\frac12}d\eta=$$
$$=C_s\int(1+|\eta|^2)^{|s|/2}|\tilde{a}(\eta)|\left(\int(1+|\xi-\eta|^2)^s|\tilde{u}(\xi-\eta)|^2d\xi\right)^{\frac12}d\eta=$$
$$=C_s\|u\|_s\int(1+|\eta|^2)^{|s|/2}|\tilde{a}(\eta)|\,d\eta=C_{1,s}\|u\|_s\,.$$

Finally, an other example of operator which maps $\mathcal{S}$ *into* $\mathcal{S}$.

Let $\zeta_\sigma(D)u = \mathcal{F}^{-1}(\zeta(\xi)|\xi|^\sigma\tilde{u}(\xi))$, $\forall u \in \mathcal{S}$, where $\zeta(\xi) \in C^\infty(R^n)$ is $=0$ for $|\xi| < \frac{1}{2}$, and is $=1$ for $|\xi| \geqslant 1$. Then obviously, $\zeta(\xi)|\xi|^\sigma \in C^\infty$; furthermore, as $\xi \to \infty$, it increases polynomially and we remark also that all derivatives $\partial^\beta(\zeta(\xi)|\xi|^\sigma)$ have the same property.

This shows that $\zeta(\xi)|\xi|^\sigma\tilde{u}(\xi) \in \mathcal{S}$ if $\tilde{u} \in \mathcal{S}$; consequently, $\zeta_\sigma(D)$ maps $\mathcal{S}$ into $\mathcal{S}$.

This operator is useful in the successive study of pseudo-differential operators of a more general form (see [1]).

We see also that $|\zeta_\sigma(\xi)| \leqslant C(1+|\xi|^2)^{\sigma/2}$, $\forall \xi \in R^n$ $(\zeta_\sigma(\xi) = \zeta(\xi)|\xi|^\sigma)$. Hence, operator $\zeta_\sigma(D)$ has order $\leqslant \sigma$.

2. – Symbols.

Let $a(x, \xi)$ be a complex valued function defined for $x \in R^n$, $\xi \in R^n - \{0\}$ and assume $a(x, \xi) \in C^\infty(R^n \times R^n - \{0\})$. Suppose that $a(x, t\xi) = a(x, \xi)$ for $t > 0$, and assume also that $\lim_{|x| \to \infty} a(x, \xi) = a(\infty, \xi)$ exists for $\xi \in R^n - \{0\}$ and $a(\infty, \xi)$ is a C^∞-function; define then $a'(x, \xi) = a(x, \xi) - a(\infty, \xi)$, and assume the estimates

$$(1+|x|^2)^p |D_x^\alpha \partial_\xi^\beta a'(x, \xi)| \leqslant C_{p,\alpha,\beta}, \qquad \forall x \in R^n, \tag{2.1}$$

and ξ such that $|\xi| = 1$ (¹); here $p = 1, 2, \dots$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$-arbitrary multi-indexes. We see some corollary of Definition (2.1), which are remarked without proof in [1].

THEOREM 1.

a) We have $|a(\infty, \xi) - a(\infty, \eta)| \leqslant C\big((|\xi - \eta|)/(|\xi| + |\eta|)\big)$, $\forall \xi, \eta$ *arbitrary in* $R^n - \{0\}$: *The estimates*

b) $(1+|\lambda|^2)^p |\tilde{a}'(\lambda, \xi)| \leqslant C_p$, $\quad \forall \lambda \in R^n$, $\xi \in R^n - \{0\}$, $p = 1, 2, \dots$;

c) $(1+|\lambda|^2)^p |\tilde{a}'(\lambda, \xi) - \tilde{a}'(\lambda, \eta)| \leqslant C_p|\xi - \eta|(|\xi| + |\eta|)^{-1}$, $\forall \lambda \in R^n$, $\xi, \eta \in R^n - \{0\}$, $p = 1, 2, \dots$ *being*

$$\tilde{a}'(\lambda, \xi) = (2\pi)^{-n/2}\int \exp(-ix\cdot\lambda)\, a'(x, \xi)\, dx\,, \qquad \forall \lambda \in R^n,\ \xi \in R^n - \{0\}$$

are verified.

PROOF OF *a*). – $a(\infty, t\xi) = a(\infty, \xi)$, $\forall t > 0$, $\xi \in R^n - \{0\}$, as easily seen. Hence $a(\infty, \xi)$ is also homogeneous of order 0, and by hypothesis is also $C^\infty(R^n - \{0\})$. Let us put $\xi/|\xi| = \zeta$, $\eta/|\eta| = \mu$; we have $|\zeta| = |\mu| = 1$, $a(\infty, \xi) = a(\infty, \zeta)$, $a(\infty, \eta) = a(\infty, \mu)$, and on the other hand

$$\frac{|\xi - \eta|}{|\xi| + |\eta|} = \frac{|\zeta|\xi| - \mu|\eta||}{|\xi| + |\eta|} = \left|\frac{|\xi|}{|\xi| + |\eta|}\,\zeta + \frac{|\eta|}{|\xi| + |\eta|}\,(-\mu)\right|. \tag{2.2}$$

(¹) Remark that $D_x^\alpha a'(x, t\xi) = D_x^\alpha a'(x, \xi)$, $\forall t > 0$. Then, from $(1+|x|^2)^p |D_x^\alpha a'(x, \xi)| \leqslant C_{p,\alpha}$ valid for $x \in R^n$, $|\xi| = 1$, it follows that same estimate is valid for $x \in R^n$, $\xi \in R^n - \{0\}$.

Immediately it can be seen, considering $\min_{0<\theta<1} |\theta\zeta + (1-\theta)(-\mu)|$, or geometrically that $|\theta\zeta + (1-\theta)(-\mu)| \geqslant \frac{1}{2}|\zeta + (-\mu)|$, for $|\zeta| = |\mu| = 1$ and hence is, as we have $|\xi|/(|\xi| + |\eta|) + |\eta|/(|\xi| + |\eta|) = 1$, the estimate

$$\frac{|\xi - \eta|}{|\xi| + |\eta|} \geqslant \frac{1}{2}|\zeta - \mu| \; ; \tag{2.3}$$

if we show here that

$$|a(\infty, \zeta) - a(\infty, \mu)| \leqslant \gamma |\zeta - \mu| \, , \qquad \forall \zeta, \mu \tag{2.4}$$

on the unit sphere in R^n, we will have shown a) for $C = 2\gamma$.

Let us suppose hence, reasoning *ad absurdum*, that there are two sequences ζ_n, μ_n, $|\zeta_n| = |\mu_n| = 1$, $n = 1, 2, \dots$ so that

$$|a(\infty, \zeta_n) - a(\infty, \mu_n)| \geqslant n |\zeta_n - \mu_n| \, , \qquad \forall n = 1, 2, \dots . \tag{2.5}$$

Now we can assume, choosing two subsequences, that

$$\lim_{n\to\infty} \zeta_n = \zeta_0 \, , \qquad \lim_{n\to\infty} \mu_n = \mu_0 \, , \qquad |\zeta_0| = |\mu_0| = 1 . \tag{2.6}$$

With use of (2.5) we shall get now:

$$|\zeta_n - \mu_n| \leqslant \frac{1}{n} 2 \sup_{|\zeta|=1} |a(\infty, \zeta)| \, .$$

This gives $\zeta_0 = \mu_0$, as the continuous function $a(\infty, \xi)$ is bounded on the unit sphere in R^n. On the other hand, it results that: $a(\infty, \zeta_n) - a(\infty, \mu_n) = (\zeta_n - \mu_n, \operatorname{grad} a(\infty, z_n))$-scalar product in R^n; here $z_n = \theta_n \zeta_n + (1-\theta_n)\mu_n$, $0 < \theta_n < 1$; this is true for n large enough.

(In fact, for these n, the vectors ζ_n and μ_n belong to same small neighbourhood: $|\zeta - \zeta_0| < \delta$ where $a(\infty, \zeta)$ is of class C^∞, the origin being outside of this neighbourhood). We have then:

$$|a(\infty, \zeta_n) - a(\infty, \mu_n)| \leqslant |\zeta_n - \mu_n| \sup_{|z-\zeta_0|<\delta} |\operatorname{grad} a(\infty, z)| \leqslant M |\zeta_n - \mu_n| \, .$$

It can be deduced that is valid the inequality

$$n|\zeta_n - \mu_n| \leqslant |a(\infty, \zeta_n) - a(\infty, \mu_n)| \leqslant M |\zeta_n - \mu_n| \, , \qquad n = 1, 2, \dots \tag{2.7}$$

which is impossible. Hence estimate a) is satisfied. More precisely: we proved that

$a(\infty, \xi)$ is in the Lipschitz class on the unit sphere, *i.e.*

$$\sup_{\substack{|\xi|=1\\|\eta|=1}} \frac{|a(\infty,\xi)-a(\infty,\eta)|}{|\xi-\eta|}=\gamma<\infty$$

Then we obtained that

$$|a(\infty,\xi)-a(\infty,\eta)|\leqslant 2\gamma|\xi-\eta|\,, \qquad \forall \xi,\ \eta\in R^n-\{0\}\,.$$

Proof of *b*). – Obviously, we have equality

$$(2.8)\qquad (1+|\lambda|^2)^p\tilde{a}'(\lambda,\xi)=(2\pi)^{-n/2}\int\exp(-ix\cdot\lambda)(I-\Delta_x)^p a'(x,\xi)\,dx\,,$$
$$\lambda\in R^n,\ \xi\in R^n-\{0\},\ \Delta_x=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$$

and therefore is verified the estimate

$$(2.9)\qquad |(1+|\lambda|^2)^p\tilde{a}'(\lambda,\xi)|\leqslant$$
$$\leqslant C\int(1+|x|^2)^q|(I-\Delta_x)^p a'(x,\xi)|(1+|x|^2)^{-q}dx\leqslant C_1\int\frac{dx}{(1+|x|^2)^q}=C_2$$

for q large enough.

Proof of *c*). – Obviously, we have the equality

$$(2.10)\qquad (1+|\lambda|^2)^p[\tilde{a}'(\lambda,\xi)-\tilde{a}'(\lambda,\eta)]=$$
$$=(2\pi)^{-n/2}\int\exp(-ix\cdot\lambda)(I-\Delta_x)^p[a'(x,\xi)-a'(x,\eta)]\,dx=$$
$$=(2\pi)^{-n/2}\int\exp(-ix\cdot\lambda)(1+|x|^2)^q(I-\Delta_x)^p[a'(x,\xi)-a'(x,\eta)](1+|x|^2)^{-q}dx\,.$$

Let us put now

$$(2.11)\qquad b_{p,q}(x,\xi)=(1+|x|^2)^q(I-\Delta_x)^p a'(x,\xi)\,,\qquad x\in R^n,\ \xi\in R^n-\{0\}\,.$$

We obtain then the estimate

$$(2.12)\qquad (1+|\lambda|^2)^p|\tilde{a}'(\lambda,\xi)-\tilde{a}'(\lambda,\eta)|\leqslant$$
$$\leqslant(2\pi)^{-n/2}\int(1+|x|^2)^{-q}|b_{p,q}(x,\xi)-b_{p,q}(x,\eta)|\,dx\,,\qquad \forall\lambda\in R^n,\ \xi,\eta\in R^n-\{0\}\,.$$

Consequently, it will be sufficient to show here that

with a constant independent of $x\in R^n$ we have, for $x\in R^n$, $\xi,\eta\in R^n-\{0\}$, the estimate

$$(2.13)\qquad |b_{p,q}(x,\xi)-b_{p,q}(x,\eta)|\leqslant C|\xi-\eta|(|\xi|+|\eta|)^{-1}\,.$$

Let us observe that $b_{p,q}(x, \xi) \in C^\infty(R^n_\xi - \{0\})$ and is also homogeneous of order 0 with respect to ξ, as follows without any difficulty from (2.11) and properties of $a'(x, \xi)$.

It will consequently be enough, by repeating the reasonings in *a*), to show that we have the inequality

$$|b_{p,q}(x, \zeta) - b_{p,q}(x, \mu)| \leqslant \gamma |\zeta - \mu| , \tag{2.14}$$

for ζ, μ on the unit sphere, and $x \in R^n$, because this will imply estimate (2.13) after use of (2.3), and then *c*) is proved if we use (2.12) for q large enough in order to have $\int (1 + |x|^2)^{-q} dx < \infty$.

In the opposite case, (i.e. if (2.14) is not true) there are three sequences $(x_k)_1^\infty$, $(\zeta_k)_1^\infty$, $(\mu_k)_1^\infty$, such that $(x_k)_1^\infty \subset R^n$, $|\zeta_k| = |\mu_k| = 1$, $k = 1, 2, \ldots$ and the following holds:

$$|b_{p,q}(x_k, \zeta_k) - b_{p,q}(x_k, \mu_k)| \geqslant k |\zeta_k - \mu_k| , \qquad \forall k = 1, 2, \ldots \tag{2.15}$$

we may suppose, by extracting subsequences, that

$$\lim_{k \to \infty} \zeta_k = \zeta_0 , \qquad \lim_{k \to \infty} \mu_k = \mu_0 , \tag{2.16}$$

exist, where $|\zeta_0| = |\mu_0| = 1$. Hence, from (2.15),

$$|\mu_k - \zeta_k| \leqslant \frac{1}{k} 2 \sup_{\substack{|\zeta|=1 \\ x \in R^n}} |b_{p,q}(x, \zeta)| \to 0$$

with $k \to \infty$ (as easily seen) and consequently $\zeta_0 = \mu_0$.

On the other hand, we have

$$b_{p,q}(x_k, \zeta_k) - b_{p,q}(x_k, \mu_k) = (\zeta_k - \mu_k, \operatorname{grad}_\xi b_{p,q}(x_k, z_k)) \tag{2.17}$$

where $z_k = \theta_k \zeta_k + (1 - \zeta_k)\mu_k$, $0 < \theta_k < 1$.

Now we remark that for ζ_k, μ_k (and hence z_k) in a small neighbourhood of ζ_0 we have

$$|\operatorname{grad}_\xi b_{p,q}(x_k, z_k)| \leqslant C . \tag{2.18}$$

In fact, first of all, we see that, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$|\partial_\xi^\alpha b_{p,q}(x, \xi)| \leqslant C , \qquad \forall x \in R^n, \ |\xi| = 1, \text{ holds}. \tag{2.19}$$

Thereafter, for any $\xi \in R^n - \{0\}$, we get:

$$|\partial_{\xi_i} b_{p,q}(x, \xi)| = \left| \partial_{\xi_i} b_{p,q}\left(x, \frac{\xi}{|\xi|} |\xi|\right) \right| = \left| \partial_{\xi_i} b_{p,q}\left(x, \frac{\xi}{|\xi|}\right) \right| \frac{1}{|\xi|} \leqslant \frac{C}{|\xi|} \leqslant C_1 \qquad \text{if } |\xi| > \delta > 0$$

(as in the neighbourhood of ζ_0 which we have considered).

We have used here the fact that $\partial_{\xi_i} b_{p,q}(x, \xi)$ is homogeneous of order -1 in respect to ξ. Having then, from (2.15), (2.17), (2.18), the estimates

$$k|\zeta_k - \mu_k| \leqslant |b_{p,q}(x_k, \zeta_k) - b_{p,q}(x_k, \mu_k)| \leqslant C|\zeta_k - \mu_k| \tag{2.20}$$

we arrive at a contradiction, q.e.d.

COROLLARY. – With an absolute constant, we have:

$$|a(x, \xi) - a(x, \eta)| \leqslant C \frac{|\xi - \eta|}{|\xi| + |\eta|} \tag{2.21}$$

for $x \in R^n$, $\xi, \eta \in R^n - \{0\}$. In fact, we have

$$\begin{aligned} a(x, \xi) - a(x, \eta) &= a(\infty, \xi) - a(\infty, \eta) + a'(x, \xi) - a'(x, \eta) = \\ &= a(\infty, \xi) - a(\infty, \eta) + (2\pi)^{-n/2}\int \exp(ix\cdot\lambda)[\tilde{a}'(\lambda, \xi) - \tilde{a}'(\lambda, \eta)]\, d\lambda \end{aligned} \tag{2.22}$$

from where we get the inequalities

$$\begin{aligned} |a(x, \xi) - a(x, \eta)| &\leqslant C \frac{|\xi - \eta|}{|\xi| + |\eta|} + C_1 \frac{|\xi - \eta|}{|\xi| + |\eta|} \int (1 + |\lambda|^2)^{-p}\, d\lambda \leqslant \\ &\leqslant C_2 |\xi - \eta| (|\xi| + |\eta|)^{-1}, \qquad \forall x \in R^n,\ \xi, \eta \in R^n - \{0\}: \qquad \text{q.e.d.} \end{aligned} \tag{2.23}$$

OBSERVATION. – We have implicitly proved, considering in (2.13) $b_{p,q}(x, \xi)$ with $p = 0$, that the following inequality

$$\begin{aligned} (1 + |x|^2)^q |a'(x, \xi) - a'(x, \eta)| &\leqslant C|\xi - \eta| (|\xi| + |\eta|)^{-1}, \\ &\forall x \in R^n,\ \xi, \eta \in R^n - \{0\},\ q = 1, 2, \ldots \end{aligned} \tag{2.24}$$

is also satisfied.

3. – The operator $A(x, D)$.

Let $a(x, \xi) = a(\infty, \xi) + a'(x, \xi)$ be a symbol, and, as previously, $\forall \lambda \in R^n$, $\xi \in R^n - \{0\}$ $\tilde{a}'(\lambda, \xi) = (2\pi)^{-n/2}\int \exp(-ix\cdot\lambda) a'(x, \xi)\, dx$. Obviously, it results that $\tilde{a}'(\lambda, \xi) \in \mathcal{S}(R^n_\lambda)$ uniformly for $\xi \in R^n - \{0\}$ [1].

Let us define, for any $u \in \mathcal{S}$ and $x \in R^n$, a function $v(x) = (A(x, D)u)(x)$, by

$$A(x, D)u = (2\pi)^{-n/2}\int \exp(ix\cdot\xi) G(\xi)\, d\xi \tag{3.1}$$

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[1] Use for that the formula

$$(1 + |\lambda|^2) D^\alpha_\lambda \tilde{a}'(\lambda, \xi) = (2\pi)^{-n/2}\int \exp(-ix\cdot\lambda)(I - \Delta_x)^p ((-ix)^\alpha a'(x, \xi))\, dx,$$

and the definition of a symbol.

where, $\forall \xi \in R^n - \{0\}$, the function $G(\xi)$ is given by

$$G(\xi) = a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta . \tag{3.2}$$

Evidently, it has to be proved that $G(\xi)$ is Fourier transformable; in fact, we have $G(\xi) \in L^1(R^n)$ as

$$|a(\infty, \xi)\tilde{u}(\xi)| \leqslant \max_{|\xi|=1} |a(\infty, \xi)|\, |\tilde{u}(\xi)| \in L^1 ,$$

then obviously, it is sufficient to show that

$$\iint |\tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)|\, d\eta\, d\xi < \infty ;$$

we have in fact, $\forall p = 1, 2, \ldots$

$$\int |\tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)|\, d\eta \leqslant C_p \int (1 + |\xi - \eta|^2)^{-p} |\tilde{u}(\eta)|\, d\eta .$$

This last expression is the convolution between $(1 + |\xi|^2)^{-p}$ and $|\tilde{u}(\xi)|$ both integrable for p sufficiently large.

Hence $A(x, D)u$ is continuous and bounded on R^n; we can say then that

$$\widetilde{A(x, D)u} = a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int a'(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta \tag{3.3}$$

is verified the Fourier transform being taken in S'.

Another formula of representation is given in

PROPOSITION 1. – *If $a(x, \xi)$ is a symbol, we have*

$$\big(A(x, D)u\big)(x) = (2\pi)^{-n/2}\int \exp(ix\cdot\xi)\Big((2\pi)^{-n/2}\int \exp(-iy\cdot\xi)\, a(y, \xi)\, u(y)\, dy\Big)\, d\xi \tag{3.4}$$

for every $u \in \mathcal{S}$, $x \in R^n$.

It will be sufficient to show that

1) The integral $\int \exp(-ix\cdot\xi)\, a(x, \xi)\, u(x)\, dx$ is absolutely convergent.

2) We have $G(\xi) = (2\pi)^{-n/2}\int \exp(-iy\cdot\xi)\, a(y, \xi)\, u(y)\, dy$, $\forall \xi \in R^n - \{0\}$.

We have 1). In fact, as $a(x, \xi) = a(\infty, \xi) + a'(x, \xi)$, it is sufficient to prove the absolute convergence of

$$\int \exp(-ix\cdot\xi)\, a(\infty, \xi)\, u(x)\, dx = a(\infty, \xi)\int \exp(-ix\cdot\xi)\, u(x)\, dx$$

which is obvious, and gives $a(\infty, \xi)\tilde{u}(\xi)$ for $u \in \mathcal{S}$, and of $\int \exp(-ix\cdot\xi)\, a'(x, \xi)\, u(x)\, dx$, for $u \in \mathcal{S}$. As $|a'(x, \xi)| \leqslant C_p(1 + |x|^2)^{-p}$ for every p, we have

$$\int |\exp(-ix\cdot\xi)\, a'(x, \xi)\, u(x)|\, dx \leqslant C_p \int (1 + |x|^2)^{-p} |u(x)|\, dx .$$

In order to prove the 2), it is sufficient that

$$(2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta,\xi)\tilde{u}(\eta)\,d\eta = (2\pi)^{-n/2}\int \exp(-ix\cdot\xi)\,a'(x,\xi)\,u(x)\,dx \tag{3.5}$$

be verified. By Fourier's inversion formula (valid in the case which is considered here) we have

$$a'(x,\xi) = (2\pi)^{-n/2}\int \exp(ix\cdot\lambda)\tilde{a}'(\lambda,\xi)\,d\lambda\,, \qquad x\in R^n,\ \xi\in R^n-\{0\} \tag{3.6}$$

the integral being absolutely convergent.

Or, the « double » integral, for $u\in \mathcal{S}$

$$\iint \exp(-ix\cdot\xi)\,\exp(ix\cdot\lambda)\tilde{a}'(\lambda,\xi)\,u(x)\,dx\,d\lambda \tag{3.7}$$

is absolutely convergent:

$$\iint |\tilde{a}'(\lambda,\xi)|\,|u(x)|\,dx\,d\lambda < \infty\,. \tag{3.8}$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}(2\pi)^{-n/2}\int \exp(-ix\cdot\xi)\,a'(x,\xi)\,u(x)\,dx &= \\ &= (2\pi)^{-n/2}\int\Big[(2\pi)^{-n/2}\int \exp[-ix\cdot(\xi-\lambda)]\tilde{a}'(\lambda,\xi)\,d\lambda\Big]u(x)\,dx\,.\end{aligned} \tag{3.9}$$

By making in the internal integral the substiution

$$\xi_1-\lambda_1=\mu_1,\ \ldots,\ \xi_n-\lambda_n=\mu_n \tag{3.10}$$

we arrive at equality between (3.9) and

$$\begin{aligned}&(2\pi)^{-n/2}\int (2\pi)^{-n/2}\Big(\int \exp(-ix\cdot\mu)\tilde{a}'(\xi-\mu,\xi)\,d\mu\Big)u(x)\,dx = \\ &= (2\pi)^{-n/2}\int (2\pi)^{-n/2}\Big(\int \exp(-ix\cdot\mu)\,u(x)\,dx\Big)\tilde{a}'(\xi-\mu,\xi)\,d\mu = (2\pi)^{-n/2}\int \tilde{a}'(\xi-\mu,\xi)\tilde{u}(\mu)\,d\mu\end{aligned} \tag{3.11}$$

q.e.d.

A fundamental property of the operator $A(x,D)$ is given in

THEOREM 2. – *We have the inequality* $\|A(x,D)u\|_s \leqslant C_s\|u\|_s$, $\forall u\in\mathcal{S}$, *s being real arbitrary.*

We have in fact the immediate decomposition:

$$A(x,D) = A(\infty,D) + A'(x,D)\,.$$

We must remark that for $u\in\mathcal{S}$, we have by definition:

$$\widetilde{A(\infty,D)u}(\xi) = a(\infty,\xi)\tilde{u}(\xi)\,, \qquad \widetilde{A'(x,D)u}(\xi) = (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta,\xi)\tilde{u}(\eta)\,d\eta\,.$$

Then we see first of all

$$\|A(\infty, D)u\|_s^2 = \int (1+|\xi|^2)^s |a(\infty,\xi)\tilde{u}(\xi)|^2 d\xi \leqslant \big(\sup_{|\xi|=1} |a(\infty,\xi)|\big)^2 \|u\|_s^2, \tag{3.12}$$

$$\|A(\infty, D)u\|_s \leqslant \big(\sup_{|\xi|=1} |a(\infty,\xi)|\big)\|u\|_s. \tag{3.13}$$

Less trivial is the estimate for $A'(x, D)u$. Its Fourier transform (in S') equals

$$(2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)\tilde{u}(\eta)\, d\eta\,.$$

And then (using the definition of H_s), we will have to estimate the norm L^2 of the expression

$$(2\pi)^{-n/2}(1+|\xi|^2)^{s/2}\int \tilde{a}'(\xi-\eta,\xi)\tilde{u}(\eta)\,d\eta \tag{3.14}$$

which is equal to

$$(2\pi)^{-n/2}\int (1+|\xi|^2)^{s/2}(1+|\eta|^2)^{-s/2}\tilde{a}'(\xi-\eta,\xi)(1+|\eta|^2)^{s/2}\tilde{u}(\eta)\,d\eta = U_s(\xi)\,. \tag{3.15}$$

Now, the proof is similar to that given in Preliminaries for a more special case. Again we shall use the estimate (some time credited to J. PEETRE)

$$(1+|\xi|^2)^t(1+|\eta|^2)^{-t} \leqslant 2^{|t|}(1+|\xi-\eta|^2)^{|t|}, \qquad \forall \text{ real } t,\ \xi,\eta\in R^n. \tag{3.15 bis}$$

We observe first of all that

$$\begin{aligned}|U_s(\xi)| &\leqslant (2\pi)^{-n/2}2^{|s|/2}\int (1+|\xi-\eta|^2)^{|s|/2}|\tilde{a}'(\xi-\eta,\xi)|(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta \leqslant \\ &\leqslant C_{p,s}\int (1+|\xi-\eta|^2)^{|s|/2-p}(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta\,.\end{aligned} \tag{3.16}$$

Then, making the substitution $\xi-\eta=\eta'$ we arrive at the inequality

$$|U_s(\xi)| \leqslant C_{p,s}\int (1+|\eta'|^2)^{|s|/2-p}(1+|\xi-\eta'|^2)^{s/2}|\tilde{u}(\xi-\eta')|\,d\eta' = C_{p,s}\int K(\xi,\eta')\,d\eta'$$

where

$$K(\xi,\eta') = (1+|\eta'|^2)^{|s|/2-p}(1+|\xi-\eta'|^2)^{s/2}|\tilde{u}(\xi-\eta')|\,. \tag{3.18}$$

Hence $|U_s(\xi)|^2 \leqslant C_1\big(\int K(\xi,\eta')d\eta'\big)^2$ and $\big(\int |U_s(\xi)|^2 d\xi\big)^{\frac12} \leqslant C_{p,s}\big(\int\big(\int K(\xi,\eta')\,d\eta'\big)^2 d\xi\big)^{\frac12}$ which is, by Minkowski's inequality for integrals, estimated in

$$\begin{aligned}\|U_s\|_0 &\leqslant C_{p,s}\int\Big(\int K^2(\xi,\eta')\,d\xi\Big)^{\frac12} d\eta' = \\ &= C_{p,s}\int\Big(\int (1+|\eta'|^2)^{|s|-2p}(1+|\xi-\eta'|^2)^s|\tilde{u}(\xi-\eta')|^2 d\xi\Big)^{\frac12} d\eta' = \\ &= C_{p,s}\int (1+|\eta'|^2)^{|s|/2-p}\Big(\int (1+|\xi-\eta'|^2)^s|\tilde{u}(\xi-\eta')|^2 d\xi\Big)^{\frac12} d\eta' = C^1_{p,s}\|u\|_s\end{aligned} \tag{3.19}$$

(when we take p sufficiently large).

Now Theorem 2 is a consequence of the relation

$$\|(A(\infty, D) + A'(x, D))u\|_{H^s} \leqslant \|A(\infty, D)u\|_{H^s} + \|A'(x, D)u\|_{H^s} \leqslant C_1\|u\|_s .$$

It proves that the operator $A(x, D)$ is of order $\leqslant 0$.

By density arguments we may extend $A(x, D)$ to a linear continuous map of H^s in H^s, and this for any real s.

In the next Chapter we define a similar, but different operator associated to a given symbol $a(x, \xi)$; we study its properties and relationship with $A(x, D)$.

4. – The operator $\mathcal{A}(x, D)$.

Let $a(x, \xi)$ be a symbol; we define an operator $\mathcal{A}(x, D)$ of S in S' by means of the formula

$$\mathcal{A}(x, D)u = (2\pi)^{-n/2}\int \exp(ix\cdot\xi)H(\xi)\,d\xi \tag{4.1}$$

where, for $u \in \mathcal{S}$, the function $H(\xi)$ is defined by the relation

$$H(\xi) = a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \eta)\tilde{u}(\eta)\,d\eta\,, \qquad \xi \in R^n - \{0\},\ u \in \mathcal{S}. \tag{4.2}$$

With the same proof used for $A(x, D)$ we have: the function $\mathcal{A}(x, D)u$ is continuous and bounded, for $x \in R^n$. Besides, we see that if the symbol $a(x, \xi)$ does not depend on x, we have $A(D) = \mathcal{A}(D)$.

Another formula of representation is given in

PROPOSITION 2. – We have:

$$\mathcal{A}(x, D)u = (2\pi)^{-n/2}\int \exp(ix\cdot\eta)a(x, \eta)\tilde{u}(\eta)\,d\eta\,, \qquad \forall u \in \mathcal{S}.$$

PROOF. – As $a(x, \eta) = a(\infty, \eta) + a'(x, \eta)$ and $\tilde{u}(\eta) \in \mathcal{S}$, the integral is absolutely convergent.

We have, then:

$$(2\pi)^{-n/2}\int \exp(ix\cdot\xi)\left[(2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \eta)\tilde{u}(\eta)\,d\eta\right]d\xi \tag{4.3}$$

is absolutely convergent because

$$\begin{aligned}\iint |\tilde{a}'(\xi-\eta, \eta)|\,|\tilde{u}(\eta)|\,d\eta\,d\xi &\leqslant \\ \leqslant C_p\iint (1+|\xi-\eta|^2)^{-p}|\tilde{u}(\eta)|\,d\eta\,d\xi &= C_p\int |\tilde{u}(\eta)|\left(\int (1+|\xi-\eta|^2)^{-p}d\xi\right)d\eta < \infty\end{aligned} \tag{4.4}$$

for p large enough.

Furthermore we see that (4.3) equals

$$(4.5)\qquad (2\pi)^{-n}\int \exp\left(ix\cdot(\xi-\eta)\right)\exp\left(ix\cdot\eta\right)\left(\int \tilde{a}'(\xi-\eta,\eta)\tilde{u}(\eta)\,d\eta\right)d\xi =$$

$$= (2\pi)^{-n}\int\left(\int \exp\left(ix\cdot(\xi-\eta)\right)\tilde{a}'(\xi-\eta,\eta)\,d\xi\right)\exp\left(ix\cdot\eta\right)\tilde{u}(\eta)\,d\eta =$$

$$= (2\pi)^{-n}\int\left(\int \exp\left(ix\cdot\lambda\right)\tilde{a}'(\lambda,\eta)\,d\lambda\right)\exp\left(ix\cdot\eta\right)\tilde{u}(\eta)\,d\eta =$$

$$= (2\pi)^{-n/}\int {}^{2}a'(x,\eta)\cdot\exp\left(ix\cdot\eta\right)\tilde{u}(\eta)\,d\eta\,.$$

This will prove Proposition 2.

EXAMPLE. – As an useful application of Prop. 2, let us take a fixed function $u(x)\in C_0^\infty$, and then the sequence

$$u_\nu(x) = \nu^{n/4}u\left((x-x_0)\nu^{\frac{1}{2}}\right)\exp\left(i(x\cdot\xi_0)\nu\right), \qquad \nu = 1, 2, \ldots,$$

where $x_0\in\operatorname{Supp} u$, and $|\xi_0| = 1$. Then it follows

$$\left(\mathcal{A}(x, D)u_\nu\right)(x) = \nu^{n/4}v_\nu\left((x-x_0)\nu^{\frac{1}{2}}\right)\exp\left(i(x\cdot\xi_0)\nu\right),$$

where $v_\nu(x)$ are defined by

$$v_\nu(x) = (2\pi)^{-n/2}\int a(x_0+\nu^{-\frac{1}{2}}x,\ \nu\xi_0+\eta\nu^{\frac{1}{2}})\,\tilde{u}(\eta)\exp\left(ix\cdot\eta\right)d\eta\,.$$

We see that $\left(v_\nu(x)\right)_{\nu=1}^\infty$ is an uniformly bounded sequence, and it can be proved that

$$\lim_{\nu\to\infty} v_\nu(x) = a(x_0, \xi_0)u(x)$$

holds, uniformly on bounded sets in R^n.

In fact, we get easily that

$$v_\nu(x) - a(x_0,\xi_0)u(x) = (2\pi)^{-n/2}\int\left[a\left(x_0+\frac{x}{\sqrt{\nu}},\ \nu\xi_0+\eta\sqrt{\nu}\right)-a(x_0,\xi_0)\right]\tilde{u}(\eta)\exp\left(ix\cdot\eta\right)d\eta\,.$$

Moreover we have, being $a(x_0,\xi_0) = a(x_0,\nu\xi_0)$, $\nu = 1, 2, \ldots$, the estimate

$$\left|a\left(x_0+\frac{x}{\sqrt{\nu}},\ \nu\xi_0+\eta\sqrt{\nu}\right)-a(x_0,\xi_0)\right| \leqslant \left|a\left(x_0+\frac{x}{\sqrt{\nu}},\ \nu\xi_0+\eta\sqrt{\nu}\right)-a\left(x_0,\nu\xi_0+\eta\sqrt{\nu}\right)\right| +$$

$$+\ \left|\left(a\left(x_0,\nu\xi_0+\eta\sqrt{\nu}\right)-a(x_0,\nu\xi_0)\right|\leqslant\frac{|x|}{\sqrt{\nu}}\sup_{x,\xi}|\operatorname{grad}_x a| + \frac{C|\eta|}{\sqrt{\nu}}\,,$$

(we use here (2.21) and (2.1)).

Consequently we have

$$|v_\nu(x) - a(x_0, \xi_0)\,u(x)| \leq \frac{C|x|}{\sqrt{\nu}} \int |\tilde{u}(\eta)|\,d\eta + \frac{C}{\sqrt{\nu}} \int |\eta|\,|\tilde{u}(\eta)|\,d\eta$$

which proves the result.

It can be shown, exactly as with the operator $A(x, D)$ that, $\forall$ real s, the estimate

$$\|\mathcal{A}(x, D)\,u\|_s \leq C_s \|u\|_s\,, \qquad u \in \mathcal{S},$$

is verified.

Considering only the case $s = 0$, and by the density of $\mathcal{S}$ in L^2, we can extend $A(x, D)$ and $\mathcal{A}(x, D)$ by continuity, to linear operators of L^2 in L^2. Now we have

PROPOSITION 3. – *Let $a(x, \xi)$ be a symbol, and $\bar{a}(x, \xi)$ its complex conjugate, operator $A(x, D)$ associated to $a(x, \xi)$, operator $\bar{\mathcal{A}}(x, D)$ associated to $\bar{a}(x, \xi)$. Then we have the equality*:

$$(A(x, D)\,u, v)_{L^2} = (u, \bar{\mathcal{A}}(x, D)v)_{L^2}\,, \qquad \forall u, v \in L^2.$$

It will be sufficient to show that for $u, v \in \mathcal{S}$. We have first of all:

$$\bar{\mathcal{A}}(x, D)\,v = (2\pi)^{-n/2}\int \exp\,(ix\cdot\eta)\,\bar{a}(x, \eta)\,\tilde{v}(\eta)\,d\eta\,, \qquad \forall v \in \mathcal{S} \text{ (Prop. 2)}. \tag{4.6}$$

Hence we get, when $(u, v)_{L^2} = \int u(x)\,\bar{v}(x)\,dx$, the equality

$$(u, \bar{\mathcal{A}}(x, D)v) = (2\pi)^{-n/2}\int u(x)\Big(\int \exp\,(-ix\cdot\eta)\,a(x, \eta)\,\bar{\tilde{v}}(\eta)\,d\eta\Big)\,dx = \tag{4.7}$$
$$= (2\pi)^{-n/2}\int\!\!\int \exp\,(-ix\cdot\eta)\,a(x, \eta)\,u(x)\,\bar{\tilde{v}}(\eta)\,dx\,d\eta\,.$$

Now, by PLANCHEREL's formula we obtain, using also Proposition 1

$$(A(x, D)u, v)_{L^2} = (\widetilde{A(x, D)u}, \tilde{v})_{L^2} = \int \widetilde{A(x, D)u}(\xi)\,\bar{\tilde{v}}(\xi)\,d\xi = \tag{4.8}$$
$$= (2\pi)^{-n/2}\int\Big(\int \exp\,(-iy\cdot\xi)\,a(y, \xi)\,u(y)\,dy\Big)\,\bar{\tilde{v}}(\xi)\,d\xi = (2\pi)^{-n/2}\int\!\!\int \exp\,(-iy\cdot\xi)\,\cdot$$
$$\cdot\, a(y, \xi)\,u(y)\,\bar{\tilde{v}}(\xi)\,d\xi\,dy$$

which is exactly (4.7).

REMARK. – Let $a(x, \xi)$ be a symbol of special type:

$$a(x, \xi) = a(x)\,b(\xi)\,.$$

Then we have

$$\mathcal{A}(x, D)\,u = a(x)\,b(D)\,u\,, \qquad A(x, D)\,u = b(D)\big(a(x)\,u(x)\big)\,, \qquad \forall u \in \mathcal{S}. \tag{4.9}$$

In fact, we see that

$$\mathcal{A}(x, D)u = (2\pi)^{-n/2}\int \exp(ix\cdot\eta)a(x)b(\eta)\tilde{u}(\eta)\,d\eta = a(x)b(D)u$$

$$\widetilde{A(x, D)u} = (2\pi)^{-n/2}\int \exp(-iy\cdot\xi)a(y)b(\xi)u(y)\,dy = b(\xi)\widetilde{au}(\xi) = \widetilde{b(D)(au)}(\xi) \qquad \forall u \in \mathcal{S},$$

and this gives the remark.

Now we are able to prove the following

PROPOSITION 4. – *We have the relation*

$$\|(A(x, D) - \mathcal{A}(x, D))u\|_s \leqslant C_s\|u\|_{s-1}, \qquad \forall u \in \mathcal{S}. \tag{4.10}$$

It is known that $A(x, D)u \in \mathcal{S}'$ and that

$$\widetilde{Au}(\xi) = a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta$$

(Fourier transform in $\mathcal{S}'$). The same is valid for $\mathcal{A}(x, D)u$ and

$$\widetilde{\mathcal{A}(x, D)u}(\xi) = a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \eta)\tilde{u}(\eta)\,d\eta\,.$$

Hence, we obtain, with Fourier transform in $\mathcal{S}'$

$$\widetilde{(A-\mathcal{A})u}(\xi) = (2\pi)^{-n/2}\int\big(\tilde{a}'(\xi-\eta, \xi) - \tilde{a}'(\xi-\eta, \eta)\big)\tilde{u}(\eta)\,d\eta\,. \tag{4.11}$$

Therefore, we will have to estimate the norm L^2 of the expression

$$\begin{aligned} U_s(\xi) &= (2\pi)^{-n/2}(1+|\xi|^2)^{s/2}\int\big(\tilde{a}'(\xi-\eta, \xi) - \tilde{a}'(\xi-\eta, \eta)\big)\tilde{u}(\eta)\,d\eta = \\ &= (2\pi)^{-n/2}\int(1+|\xi|^2)^{s/2}(1+|\eta|^2)^{-s/2}\big(\tilde{a}'(\xi-\eta, \xi) - \tilde{a}'(\xi-\eta, \eta)\big)(1+|\eta|^2)^{s/2}\tilde{u}(\eta)\,d\eta\,. \end{aligned} \tag{4.12}$$

We have

$$\begin{aligned} |U_s(\xi)| &\leqslant C_s\int(1+|\xi-\eta|^2)^{|s|/2}|\tilde{a}'(\xi-\eta, \xi) - \tilde{a}'(\xi-\eta, \eta)|(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta \leqslant \\ &\leqslant C_{s,p}\int(1+|\xi-\eta|^2)^{|s|/2}(1+|\xi-\eta|^2)^{-p}\frac{|\xi-\eta|}{|\xi|+|\eta|}(1+|\eta|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta \\ &\qquad \forall p = 1, 2, \ldots,\ \xi, \eta \in R^n - \{0\} \end{aligned} \tag{4.13}$$

(we used here Theorem 1, c)).

We have now the following

LEMMA. – *For every $\xi, \eta \in R^n - \{0\}$ we have the inequality*:

$$|\xi-\eta|(|\xi| + |\eta|)^{-1} \leqslant C(1+|\xi-\eta|^2)^{\frac{1}{2}}(1+|\eta|^2)^{-\frac{1}{2}}\,. \tag{4.14}$$

In fact, we have, for $\xi, \eta \in R^n - \{0\}$, the evident relation $|\xi-\eta| + |\xi-\eta||\eta| \leqslant \leqslant |\xi| + |\eta| + |\eta||\xi-\eta| + |\xi||\xi-\eta|$, which is equivalent to

$$\frac{|\xi-\eta|}{|\xi|+|\eta|} \leqslant \frac{1+|\xi-\eta|}{1+|\eta|}, \qquad \xi, \eta \in R^n - \{0\}.$$

Now, it will be sufficient to observe that, for $0 < c < C$, we have

$$c < (1+|\zeta|)(1+|\zeta|^2)^{-\frac{1}{2}} \leqslant C, \qquad \forall \zeta \in R^n$$

and to substitute

$$(1+|\xi-\eta|) \leqslant C(1+|\xi-\eta|^2)^{\frac{1}{2}}, \qquad (1+|\eta|) \geqslant c(1+|\eta|^2)^{\frac{1}{2}}.$$

Continuing now the estimates, from (4.13) we have for $\xi \in R^n - \{0\}$, that

$$(4.15) \qquad |U_s(\xi)| \leqslant C^1_{s,p} \int (1+|\xi-\eta|^2)^{|s|/2-p+\frac{1}{2}} (1+|\eta|^2)^{s/2-\frac{1}{2}} |\tilde{u}(\eta)| \, d\eta$$

and reasoning as in the proof of Theorem 2, we deduce the result.

REMARK 1. – The result above means that $A - \mathcal{A}$ is an operator of order $\leqslant -1$; for any real s, $A - \mathcal{A}$ extends to a linear continuous map of H^s into H^{s+1}; this implies that $A - \mathcal{A}$ has a certain « regularizing » effect. The property is also useful in the following way: suppose to have an estimate for operator A; then we can get same kind of estimate for the operator $\mathcal{A}$ by writing that $\mathcal{A} = \mathcal{A} - A + A$, applying (4.10) and the known estimate for A. Finally, sometimes we may neglect operators of order $\leqslant -1$. Then we can say that $A \equiv \mathcal{A}$ (mod operators of order $\leqslant -1$).

REMARK 2. – Proposition 3 means that $\bar{\mathcal{A}}$ is the L^2-adjoint of A; for real symbols $\mathcal{A} = A^*$. Hence $A = A^*$ iff $A = \mathcal{A}$; this happens for special symbols like $a(\xi)$ or $b(x)$; we don't know a necessary and sufficient condition on $a(x, \xi)$ in order that $A(x, D) = A^*(x, D)$.

Let us give now another proof of Proposition 3. We will use the definition (in case $a(\infty, \xi) = 0$):

$$(4.16) \qquad \widetilde{Au}(\xi) = (2\pi)^{-n/2} \int \tilde{a}(\xi-\eta, \xi) \tilde{u}(\eta) \, d\eta, \qquad \widetilde{\mathcal{A}u}(\xi) = (2\pi)^{-n/2} \int \tilde{a}(\xi-\eta, \eta) \tilde{u}(\eta) \, d\eta$$

and the relation to be proved becomes, when we use Plancherel's theorem again

$$(4.17) \qquad \int \Big(\int \tilde{a}(\xi-\eta, \xi) \tilde{u}(\eta) \, d\eta \Big) \bar{\tilde{v}}(\xi) \, d\xi = \int \tilde{u}(\xi) \overline{\Big(\int \tilde{\bar{a}}(\xi-\eta, \eta) \tilde{v}(\eta) \, d\eta \Big)} \, d\xi$$

or

$$(4.18) \qquad \iint \tilde{a}(\xi-\eta, \xi) \tilde{u}(\eta) \bar{\tilde{v}}(\xi) \, d\xi \, d\eta = \iint \tilde{u}(\xi) \bar{\tilde{\bar{a}}}(\xi-\eta, \eta) \bar{\tilde{v}}(\eta) \, d\xi \, d\eta.$$

Let us observe here that:

$$\tilde{\bar{a}}(\lambda, \eta) = (2\pi)^{-n/2}\int \exp(-ix\cdot\lambda)\bar{a}(x, \eta)\,dx\,,$$

(4.19)

$$\bar{\tilde{\bar{a}}}(\lambda, \eta) = (2\pi)^{-n/2}\int \exp(ix\cdot\lambda)a(x, \eta)\,dx = \tilde{a}(-\lambda, \eta)\,.$$

Therefore, the relation to be proved becomes:

$$\iint \tilde{a}(\xi-\eta, \xi)\tilde{u}(\eta)\bar{\tilde{v}}(\xi)\,d\xi\,d\eta = \iint \tilde{a}(\eta-\xi, \eta)\tilde{u}(\xi)\bar{\tilde{v}}(\eta)\,d\eta\,d\xi \tag{4.20}$$

changing the variable: $\xi = \eta$, $\eta = \xi$, it becomes obvious.

The case $a(\infty, \xi) \neq 0$ does not introduce any new difficulty. Let $\zeta(\xi) \in C^\infty = 0$, for $|\xi| < \frac{1}{2}$, $=1$ for $|\xi| \geqslant 1$, and $\zeta(D) = \mathcal{F}^{-1}(\zeta(\xi)\mathcal{F})$, the associated operator.

Define two new operators:

$$A_\zeta(x, D) = \zeta(D)\,A(x, D)$$

and

$$\mathcal{A}_\zeta(x, D) = \mathcal{A}(x, D)\zeta(D)\,, \qquad A_\zeta(x, D) - A(x, D) = \big(\zeta(D) - E\big)A(x, D)$$

where $\zeta(D) - E$ has true order $= -\infty$; similarly $\mathcal{A}_\zeta(x, D) - \mathcal{A}(x, D)$ is an operator of order $= -\infty$. It follows that

$$A_\zeta(x, D) - \mathcal{A}_\zeta(x, D) = A(x, D) - \mathcal{A}(x, D) + T\,,$$

where T has order $-\infty$. By (4.10) we deduce, $\forall u \in \mathcal{S}$, relation

$$\|(A_\zeta - \mathcal{A}_\zeta)u\|_s \leqslant c_s\|u\|_{s-1} + \|Tu\|_s \leqslant c'_s\|u\|_{s-1}\,.$$

Furthermore, the L^2-adjoint of $A_\zeta(x, D)$ is $A^*(x, D)\zeta(D) = \bar{\mathcal{A}}(x, D)\zeta(D) = \bar{\mathcal{A}}_\zeta(x, D)$; this because $\zeta(D)$ is self-adjoint for real-valued $\zeta(\xi)$.

5. – Product and commutators.

PROPOSITION. – *Let $a(x, \xi)$, $b(x, \xi)$ be two symbols. Then $c(x, \xi) = a(x, \xi)b(x, \xi)$ is a symbol too.*

Obviously, $c(x, \xi) \in C^\infty(R^n \times R^n - \{0\})$ as $a(x, \xi)$ and $b(x, \xi)$ are in this space. Besides, $\forall t > 0$,

$$c(x, t\xi) = a(x, t\xi)b(x, t\xi) = a(x, \xi)b(x, \xi) = c(x, \xi)\,, \quad x \in R^n,\ \xi \in R^n - \{0\}.$$

As

$$\lim_{|x|\to\infty} a(x, \xi) = a(\infty, \xi)\,, \qquad \lim_{|x|\to\infty} b(x, \xi) = b(\infty, \xi)$$

exist, for $\xi \in R^n - \{0\}$ the same is valid for $c(x, \xi)$;

$$\lim_{|x| \to \infty} c(x, \xi) = c(\infty, \xi) = a(\infty, \xi) b(\infty, \xi)$$

which exists for $\xi \in R^n - \{0\}$.

Hence: if we put $c'(x, \xi) = c(x, \xi) - c(\infty, \xi)$, we have:

$$c(x, \xi) = \big(a'(x, \xi) + a(\infty, \xi)\big)\big(b'(x, \xi) + b(\infty, \xi)\big) =$$
$$= a'(x, \xi) b'(x, \xi) + a(\infty, \xi) b'(x, \xi) + b(\infty, \xi) a'(x, \xi) + a(\infty, \xi) b(\infty, \xi) = c'(x, \xi) + c(\infty, \xi)$$

where

$$c'(x, \xi) = a'(x, \xi) b'(x, \xi) + a(\infty, \xi) b'(x, \xi) + b(\infty, \xi) a'(x, \xi) .$$

Obviously $c(\infty, \xi) \in C^\infty(R^n - \{0\})$.

Let us now remark now that:

$$(1 + |x|^2)^p |D_x^\alpha \partial_\xi^\beta c'(x, \xi)| \leqslant C_{p,\alpha,\beta} , \tag{5.1}$$
$$\forall x \in R^n,\ \xi \in R^n - \{0\},\ p = 1, 2, \ldots,\ \alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n)$$

(consequence of Leibniz's theorem).

Let $C(x, D)$, $A(x, D)$, $B(x, D)$ be the operators corresponding to $c(x, \xi)$, $a(x, \xi)$, $b(x, \xi)$, respectively. We have

$$A(x, D) = A(\infty, D) + A'(x, D) , \qquad B(x, D) = B(\infty, D) + B'(x, D) ;$$

$$A(x, D) B(x, D) = A(\infty, D) B(\infty, D) + \tag{5.2}$$
$$+ A'(x, D) B(\infty, D) + A(\infty, D) B'(x, D) + A'(x, D) B'(x, D) .$$

We denote $a(\infty, \xi) b(\infty, \xi) = \gamma(\xi) = c(\infty, \xi)$; $a'(x, \xi) b'(x, \xi) = k(x, \xi)$, $a(\infty, \xi) b'(x, \xi) = k_1(x, \xi)$, $b(\infty, \xi) a'(x, \xi) = k_2(x, \xi)$. Then,

$$C(x, D) = \gamma(D) + K(x, D) + K_1(x, D) + K_2(x, D) . \tag{5.3}$$

Hence; we have some simple results:

LEMMA 1. – *We have* $\gamma(D) u = A(\infty, D) B(\infty, D) u$ *for* $u \in S$.

In fact,

$$\widetilde{\gamma(D) u}(\xi) = \gamma(\xi) \tilde{u}(\xi) = a(\infty, \xi) b(\infty, \xi) \tilde{u}(\xi) =$$
$$= a(\infty, \xi) \big(\widetilde{B(\infty, D) u}\big)(\xi) = \widetilde{A(\infty, D)\big(B(\infty, D) u}\big)(\xi) ;$$

hence, by Fourier's inversion formula, valid in S', we arrive at Lemma 1.

LEMMA 2. – *We have* $K_1(x, D) = A(\infty, D)B'(x, D)$.

In fact,

$$\widetilde{K_1(x, D)u}(\xi) = (2\pi)^{-n/2}\int a(\infty, \xi)\tilde{b}'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta$$

(as $k_1(\infty, \xi) = 0$, and $\tilde{k}_1(\lambda, \xi) = a(\infty, \xi)\tilde{b}'(\lambda, \xi)$). Hence

$$\widetilde{K_1(x, D)u}(\xi) = a(\infty, \xi)\widetilde{B'(x, D)u}(\xi) = \widetilde{A(\infty, D)(B'(x, D)u)}(\xi) \tag{5.4}$$

and this is true for any $u \in \mathcal{S}$; whence the Lemma follows.

LEMMA 3. – *We have* $K_2(x, D) = B(\infty, D)A'(x, D)$.

The proof is the same, as in Lemma 2. Let us examine here the difference

$$\begin{aligned} A(x, D)B(x, D) - C(x, D) &= A(\infty, D)B(\infty, D) + A'(x, D)B(\infty, D) + \\ &+ A(\infty, D)B'(x, D) + A'(x, D)B'(x, D) - A(\infty, D)B(\infty, D) - A(\infty, D)B'(x, D) - \\ &- B(\infty, D)A'(x, D) - K(x, D) = [A'(x, D), B(\infty, D)] + A'(x, D)B'(x, D) - K(x, D) \end{aligned}$$

where [ ] means the commutator between the two operators, and $K(x, D)$ is the pseudo-differential operator associated with $k(x, \xi) = a'(x, \xi)b'(x, \xi)$.

Let us begin by proving the

PROPOSITION 5. – *We have the relation* (*)

$$\|[A'(x, D), B(\infty, D)]u\|_s \leqslant C_s\|u\|_{s-1}, \qquad \forall u \in \mathcal{S},\ \forall \text{ real } s. \tag{5.5}$$

In fact, we apply the formula

$$\begin{aligned} \widetilde{A'(x, D)B(\infty, D)u}(\xi) &= (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)\widetilde{B(\infty, D)u}(\eta)\,d\eta = \\ &= (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)b(\infty, \eta)\tilde{u}(\eta)\,d\eta\,. \end{aligned} \tag{5.6}$$

Besides,

$$\widetilde{B(\infty, D)A'(x, D)u}(\xi) = b(\infty, \xi)\widetilde{A'(x, D)u}(\xi) = b(\infty, \xi)(2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta$$

and hence

$$\widetilde{[A'(x, D), B(\infty, D)]u}(\xi) = (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta, \xi)\big(b(\infty, \eta) - b(\infty, \xi)\big)\,\tilde{u}(\eta)\,d\eta\,.$$

As $b(\infty, \xi)$ is homogeneous of order 0 in ξ and $C^\infty(R^n - \{0\})$, we have, as seen at the beginning, for $\xi, \eta \in R^n - \{0\}$

$$|b(\infty, \xi) - b(\infty, \eta)| \leqslant C|\xi-\eta|(|\xi| + |\eta|)^{-1} \leqslant C(1 + |\xi-\eta|^2)^{\frac{1}{2}}(1 + |\eta|^2)^{-\frac{1}{2}}\,. \tag{5.7}$$

(*) Obviously the same holds if we replace $A'(x, D)$ by $\mathcal{A}'(x, D)$ and $B(\infty, D)$ by $\mathcal{B}(\infty, D) = B(\infty, D)$.

Hence, we are obliged to estimate the norm L^2 of the expression

$$U_s(\xi)=(1+|\xi|^2)^{s/2}(2\pi)^{-n/2}\int\tilde{a}'(\xi-\eta,\xi)\big(b(\infty,\xi)-b(\infty,\eta)\big)\tilde{u}(\eta)\,d\eta\,. \tag{5.8}$$

We have:

$$|U_s(\xi)|\leqslant C_{s,p}\int(1+|\xi-\eta|^2)^{|s|/2}(1+|\xi-\eta|^2)^{-p+\frac12}(1+|\eta|^2)^{(s-1)/2}|\tilde{u}(\eta)|\,d\eta=$$
$$=C_{s,p}\int(1+|\xi-\eta|^2)^{|s|/2+\frac12-p}(1+|\eta|^2)^{(s-1)/2}|\tilde{u}(\eta)|\,d\eta$$

from where we arrive, as before, at the desired estimate.

A more refined technique is necessary in order to prove (*)

THEOREM 3. – *We have the relation*

$$\|(A'(x,D)B'(x,D)-K(x,D))u\|_s\leqslant C_s\|u\|_{s-1}\,,\qquad \forall u\in\mathcal{S},\ \forall\ \text{real } s. \tag{5.9}$$

Let us consider the operator $K(x,D)$ associated with $k(x,\xi)$:

$$\widetilde{K(x,D)u}(\xi)=(2\pi)^{-n/2}\int\tilde{k}(\xi-\eta,\xi)\tilde{u}(\eta)\,d\eta\,;$$

but we have, for $k(x,\xi)=a'(x,\xi)b'(x,\xi)$ that

$$\tilde{k}(\lambda,\xi)=(2\pi)^{-n/2}\int\tilde{a}'(\lambda-\mu,\xi)\tilde{b}'(\mu,\xi)\,d\mu \tag{5.10}$$

whence we arrive at

$$\widetilde{K(x,D)u}(\xi)=(2\pi)^{-n}\int\tilde{u}(\eta)\Big(\int\tilde{a}'(\xi-\eta-\mu,\xi)\tilde{b}'(\mu,\xi)\,d\mu\Big)d\eta= \tag{5.11}$$
$$=(2\pi)^{-n}\int\Big(\int\tilde{a}'(\xi-\eta-\mu,\xi)\tilde{u}(\eta)\,d\eta\Big)\tilde{b}'(\mu,\xi)\,d\mu\,.$$

In the interior integral, we make: $\eta+\mu=\tau$; $d\eta=d\tau$; it follows

$$\widetilde{Ku}(\xi)=(2\pi)^{-n}\int\Big(\int\tilde{a}'(\xi-\tau,\xi)\tilde{u}(\tau-\mu)\,d\tau\Big)\tilde{b}'(\mu,\xi)\,d\mu= \tag{5.12}$$
$$=(2\pi)^{-n}\int\Big(\int\tilde{a}'(\xi-\tau,\xi)\tilde{b}'(\mu,\xi)\tilde{u}(\tau-\mu)\,d\mu\Big)d\tau=$$
$$=(2\pi)^{-n}\int\tilde{a}'(\xi-\tau,\xi)\Big(\int\tilde{b}'(\mu,\xi)\tilde{u}(\tau-\mu)\,d\mu\Big)d\tau\,.$$

And once more, in the interior integral, we make: $\tau-\mu=\nu$, $d\mu=d\nu$.

(*) Same estimate holds for operator $\mathcal{A}'(x,D)\mathcal{B}'(x,D)-\mathcal{K}(x,D)$ which equals

$$(\mathcal{A}'(x,D)-A'(x,D))\mathcal{B}'(x,D)+A'(x,D)B'(x,D)-K(x,D)+$$
$$+A'(x,D)(\mathcal{B}'(x,D)-B'(x,D))+K(x,D)-\mathcal{K}(x,D)$$

as easily seen.

We have now

$$\widetilde{Ku}(\xi) = (2\pi)^{-n}\int \tilde{a}'(\xi-\tau,\xi)\Big(\int \tilde{b}'(\tau-\nu,\xi)\tilde{u}(\nu)\,d\nu\Big)\,d\tau = \tag{5.13}$$
$$= (2\pi)^{-n}\iint \tilde{a}'(\xi-\tau,\xi)\tilde{b}'(\tau-\nu,\xi)\tilde{u}(\nu)\,d\nu\,d\tau\,.$$

Hence, we arrive at

$$\widetilde{Ku}(\xi) = (2\pi)^{-n}\iint \tilde{a}'(\xi-\tau,\xi)\tilde{b}'(\tau-\eta,\xi)\tilde{u}(\eta)\,d\eta\,d\tau\,. \tag{5.14}$$

On the other hand, we have

$$\widetilde{A'(x,D)B'(x,D)u}(\xi) = (2\pi)^{-n/2}\int \tilde{a}'(\xi-\eta,\xi)\widetilde{B'(x,D)u}(\eta)\,d\eta \tag{5.15}$$

and besides:

$$\widetilde{B'(x,D)u}(\eta) = (2\pi)^{-n/2}\int \tilde{b}'(\eta-\tau,\eta)\tilde{u}(\tau)\,d\tau\,; \tag{5.16}$$

and hence we shall obtain

$$\widetilde{A'(x,D)B'(x,D)u}(\xi) = (2\pi)^{-n}\int \tilde{a}'(\xi-\eta,\xi)\Big(\int \tilde{b}'(\eta-\tau,\eta)\tilde{u}(\tau)\,d\tau\Big)\,d\eta = \tag{5.17}$$
$$= (2\pi)^{-n}\iint \tilde{a}'(\xi-\eta,\xi)\tilde{b}'(\eta-\tau,\eta)\tilde{u}(\tau)\,d\tau\,d\eta\,.$$

By making substitution $\eta=\tau$, $\tau=\eta$, we arrive at

$$\widetilde{A'(x,D)B'(x,D)u}(\xi) = (2\pi)^{-n}\iint \tilde{a}'(\xi-\tau,\xi)\tilde{b}'(\tau-\eta,\tau)\tilde{u}(\eta)\,d\eta\,d\tau\,. \tag{5.18}$$

The absolute convergence of the « double » integrals here considered results from the estimates

$$|\tilde{a}'(\xi-\tau,\xi)| \leqslant C_p(1+|\xi-\tau|^2)^{-p}\,,\quad |\tilde{b}'(\tau-\eta,\tau)| \leqslant C\,,\quad |\tilde{u}(\eta)| \leqslant C_p(1+|\eta|^2)^{-p}\,, \tag{5.19}$$
$$\forall p = 1, 2, \ldots .$$

Therefore, we can express the difference $\big(\widetilde{A'(x,D)B'(x,D) - K(x,D)\big)u}(\xi)$ by the « double » integral

$$(2\pi)^{-n}\iint \tilde{a}'(\xi-\tau,\xi)\big(\tilde{b}'(\tau-\eta,\tau) - \tilde{b}'(\tau-\eta,\xi)\big)\tilde{u}(\eta)\,d\eta\,d\tau\,. \tag{5.20}$$

Let us examine here the norm L^2 of the expression

$$U_s(\xi) = (2\pi)^{-n}\iint (1+|\xi|^2)^{s/2}\tilde{a}'(\xi-\tau,\xi)\big(\tilde{b}'(\tau-\eta,\xi) - \tilde{b}'(\tau-\eta,\tau)\big)\tilde{u}(\eta)\,d\eta\,d\tau\,. \tag{5.21}$$

We have, first of all, the pointwise estimate, $\forall \xi \in R^n - \{0\}$

$$(5.22)\quad |U_s(\xi)| \leqslant C\iint (1+|\xi|^2)^{s/2}(1+|\xi-\tau|^2)^{-p}(1+|\tau-\eta|^2)^{-p}|\xi-\tau|(|\xi|+|\tau|)^{-1}|\tilde{u}(\eta)|\,d\eta\,d\tau \leqslant$$
$$\leqslant C_{s,p}\iint (1+|\xi|^2)^{s/2}(1+|\xi-\tau|^2)^{-p}(1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{\frac{1}{2}}(1+|\tau|^2)^{-\frac{1}{2}}|\tilde{u}(\eta)|\,d\eta\,d\tau =$$
$$= C_{s,p}\iint (1+|\xi|^2)^{s/2}(1+|\xi-\tau|^2)^{-p+\frac{1}{2}}(1+|\tau-\eta|^2)^{-p}(1+|\tau|^2)^{-\frac{1}{2}}|\tilde{u}(\eta)|\,d\eta\,d\tau\,.$$

Let us denote now:

$$(5.23)\quad H(\xi,\eta,\tau) = (1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{-p+\frac{1}{2}}(1+|\tau|^2)^{-\frac{1}{2}}$$

$$(5.24)\quad K_s(\xi,\eta) = \frac{(1+|\xi|^2)^{s/2}}{(1+|\eta|^2)^{(s-1)/2}}\int H(\xi,\eta,\tau)\,d\tau\,.$$

We remark that it follows, $\forall \xi \in R^n - \{0\}$

$$(5.25)\quad |U_s(\xi)| \leqslant C_{s,p}\iint (1+|\xi|^2)^{s/2} H(\xi,\eta,\tau)\,|\tilde{u}(\eta)|\,d\eta\,d\tau\,.$$

Therefore, we have only to prove the inequality

$$(5.26)\quad \left(\int\left(\iint (1+|\xi|^2)^{s/2} H(\xi,\eta,\tau)\,|\tilde{u}(\eta)|\,d\eta\,d\tau\right)^2 d\xi\right)^{\frac{1}{2}} \leqslant C_s\|u\|_{s-1}\,, \qquad \forall u \in \mathcal{S}.$$

In order to do that we shall prove here a more general result, which is given in

LEMMA 1. – *Let* $r(\xi,\eta,\tau) > 0$ *be a function such that* $\int r(\xi,\eta,\tau)\,d\tau < \infty$ *for every* ξ, η *fixed in* $R^n - \{0\}$.

We denote

$$\varrho_s(\xi,\eta) = \frac{(1+|\xi|^2)^{s/2}}{(1+|\eta|^2)^{(s-1)/2}}\int r(\xi,\eta,\tau)\,d\tau\,,$$

and we suppose

$$\int \varrho_s(\xi,\eta)\,d\xi \leqslant L\,, \qquad \int \varrho_s(\xi,\eta)\,d\eta \leqslant L\,, \qquad \xi,\eta \in R^n - \{0\}.$$

Then, there is a constant C_s *such that the inequality*

$$(5.27)\quad \left(\int\left(\iint (1+|\xi|^2)^{s/2} r(\xi,\eta,\tau)\,|\tilde{u}(\eta)|\,d\eta\,d\tau\right)^2 d\xi\right)^{\frac{1}{2}} \leqslant C_s\|u\|_{s-1}\,, \qquad \forall u \in \mathcal{S},\ \forall \text{ real } s$$

is verified.

PROOF OF LEMMA 1. – We remark that in fact, we have

$$(5.28)\quad \iint r(\xi,\eta,\tau)(1+|\xi|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta\,d\tau = \int \varrho_s(\xi,\eta)(1+|\eta|^2)^{(s-1)/2}|\tilde{u}(\eta)|\,d\eta\,.$$

Let us put $v(\eta)=(1+|\eta|^2)^{(s-1)/2}|\tilde{u}(\eta)|$. We remark that (*)

$$(5.29)\qquad \int\varrho_s(\xi,\eta)v(\eta)\,d\eta=\int\sqrt{\varrho_s(\xi,\eta)}\sqrt{\varrho_s(\xi,\eta)}v(\eta)\,d\eta\leqslant$$
$$\leqslant\Big(\int\varrho_s(\xi,\eta)\,d\eta\Big)^{\frac12}\Big(\int\varrho_s(\xi,\eta)v^2(\eta)\,d\eta\Big)^{\frac12}\leqslant\sqrt{L}\Big(\int\varrho_s(\xi,\eta)v^2(\eta)\,d\eta\Big)^{\frac12},$$

$$(5.30)\qquad \iint r(\xi,\eta,\tau)(1+|\xi|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta\,d\tau\leqslant\sqrt{L}\Big(\int\varrho_s(\xi,\eta)v^2(\eta)\,d\eta\Big)^{\frac12},\qquad \xi\in R^n-\{0\}.$$

Hence we have also:

$$\Big\|\iint r(\xi,\eta,\tau)(1+|\xi|^2)^{s/2}|\tilde{u}(\eta)|\,d\eta\,d\tau\Big\|_{L^2(R^n)}\leqslant\sqrt{L}\Big(\int\Big(\int\varrho_s(\xi,\eta)v^2(\eta)\,d\eta\Big)\,d\xi\Big)^{\frac12}=$$
$$=\sqrt{L}\Big(\int\Big(\int\varrho_s(\xi,\eta)\,d\xi\Big)v^2(\eta)\,d\eta\Big)^{\frac12}\leqslant L\Big(\int v^2(\eta)\,d\eta\Big)^{\frac12}=L\|u\|_{s-1}.$$

We shall apply Lemma 1 taking $r(\xi,\eta,\tau)=H(\xi,\eta,\tau)$ and $\varrho_s(\xi,\eta)=K_s(\xi,\eta)$.

We see readily that (1) $\int H(\xi,\eta,\tau)\,d\tau<\infty$, and it remains to prove

LEMMA 2. – *We have*

$$(5.31)\qquad \int K_s(\xi,\eta)\,d\xi\leqslant L,\qquad \int K_s(\xi,\eta)\,d\eta\leqslant L.$$

In fact,

$$K_s(\xi,\eta)=\frac{(1+|\xi|^2)^{s/2}}{(1+|\eta|^2)^{(s-1)/2}}\int(1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{-p+\frac12}(1+|\tau|^2)^{-\frac12}\,d\tau.$$

Because we have the known estimate $(1+|\tau|^2)^{-\frac12}\leqslant 2^{\frac12}(1+|\xi|^2)^{-\frac12}(1+|\xi-\tau|^2)^{\frac12}$, we obtain

$$(5.32)\qquad K_s(\xi,\eta)\leqslant\Big(\frac{1+|\xi|^2}{1+|\eta|^2}\Big)^{(s-1)/2}2^{\frac12}\int(1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{-p+1}\,d\tau\leqslant$$
$$\leqslant C_s(1+|\xi-\eta|^2)^{|s-1|/2}\int(1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{-p+1}\,d\tau=$$
$$=C_s\int(1+|\xi-\eta|^2)^{|s-1|/2}(1+|\tau-\eta|^2)^{-p}(1+|\xi-\tau|^2)^{-p+1}\,d\tau.$$

Now we have, $(1+|\xi-\eta|^2)^{|s-1|/2}\leqslant C(1+|\xi-\tau|^2)^{|s-1|/2}(1+|\tau-\eta|^2)^{|s-1|/2}$, and hence

$$(5.33)\qquad K_s(\xi,\eta)\leqslant C_s\int(1+|\xi-\tau|^2)^{-p+1+|s-1|/2}(1+|\tau-\eta|^2)^{-p+|s-1|/2}\,d\tau.$$

(*) This proof, quite well-known in fact, was communicated to us some years ago by the colleague S. TAKAHASHI (see however *Seeley's lectures in Stresa*, C.I.M.E., 1968).

(1) For sufficiently large p.

We denote at this stage:

$$\lambda(t) = \int (1+|t-u|^2)^{-p+|s-1|/2}(1+|u|^2)^{-p+1+|s-1|/2}du\,, \qquad t\in R^n,$$

where p is large enough.

We see that $\lambda(t)\in L^1$ as convolution of two integrable functions; hence, we have:

$$\lambda(\xi-\eta) = \int (1+|\xi-\eta-u|^2)^{-p+|s-1|/2}(1+|u|^2)^{-p+1+|s-1|/2}du =$$
$$= (\text{by substituting } u=\xi-\tau) = \int (1+|\tau-\eta|^2)^{-p+|s-1|/2}(1+|\xi-\tau|^2)^{-p+1+|s-1|/2}d\tau\,.$$

Hence, we get

$$K_s(\xi,\eta)\leqslant C_s\lambda(\xi-\eta)$$

and obviously:

$$\int\lambda(\xi-\eta)\,d\xi<\infty\,, \qquad \int\lambda(\xi-\eta)\,d\eta<\infty$$

which proves the Lemma 2.

Hence, for Lemma 1 we have that

$$\|U_s(\xi)\|_{L^2}\leqslant C\|u\|_{s-1}\,, \qquad \forall u\in \mathcal{S} \tag{5.34}$$

and this proves Theorem 3.

COROLLARY. – *If $A(x,D)$, $B(x,D)$ are two pseudo-differential operators, the commutator $[A(x,D), B(x,D)]$ is of order $\leqslant -1$.*

In fact, we have that

$$A(x,D)B(x,D)-(ab)(x,D) = [A'(x,D), B(x,D)] + A'(x,D)B'(x,D) - K(x,D)$$

is of order $\leqslant -1$ as by Theorem 3 and Proposition 5.

In the same way, we can prove that $B(x,D)A(x,D)-(ab)(x,D)$ is of order $\leqslant -1$. Hence we arrive at the desired result (*).

REMARK. 1. – Let $a(x,\xi)$ be a symbol such that $|a(x,\xi)|>\alpha>0$, $\forall x\in R^n$, $\forall\xi\in R^n-\{0\}$. Then one can see that $b(x,\xi)=(a(x,\xi))^{-1}$ is again a symbol. Hence $a(x,\xi)b(x,\xi)=1$ $\forall x\in R^n$, $\xi\in R^n-\{0\}$. The operator $\mathcal{A}$ associated to $c(x,\xi)\equiv 1$ is the identity operator. Hence we get

$$\|(I-\mathcal{B}\mathcal{A})u\|_0\leqslant c\|u\|_{-1}\,, \qquad \forall u\in\mathcal{S}\,.$$

Furthermore we have

$$u = u-\mathcal{B}\mathcal{A}u+\mathcal{B}\mathcal{A}u\,, \qquad \forall u\in\mathcal{S}\,.$$

(*) Same result holds for the commutator $[\mathcal{A}(x,D), \mathcal{B}(x,D)]$ as follows from footnotes to Prop. 5 and Th. 3.

We derive inequality

$$\|u\|_0 \leqslant c\|u\|_{-1} + \|\mathcal{B}\|_{\mathfrak{L}(L^2;L^2)} \cdot \|\mathcal{A}u\|_0 \leqslant c_1(\|\mathcal{A}u\|_0 + \|u\|_{-1}), \qquad \forall u \in L^2 .$$

Same estimate holds when we replace $\mathcal{A}$ by A.

We have also the interesting

REMARK 2. – Let $a(x, \xi)$ be a symbol and $A(x, D)$ the associated p.d.o. Assume that $\lambda_0 \in \mathbb{C}$ is an eigen-value of $A(x, D)$ (in $L^2(R^n)$), such that $|a(x, \xi) - \lambda_0| > \alpha > 0$ $\forall x \in R^n$, $|\xi| = 1$. Then, any eigen-vector $u_0(x)$ corresponding to λ_0 is a C^∞-function.

In fact, $b(x, \xi) = (a(x, \xi) - \lambda_0)^{-1}$ is a symbol. If $B(x, D)$ is associated to it we get, as in Remark 1, that $B(x, D)(A(x, D) - \lambda_0 E) = E + T$, where E is the identity operator and T has order $\leqslant -1$. It follows: $\theta = B(A - \lambda_0 E)u_0 = u_0 + Tu_0$, i.e. $u_0 = -Tu_0$. Being $u_0 \in L^2$, it follows that $Tu_0 \in H^1$ and $u_0 \in H^1$ too.

In the same way we get that $u_0 \in \bigcap_{p=0}^{\infty} H^p$, which implies, as wellknown, that $u_0(x) \in C^\infty(R^n)$.

Let us consider now the operator $I_s = (1 + |D|^2)^{s/2}$, defined by $\widetilde{I_s u}(\xi) = (1 + |\xi|^2)^{s/2}\tilde{u}(\xi)$, $\forall u \in \mathcal{S}$. A useful result is given in

THEOREM 4. – *Let $a(x, \xi)$ be a symbol, $A(x, D)$ the associated pseudo-differential operator. We have:*

$$\|[A(x, D), I_s]\|_{L^2(R^n)} \leqslant C\|u\|_{H^{s-1}}, \qquad \forall u \in \mathcal{S}. \tag{5.35}$$

In fact, we have:

$$\widetilde{A(x, D)I_s u}(\xi) = a(\infty, \xi)(1 + |\xi|^2)^{s/2}\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi - \eta, \xi)(1 + |\eta|^2)^{s/2}\tilde{u}(\eta)\, d\eta , \tag{5.36}$$

and also

$$\widetilde{I_s A(x, D)u}(\xi) = (1 + |\xi|^2)^{s/2}\widetilde{A(x, D)u}(\xi) = (1 + |\xi|^2)^{s/2} a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int \tilde{a}'(\xi - \eta, \xi)(1 + |\xi|^2)^{s/2}\tilde{u}(\eta)\, d\eta \tag{5.37}$$

and hence it can be deduced that

$$\widetilde{[A(x, D), I_s]u}(\xi) = (2\pi)^{-n/2}\int \tilde{a}'(\xi - \eta, \xi)[(1 + |\eta|^2)^{s/2} - (1 + |\xi|^2)^{s/2}]\tilde{u}(\eta)\, d\eta = U_s(\xi), \quad \xi \in R^n - \{0\}. \tag{5.38}$$

By estimating the norm L^2 of $U_s(\xi)$ we have first of all the point-wise estimate

$$|U_s(\xi)| \leqslant C_p \int (1 + |\xi - \eta|^2)^{-p} |(1 + |\eta|^2)^{s/2} - (1 + |\xi|^2)^{s/2}| |\tilde{u}(\eta)|\, d\eta . \tag{5.39}$$

Let us remark here the elementary inequality, for $0 < \theta < 1$

$$(1 + |\eta + \theta(\xi - \eta)|^2)^{(s-1)/2} \leqslant 2^{|s-1|/2}(1 + |\eta|^2)^{(s-1)/2}(1 + |\theta(\xi - \eta)|^2)^{|s-1|/2} \tag{5.40}$$

and therefore, as $|s-1|/2>0$, $0<\theta<1$: $(1+(\theta(\xi-\eta)|^2)^{|s-1|/2}\leqslant(1+|\xi-\eta|^2)^{|s-1|/2}$ whence

$$(5.41)\qquad (1+|\eta+\theta(\xi-\eta)|^2)^{(s-1)/2}\leqslant 2^{|s-1|/2}(1+|\eta|^2)^{(s-1)/2}(1+|\xi-\eta|^2)^{|s-1|/2}.$$

By Taylor's formula, we have

$$(5.42)\qquad (1+|\xi|^2)^{s/2}-(1+|\eta|^2)^{s/2}=\left((\xi-\eta),\operatorname{grad}(1+|\xi|^2)^{s/2}_{\xi=\zeta}\right),\qquad \zeta=\eta+\theta(\xi-\eta)$$

$$(5.43)\qquad |(1+|\xi|^2)^{s/2}-(1+|\eta|^2)^{s/2}|\leqslant|\xi-\eta||\operatorname{grad}(1+|\xi|^2)^{s/2}_{\xi=\zeta}|.$$

As we have

$$\frac{\partial}{\partial\xi_i}(1+|\xi|^2)^{s/2}=\xi_i s(1+|\xi|^2)^{s/2-1},$$

it follows that

$$(5.44)\qquad |\operatorname{grad}(1+|\xi|^2)^{s/2}|=|\xi||s|(1+|\xi|^2)^{s/2-1}\leqslant|s|(1+|\xi|^2)^{(s-1)/2}$$

and hence

$$(5.45)\qquad \begin{aligned}|(1+|\xi|^2)^{s/2}-(1+|\eta|^2)^{s/2}|&\leqslant|\xi-\eta||s|(1+|\eta+\theta(\xi-\eta)|^2)^{(s-1)/2}\leqslant\\ &\leqslant|s|(1+|\xi-\eta|^2)^{\frac12}(1+|\eta+\theta(\xi-\eta)|^2)^{(s-1)/2}\leqslant\\ &\leqslant|s|(1+|\xi-\eta|^2)^{\frac12}2^{|s-1|/2}(1+|\eta|^2)^{(s-1)/2}(1+|\xi-\eta|^2)^{|s-1|/2}.\end{aligned}$$

Introducing (5.45) in (5.39) we shall obtain

$$(5.46)\qquad \begin{aligned}|U_s(\xi)|&\leqslant C_{p,s}\int(1+|\xi-\eta|^2)^{-p}(1+|\xi-\eta|^2)^{(|s-1|+1)/2}(1+|\eta|^2)^{(s-1)/2}|\tilde u(\eta)|\,d\eta=\\ &=C_{p,s}\int(1+|\xi-\eta|^2)^{-p+(|s-1|+1)/2}(1+|\eta|^2)^{(s-1)/2}|\tilde u(\eta)|\,d\eta.\end{aligned}$$

From here on the proof finishes as in Theorem 2, when we take large enough p.

REMARK. – Same proof works for the commutator $[\mathcal{A}(x,D),I_s]$ (just replace in (5.38) $\tilde a'(\xi-\eta,\xi)$ by $\tilde a'(\xi-\eta,\eta)$).

6. – Some inequalities.

We want to prove the following (*)

THEOREM 5. – *Let $A(x,D)$, $L^2\to L^2$ be a pseudo-differential operator associated with the symbol $a(x,\xi)$, such that $a=\bar a$ and*

$$(6.1)\qquad a(x,\xi)\geqslant\gamma$$

(*) Same result holds for the operator $\mathcal{A}(x,D)$; also, nonreal valued symbols $a(x,\xi)$ such that $\operatorname{Re}a(x,\xi)\geqslant\gamma$ can be considered. The proof uses (6.2) and (4.10), (cfr. with our paper [4], Th. 3).

for $|\xi| = 1$, $x \in R^n$. *Then for every* $\varepsilon > 0$ *there is a constant* $C'(\varepsilon)$ *such that, for* $u \in \mathcal{S}$

$$\mathrm{Re}\,(A(x, D)u, u)_{L^2} + C'(\varepsilon)\|u\|^2_{H^{-\frac{1}{2}}} \geq (\gamma - \varepsilon)\|u\|^2_{L^2} \tag{6.2}$$

is verified.

PROOF. – In fact, we have obviously, for arbitrary $\varepsilon > 0$, the inequality

$$a(x, \xi) - \gamma + \varepsilon \geq \varepsilon\,, \tag{6.3}$$

for $|\xi| = 1$, $x \in R^n$.

Let be $b(x, \xi) = (a(x, \xi) - \gamma + \varepsilon)^{\frac{1}{2}}$, $x \in R^n$, $|\xi| = 1$; for arbitrarily $\xi \in R^n - \{0\}$ we put $b(x, \xi) = b(x, \xi/|\xi|)$. Hence $b(x, \xi)$ is homogeneous of order 0. It is « easy » to verify that, when $x \in R^n$ and $|\xi| = 1$ we have

$$|(1 + |x|^2)^p D_x^\alpha \partial_\xi^\beta b'(x, \xi)| \leq C_{p,\alpha,\beta}\,, \tag{6.4}$$

if we are based on the same property valid for $a'(x, \xi)$ and upon the fact that $a(x, \xi) - \gamma + \varepsilon \geq \varepsilon$ for $\xi \in R^n - \{0\}$, $x \in R^n$.

Hence, $b(x, \xi)$ is a symbol in the sense of KOHN-NIRENBERG. We consider hence the operators $B(x, D)$ and $\mathcal{B}(x, D)$ associated with the symbol $b(x, \xi)$. We have

1) *The operator* $A - (\gamma - \varepsilon)I - \mathcal{B}\cdot B$ *is of order* ≤ -1.

In fact, $B(x, D)B(x, D) - b^2(x, D)$ is of order ≤ -1, as shown in Chapter 5. Being $b^2(x, \xi) = a(x, \xi) - \gamma + \varepsilon$, we have that $b^2(x, D) = A(x, D) - (\gamma - \varepsilon)I$, and hence we deduce that $B(x, D)B(x, D) - A(x, D) + (\gamma - \varepsilon)I$ is of order ≤ -1.

Hence: $B\cdot B = A - (\gamma - \varepsilon)I + T_{-1}$ and $\mathcal{B}B = (\mathcal{B} - B)B + B\cdot B$; here T_{-1} is an operator of order ≤ -1; whence we get

$$\begin{aligned} A - (\gamma - \varepsilon)I - \mathcal{B}\cdot B &= A - (\gamma - \varepsilon)I - B\cdot B + (B - \mathcal{B})B = \\ &A - (\gamma - \varepsilon)I - A + (\gamma - \varepsilon)I - T_{-1} + (B - \mathcal{B})B = T_{-1} + U_{-1} \end{aligned} \tag{6.5}$$

as $B - \mathcal{B}$ being of order ≤ -1 and B of order 0 their product is of order ≤ -1.

Hence, we have also

2) *Let* T *be an operator of* S *in* S' *such that* $\|Tu\|_s \leq C\|u\|_{s-1}$ (¹). *Then* T *is continuous of* L^2 *in* L^2, *and we have*

$$\mathrm{Re}\,(Tu, u)_0 \geq -C'\|u\|^2_{-\frac{1}{2}}\,, \qquad \forall u \in \mathcal{S}. \tag{6.6}$$

In fact, we obtain obviously the estimate

$$|\mathrm{Re}\,(Tu, u)_0| \leq |(Tu, u)_0| \leq \|Tu\|_s\|u\|_{-s} \tag{6.7}$$

(¹) For any real s.

by SCHWARZ's inequality (generalized)

(6.8) $$|(u,v)_0| \leqslant \|u\|_s \|v\|_{-s}, \qquad \forall u, v \in \mathcal{S}.$$

Hence:

(6.8) $$|\mathrm{Re}\,(Tu,u)_0| \leqslant C_s \|u\|_{s-1} \|u\|_{-s}, \qquad \forall \text{ real } s,\ u \in \mathcal{S};$$

we take $s = \frac{1}{2}$ and we obtain

(6.9) $$|\mathrm{Re}\,(Tu,u)_0| \leqslant C' \|u\|^2_{-\frac{1}{2}}$$

therefore is

(6.10) $$\mathrm{Re}\,(Tu,u)_0 \geqslant -C' \|u\|^2_{-\frac{1}{2}}.$$

By combining 1) and 2), we deduce that

(6.11) $$\mathrm{Re}\,\big((A-(\gamma-\varepsilon)I - \mathcal{B}\cdot B)u, u\big)_0 \geqslant -C' \|u\|^2_{-\frac{1}{2}}, \qquad \forall u \in \mathcal{S},$$

or

(6.12) $$\mathrm{Re}\,(Au,u)_0 - (\gamma-\varepsilon)\|u\|_0^2 - \mathrm{Re}\,(\mathcal{B}\cdot Bu, u)_0 \geqslant -C' \|u\|^2_{-\frac{1}{2}};$$

as $b(x,\xi) = \bar{b}(x,\xi)$, it follows that $\mathcal{B}$ is the L^2 adjoint of B whence

(6.13) $$\mathrm{Re}\,(Au,u)_0 - (\gamma-\varepsilon)\|u\|_0^2 - \|Bu\|_0^2 \geqslant -C' \|u\|^2_{-\frac{1}{2}}$$

and therefore

(6.14) $$\mathrm{Re}\,(Au,u)_0 + C'\|u\|^2_{-\frac{1}{2}} \geqslant (\gamma-\varepsilon)\|u\|_0^2, \qquad \forall u \in \mathcal{S}.$$

By using this result, we arrive at the following main

THEOREM 6. – *Let $a(x,\xi)$ be a symbol, $A(x,D)$ the associated pseudo-differential operator. Let be $K = \max\limits_{\substack{x\in R^n\\ |\xi|=1}} |a(x,\xi)|$. We have that $\forall \varepsilon > 0$ there is a constant C_ε such that the inequality*

(6.15) $$\|A(x,D)u\|_0 \leqslant (K+\varepsilon)\|u\|_0 + C_\varepsilon \|u\|_{-1},$$

for $u \in \mathcal{S}$, is verified.

REMARK. – Let be $K = \max\limits_{\substack{x\in R^n\\ |\xi|=1}} |a(x,\xi)|$ and, $\forall N = 1, 2, \ldots$

$$K_N = \max_{\substack{|x|\leqslant N\\ |\xi|=1}} |a(x,\xi)|.$$

Then obviously we have $K_1 \leqslant K_2 \leqslant \ldots \leqslant K$.

Furthermore we can see that $\lim\limits_{N\to\infty} K_N = K$.

PROOF. – In fact, let be $b=\bar{a}\cdot a=|a|^2$; we see that $b(x,\xi)$ is a symbol too. We put $B(x,D)$ as the associated pseudo-differential operator; then consider $\bar{\mathcal{A}}(x,D)$ associated with $\bar{a}(x,\xi)$; $\bar{\mathcal{A}}(x,D)$ is the L^2-adjoint of $A(x,D)$.

We have $B-\bar{\mathcal{A}}A$ is of order $\leqslant -1$.

In fact, $B-\bar{A}A=T_{-1}$ is of order $\leqslant -1$; hence:

$$B-\bar{\mathcal{A}}A=\bar{A}A-\bar{\mathcal{A}}A+T_{-1}=(\bar{A}-\bar{\mathcal{A}})A+T_{-1}$$

is again of order $\leqslant -1$.

Hence, by 2) of Theorem 5 we deduce

$$\operatorname{Re}\left((B-\bar{\mathcal{A}}A)u\,,u\right)_{L^2}\geqslant -c'\|u\|^2_{H^{-\frac{1}{2}}}\,, \qquad \forall u\in \mathcal{S} \tag{6.16}$$

and therefore:

$$\operatorname{Re}(Bu,u)_{L^2}-\operatorname{Re}(\bar{\mathcal{A}}\cdot Au,u)_{L^2}=\operatorname{Re}(Bu,u)_0-\|Au\|_0^2\geqslant -c'\|u\|^2_{-\frac{1}{2}}\,, \qquad \forall u\in\mathcal{S}. \tag{6.17}$$

Let us consider now the symbol $\alpha(x,\xi)=K^2-\bar{a}(x,\xi)a(x,\xi)$ which satisfies obviously the conditions of Theorem 5. Hence, we obtain, taking $\gamma=0$ in Theorem 5, that $\forall\varepsilon'>0$, $\exists c'(\varepsilon')$ such that, for $u\in\mathcal{S}$

$$\operatorname{Re}\left((K^2-B)u,u\right)_0+c'(\varepsilon')\|u\|^2_{-\frac{1}{2}}\geqslant -\varepsilon'\|u\|_0^2\,, \tag{6.18}$$

is verified.

By adding (6.17) and (6.18) we arrive at the inequality

$$K^2\|u\|_0^2-\|Au\|_0^2+c'(\varepsilon')\|u\|^2_{-\frac{1}{2}}\geqslant -c'\|u\|^2_{-\frac{1}{2}}-\varepsilon'\|u\|_0^2 \tag{6.19}$$

$$\|Au\|_0^2-(K^2+\varepsilon')\|u\|_0^2\leqslant C_1(\varepsilon')\|u\|^2_{-\frac{1}{2}} \tag{6.20}$$

that is

$$\|Au\|_0^2\leqslant (K^2+\varepsilon')\|u\|_0^2+C_1(\varepsilon')\|u\|^2_{-\frac{1}{2}}\,, \qquad \forall u\in\mathcal{S},\ \forall\varepsilon'>0 \tag{6.21}$$

and we may assume $C_1(\varepsilon')>0$; using now $\sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b}$, $a,b>0$ we have

$$\|Au\|_0\leqslant\left(K+\sqrt{\varepsilon'}\right)\|u\|_0+C_2(\varepsilon')\|u\|_{-\frac{1}{2}}\,. \tag{6.22}$$

On the other hand, $\forall\varepsilon''>0$, $\exists\gamma(\varepsilon'')$, such that $\|u\|_{-\frac{1}{2}}\leqslant\varepsilon''\|u\|_0+\gamma(\varepsilon'')\|u\|_{-1}$ whence we obtain, from (6.22), the estimate

$$\|Au\|_0\leqslant\left(K+\sqrt{\varepsilon'}\right)\|u\|_0+C_2(\varepsilon')\varepsilon''\|u\|_0+\gamma(\varepsilon'')C_2(\varepsilon')\|u\|_{-1}\,. \tag{6.23}$$

Let $\varepsilon>0$ be given; we take ε' such that $\sqrt{\varepsilon'}<\varepsilon/2$; and ε'' such that $C_2(\varepsilon')\varepsilon''<\varepsilon/2$; this is trivially done. We have, with a constant $\Gamma(\varepsilon',\varepsilon'')=\gamma'(\varepsilon)$

$$\|Au\|_0\leqslant(K+\varepsilon)\|u\|_0+\gamma'(\varepsilon)\|u\|_{-1}\,, \qquad \forall\varepsilon>0,\ \forall u\in\mathcal{S}. \tag{6.24}$$

COROLLARY. – *If we have*

$$K = \max_{\substack{x \in R^n \\ |\xi|=1}} |a(x, \xi)| ,$$

then for every real s and $\forall \varepsilon > 0$ *there is a constant* $C_{\varepsilon,s}$ *such that*

$$\|Au\|_s \leqslant (K + \varepsilon)\|u\|_s + C_{\varepsilon,s}\|u\|_{s-1} , \tag{6.25}$$

is verified.

In fact, we observe here that, using some previous results, we obtain

$$\|Au\|_s = \|(I + |D|^2)^s Au\|_0 \leqslant \|A(I + |D|^2)^s u\|_0 + \tag{6.26}$$
$$+ \|[A, (I + |D|^2)^s]u\|_0 \leqslant (K + \varepsilon)\|u\|_s + C_\varepsilon\|(I + |D|^2)^s u\|_{-1} +$$
$$+ C\|u\|_{s-1} = (K + \varepsilon)\|u\|_s + C^1_\varepsilon\|u\|_{s-1} .$$

We will prove now, as a consequence of the foregoing result, the following

THEOREM 7. – *Let* $a(x, \xi)$ *be a symbol;* $K = \max_{\substack{x \in R^n \\ |\xi|=1}} |a(x, \xi)|$ *and* $A(x, D)$ *the associated pseudo-differential operator. Then we have*

$$\inf_{T \in \mathcal{C}_{-1}} \|A(x, D) + T\| \leqslant K \tag{6.27}$$

where $\mathcal{C}_{-1}$ *is the class of operators of order* $\leqslant -1$, *and the norm is the one of* $\mathcal{L}(L^2(R^n); L^2(R^n))$.

In fact, we must prove that $\forall \varepsilon > 0$ there is an operator T_ε of order $\leqslant -1$ such that

$$\|(A + T_\varepsilon)u\|_0 \leqslant (K + \varepsilon)\|u\|_0 , \qquad \forall u \in L^2(R^n). \tag{6.28}$$

We build such an operator T_ε by considering a function in $C^\infty(R^n)$, $\varphi_R(\xi)$ dependent on parameter $R > 0$, such that $0 \leqslant \varphi_R(\xi) \leqslant 1$, $\varphi_R(\xi) = 1$ for $|\xi| \leqslant R$, $\varphi_R(\xi) = 0$ for $|\xi| \geqslant 2R$.

The operator $T_R = -A\varphi_R(D)$ is of order $\leqslant -1$; in fact, we have for every $u \in \mathcal{S}$, the estimates

$$\|T_R u\|_s = \|A\varphi_R(D)u\|_s \leqslant C_s\|\varphi_R(D)u\|_s = \tag{6.29}$$
$$= C_s\Big(\int(1 + |\xi|^2)^s \varphi_R^2(\xi)|\tilde{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \leqslant C_s\Big(\int_{|\xi| \leqslant 2R} (1 + |\xi|^2)^s |\tilde{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} =$$
$$= C_s\Big(\int_{|\xi| \leqslant 2R} (1 + |\xi|^2)^{s-1} |\tilde{u}(\xi)|^2 (1 + |\xi|^2)\, d\xi\Big)^{\frac{1}{2}} \leqslant (1 + 4R^2)C_s\|u\|_{s-1} = C_{s,R}\|u\|_{s-1} .$$

By applying here Theorem 6, we have, $\forall\varepsilon>0$ and $u\in\mathcal{S}$,

$$(6.30)\qquad \|(A-A\varphi_R(D))u\|_0=\|A(I-\varphi_R(D))u\|_0\leqslant \leqslant(K+\varepsilon)\|(I-\varphi_R(D))u\|_0+C_\varepsilon\|(I-\varphi_R(D))u\|_{-1}.$$

Remark that we have

$$(6.31)\qquad \|(I-\varphi_R(D))u\|_0=\Big(\int(1-\varphi_R(\xi))^2|\tilde{u}(\xi)|^2d\xi\Big)^{\frac12}\leqslant\|u\|_0,\qquad u\in\mathcal{S}$$

and also that

$$(6.32)\qquad \|(I-\varphi_R(D))u\|_{-1}=\Big(\int(1-\varphi_R(\xi))^2|\tilde{u}(\xi)|^2(1+|\xi|^2)^{-1}d\xi\Big)^{\frac12}\leqslant \leqslant\Big(\int_{|\xi|\geqslant R}(1+|\xi|^2)^{-1}|\tilde{u}(\xi)|^2d\xi\Big)^{\frac12}\leqslant\Big(\int(1+R^2)^{-\frac12}|\tilde{u}(\xi)|^2d\xi\Big)^{\frac12}$$

whence we get

$$(6.33)\qquad \|(A+T_R)u\|_0\leqslant(K+\varepsilon)\|u\|_0+C_\varepsilon(1+R^2)^{-\frac12}\|u\|_0.$$

We choose R_ε such that $C_\varepsilon/\sqrt{1+R_\varepsilon^2}<\varepsilon$; hence we get finally

$$(6.34)\qquad \|(A+T_R)u\|_0\leqslant(K+2\varepsilon)\|u\|_0,\qquad \forall u\in L^2$$

and this proves Theorem 7.

7. – Some results on compactness.

In this paragraph we will prove the following

THEOREM 8. – *Let $a(x,\xi)$ be a symbol, $A(x,D)$ and $\mathcal{A}(x,D)$ the associated pseudo-differential operators. Then $A-\mathcal{A}$ is compact linear operator, $L^2(R^n)\to L^2(R^n)$.*

Let $a(x,\xi)$, $b(x,\xi)$ and $c(x,\xi)=a(x,\xi)b(x,\xi)$ be three symbols, and $A(x,D)$, $B(x,D)$, $C(x,D)$ the associated pseudo-differential operators. Then $A(x,D)B(x,D)-C(x,D)$ is a compact operator, $L^2(R^n)\to L^2(R^n)$.

REMARK 1. – Let $a(x,\xi)$, $b(x,\xi)$ be two symbols such that $a(x,\xi)\,b(x,\xi)=0$ $\forall x\in R^n$, $\xi\in R^n-\{0\}$. Then the operator $A(x,D)B(x,D)$ is compact in L^2.

In fact $AB-C$ is compact, where C is associated to $a(x,\xi)\,b(x,\xi)\equiv 0$.

So, C is the null operator, and the result follows.

REMARK 2. – Let $\varphi(x)$, $\psi(x)$ be C^∞ functions with disjoint supports, and $a(x,\xi)$ be a symbol. Then the operator

$$\varphi(x)\,A(x,D)\,\psi(x)$$

is compact, $L^2\to L^2$.

We have in fact $\varphi(x)\psi(x)\equiv 0$. Furthermore

$$\varphi A\psi = (A\varphi)\psi + [\varphi, A]\psi = A(\varphi\psi) + [\varphi, A]\psi = [\varphi, A]\psi\,.$$

But $[\varphi, A]$ is compact, as follows also from Th. 8, because $\varphi(x)$ is a symbol.

PROOF OF THEOREM 8. – In the present case, we use the following

CRITERION OF COMPACTNESS. – *Let $S\subset L^2(R^n)$ be a set, such that*

a) $\|u\|_{H^1} = \left(\int (1+|\xi|^2)|\tilde{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leqslant C$ *for* $u\in S$ *and*

b) $\lim\limits_{|\tau|\to 0} \int\limits_{|\xi|\leqslant R} |\tilde{u}(\xi+\tau)-\tilde{u}(\xi)|^2 d\xi = 0$ *uniformly for* $u\in S$, *for any fixed* $R>0$.

The S is precompact in L^2, and therefore a subsequence of every sequence in S is convergent in L^2.

As a set S is precompact in L^2 if, and only if, the set $\tilde{S}$, of FOURIER's transforms is precompact in L^2, it will be sufficient to prove that:

Every set $\mathfrak{S}$ which is bounded in $L^2_{(1+|\xi|^2)}$ and is L^2-equicontinuous on every sphere $\{\xi; |\xi|\leqslant R\}$ is relatively compact in $L^2(R^n_\xi)$.

This last result is a consequence of the well-known criterion of M. RIESZ. *A set K in $L^2(R^n_\xi)$ is relatively compact if, and only if*

a) $\int |v(\xi)|^2 d\xi \leqslant C, \qquad \forall v\in K$

b) $\lim\limits_{|\tau|\to 0} \int |v(\xi+\tau)-v(\xi)|^2 d\xi = 0$ *uniformly for* $v\in K$

c) $\lim\limits_{R\uparrow\infty} \int\limits_{|\xi|\geqslant R} |v(\xi)|^2 d\xi = 0$ *uniformly for* $v\in K$.

Let us consider now the set $\mathfrak{S}$ which is bounded in $L^2_{(1+|\xi|^2)}$, hence it is bounded in L^2, and *a)* is verified.

Besides, as $\int (1+|\xi|^2)|v(\xi)|^2 d\xi \leqslant C$, it follows that $\int\limits_{|\xi|\geqslant R} (1+|\xi|^2)|v(\xi)|^2 d\xi\leqslant C$ and consequently $(1+R^2)\int\limits_{|\xi|\geqslant R}|v(\xi)|^2 d\xi \leqslant C$ and therefore

$$\int\limits_{|\xi|\geqslant R} |v(\xi)|^2 d\xi \leqslant C(1+R^2)^{-1}\,, \qquad \forall R>0,\ \forall v\in\mathfrak{S}; \tag{7.1}$$

hence, *c)* is verified.

We observe here the following inequality, valid for $\tau\in R^n$, $|\tau|\leqslant 1$

$$\int\limits_{|\xi|\geqslant R+1} |v(\xi+\tau)-v(\xi)|^2 d\xi \leqslant 2\int\limits_{|\xi+\tau|\geqslant R} |v(\xi+\tau)|^2 d\xi + 2\int\limits_{|\xi|>R} |v(\xi)|^2 d\xi \qquad \text{for } R>0,\ v\in\mathfrak{S}. \tag{7.2}$$

In fact; for $|\xi|\geqslant R+1$, $|\tau|\leqslant 1$ we get $|\xi+\tau|\geqslant|\xi|-|\tau|\geqslant R+1-1=R$ and besides we have $|a-b|^2\leqslant 2|a|^2+2|b|^2$, whence we derive first of all

$$\int\limits_{|\xi|\geqslant R+1}|v(\xi+\tau)-v(\xi)|^2d\xi\leqslant 2\int\limits_{|\xi|\geqslant R+1}|v(\xi+\tau|^2d\xi+2\int\limits_{|\xi|\geqslant R+1}|v(\xi)|^2d\xi\,. \tag{7.3}$$

As the set $\{\xi;|\xi|\geqslant R+1\}$ is included in $\{\xi;|\xi+\tau|\geqslant R\}$ when $|\tau|\leqslant 1$ we deduce that

$$\int\limits_{|\xi|\geqslant R+1}|v(\xi+\tau)|^2d\xi\leqslant\int\limits_{|\xi+\tau|\geqslant R}|v(\xi+\tau)|^2d\xi \tag{7.4}$$

and hence we get

$$\int\limits_{|\xi|\geqslant R+1}|v(\xi+\tau)-v(\xi)|^2d\xi\leqslant \\ \leqslant 2\int\limits_{|\xi|\geqslant R}|v(\xi)|^2d\xi+2\int\limits_{|\xi|\geqslant R+1}|v(\xi)|^2d\xi\leqslant 4\int\limits_{|\xi|\geqslant R}|v(\xi)|^2d\xi\leqslant 4C(1+R^2)^{-1}\,,\qquad \forall v\in\mathfrak{S}. \tag{7.5}$$

We have, then, for every $R>0$ and $|\tau|\leqslant 1$, the estimate

$$\int|v(\xi+\tau)-v(\xi)|^2d\xi\leqslant\int\limits_{|\xi|\leqslant R+1}|v(\xi+\tau)-v(\xi)|^2d\xi+4C(1+R^2)^{-1}\,,\ \ \forall v\in\mathfrak{S},\ \forall R>0. \tag{7.6}$$

Taken $\varepsilon>0$ let us take R_ε such that $4C(1+R_\varepsilon^2)^{-1}<\varepsilon/2$, and then $|\tau|<\delta_{R_\varepsilon,\varepsilon}$ such that

$$\int\limits_{|\xi|\leqslant R+1}|v(\xi+\tau)-v(\xi)|^2d\xi<\frac{\varepsilon}{2}\,,\qquad \forall v\in\mathfrak{S}$$

(according to the hypothesis). Hence, we have, for $|\tau|\leqslant\delta'_\varepsilon$

$$\int|v(\xi+\tau)-v(\xi)|^2d\xi\leqslant\varepsilon\,,\qquad \forall v\in\mathfrak{S}. \tag{7.7}$$

As *a*), *b*) and *c*) have been so verified, the set $\mathfrak{S}$ is precompact for the criterion of M. RIESZ.

We will now prove the

THEOREM 8*a*. – *If* $a(x,\xi)$ *is a symbol, the operator* $A-\mathcal{A}$ *is compact in* L^2.

We define $T=A-\mathcal{A}$; let Ω be a set which is bounded in $L^2(R^n)$. We will show that the set $T(\Omega)$ is relatively compact in $L^2(R^n)$; or that $\widetilde{T(\Omega)}=\{\widetilde{Tu},\,u\in\Omega\}$ is relatively compact in $L^2(R^n_\xi)$.

By a preceding result (Proposition 4) we have

$$\|(A-\mathcal{A})u\|_{H^1}\leqslant C\|u\|_0\,; \tag{7.8}$$

hence, for $u \in \Omega$, the set $\{Tu\}_{u\in\Omega}$ is bounded in H^1. Therefore the set $\widetilde{T(\Omega)}$ is bounded in $L^2_{(1+|\xi|^2)}$.

Besides, we have to prove that for every $R > 0$, it is

$$\lim_{|\tau|\to 0} \int_{|\xi|\leqslant R} |\widetilde{Tu}(\xi+\tau) - \widetilde{Tu}(\xi)|^2 d\xi = 0 \tag{7.9}$$

uniformly for $u \in \Omega$.

The first (preliminary) result is given here in

LEMMA 1. – We have, in the case of a symbol $a(x, \xi)$ such that $a(\infty, \xi) \equiv 0$

$$\lim_{|\tau|\to 0} \int_{|\xi|\leqslant R} |\widetilde{Au}(\xi+\tau) - \widetilde{Au}(\xi)|^2 d\xi = 0 \tag{7.10}$$

uniformly for $u \in \Omega \cap \mathcal{S}$.

Let us remember the formula which we have proved before (Proposition 1).

$$\widetilde{Au}(\xi) = (2\pi)^{-n/2}\int \exp(-ix\cdot\xi)\, a(x, \xi)\, u(x)\, dx\,, \qquad \forall u \in \mathcal{S} \tag{7.11}$$

(the Fourier transform in the sense of $\mathcal{S}'$, belongs to $L^2(R^n)$) and therefore we obtain

$$\widetilde{Au}(\xi+\tau) = (2\pi)^{-n/2}\int \exp\big(-ix\cdot(\xi+\tau)\big)\, a(x, \xi+\tau)\, u(x)\, dx \tag{7.12}$$

and consequently

$$\begin{aligned}\widetilde{Au}(\xi+\tau) - \widetilde{Au}(\xi) &= (2\pi)^{-n/2}\int \exp\big(-ix\cdot(\xi+\tau)\big)\, a(x, \xi+\tau)\, u(x)\, dx - \\ &- (2\pi)^{-n/2}\int \exp(-ix\cdot\xi)\, a(x, \xi)\, u(x)\, dx = (2\pi)^{-n/2}\int \big[\exp\big(-ix\cdot(\xi+\tau)\big) - \exp(-ix\cdot\xi)\big]\cdot \\ &\cdot a(x, \xi+\tau)\, u(x)\, dx + (2\pi)^{-n/2}\int \exp(-ix\cdot\xi)[a(x, \xi+\tau) - a(x, \xi)]\, u(x)\, dx = \\ &= I_1(\xi, \tau) + I_2(\xi, \tau)\,.\end{aligned} \tag{7.13}$$

Hence, we have the estimate

$$\begin{aligned}|I_1(\xi, \tau)| &\leqslant c\int |\exp(-ix\cdot\tau) - 1|\, |a(x, \xi+\tau)|\, |u(x)|\, dx \leqslant \\ &\leqslant c\Big(\int |u(x)|^2 dx\Big)^{\frac12}\Big(\int |\exp(-ix\cdot\tau) - 1|^2 |a(x, \xi+\tau)|^2 dx\Big)^{\frac12}.\end{aligned} \tag{7.14}$$

On the other hand, we have:

$$|\exp(-ix\cdot\tau) - 1|^2 = |\cos x\cdot\tau - 1 - i\sin x\cdot\tau| = 2 - 2\cos x\cdot\tau = 4\sin^2\frac{x\cdot\tau}{2}$$

as: $|\sin\alpha| \leqslant |\alpha|$ we deduce that $|\exp(-ix\cdot\tau) - 1|^2 \leqslant |x|^2|\tau|^2$ whence we obtain

$$|I_1(\xi, \tau)| \leqslant C|\tau|\, \|u\|_0 \Big(\int |x|^2 |a(x, \xi+\tau)|^2 dx\Big)^{\frac12} = C_1|\tau|\, \|u\|_0 \tag{7.15}$$

as obviously $|x|\,|a(x,\xi)|\in L^2$ uniformly with respect to $\xi\in R^n-\{0\}$ (we remember that we took $a(\infty,\xi)\equiv 0$, so $a'(x,\xi)=a(x,\xi)$).

And on the other hand we have the estimate concerning $I_2(\xi,\tau)$

$$|I_2(\xi,\tau)|\leqslant C\|u\|_0\Big(\int|a(x,\xi+\tau)-a(x,\xi)|^2dx\Big)^{\frac12}. \tag{7.16}$$

Let us remember here that in the case $a(\infty,\xi)\equiv 0$ it follows

$$(1+|x|^2)^p|a(x,\xi)|\leqslant C_p\,,\qquad x\in R^n,\ \xi\in R^n-\{0\},\ p=1,2,\ldots \tag{7.17}$$

$$(1+|x|^2)^p|a(x,\xi+\tau)-a(x,\xi)|\leqslant C_p\frac{|\tau|}{|\xi|+|\xi+\tau|},\qquad x\in R^n,\ \xi,\tau\in R^n-\{0\},\ p=1,2,\ldots \tag{7.18}$$

and therefore we have, for every fixed $R>0$

$$\begin{aligned}\int\limits_{|\xi|\leqslant R}|Au(\xi+\tau)-Au(\xi)|^2d\xi&=\int\limits_{|\xi|\leqslant R}|I_1(\xi,\tau)+I_2(\xi,\tau)|^2d\xi\leqslant\\ &\leqslant 2\int\limits_{|\xi|\leqslant R}|I_1(\xi,\tau)|^2d\xi+2\int\limits_{|\xi|\leqslant R}|I_2(\xi,\tau)|^2d\xi\leqslant\\ &\leqslant C\omega_{R,n}|\tau|^2\|u\|_0^2+2\int\limits_{|\xi|\leqslant\varrho}|I_2(\xi,\tau)|^2d\xi+2\int\limits_{\varrho\leqslant|\xi|\leqslant R}|I_2(\xi,\tau)|^2d\xi\,,\qquad\forall\varrho>0,\ \varrho<R.\end{aligned} \tag{7.19}$$

For $|\xi|\leqslant\varrho$, we estimate $I_2(\xi,\tau)$ in the following way (using (7.16) and (7.17)):

$$\begin{aligned}|I_2(\xi,\tau)|&\leqslant 2^{\frac12}C\|u\|_0\Big(\int(|a(x,\xi+\tau)|^2+|a(x,\xi)|^2)\,dx\Big)^{\frac12}\leqslant\\ &\leqslant C_1\|u\|_0 2^{\frac12}C_p\Big(\int(1+|x|^2)^{-2p}dx\Big)^{\frac12}=C_{1,p}\|u\|_0\,,\end{aligned} \tag{7.20}$$

where p is sufficiently large.

For $|\xi|\geqslant\varrho$ we use the estimate (deriving from (7.18))

$$|I_2(\xi,\tau)|\leqslant C_p\Big(\int(1+|x|^2)^{-2p}\,dx\Big)^{\frac12}\|u\|_0|\tau|(|\xi|)^{-1},\qquad\forall\xi\in R^n-\{0\},\ |\tau|\leqslant 1 \tag{7.21}$$

and hence we obtain, using (7.19), (7.20), (7.21), the inequality

$$\begin{aligned}\int\limits_{|\xi|\leqslant R}|\widetilde{Au}(\xi+\tau)-\widetilde{Au}(\xi)|^2d\xi&\leqslant C_1|\tau|^2\|u\|_0^2+C\|u\|_0^2\int\limits_{|\xi|\leqslant\varrho}d\xi+\\ &+C_1\|u\|_0^2|\tau|^2\Big(\int\limits_{\varrho\leqslant|\xi|\leqslant R}\frac{d\xi}{|\xi|^2}\Big)\leqslant C_R|\tau|^2\|u\|_0^2\Big(1+\frac{1}{\varrho^2}\Big)+C\|u\|_0^2\int\limits_{|\xi|\leqslant\varrho}d\xi\,.\end{aligned} \tag{7.22}$$

If $u\in\Omega\cap S$ we have $\|u\|_0\leqslant H$. We take $\varepsilon>0$, and choose at first $\varrho_0(\varepsilon)$ such that

$$CH^2\int\limits_{|\xi|\leqslant\varrho_0}d\xi\leqslant\frac{\varepsilon}{2}\,. \tag{7.23}$$

Once $\varrho_0(\varepsilon)$ fixed, we take $\tau_0(\varepsilon)$ such that

$$C_R|\tau_0|^2H^2\left(1+\frac{1}{\varrho_0^2(\varepsilon)}\right)<\frac{\varepsilon}{2}. \tag{7.24}$$

We arrive hence for $|\tau|\leqslant|\tau_0|$ and $\forall u\in\Omega\cap S$ at the estimate

$$\int\limits_{|\xi|\leqslant R}|\widetilde{Au}(\xi+\tau)-\widetilde{Au}(\xi)|^2d\xi\leqslant\varepsilon .$$

Lemma 1 is proved.

Hence, we can observe that:

$$\begin{aligned}\widetilde{Tu}(\xi)&=\widetilde{Au}(\xi)-\widetilde{\mathcal{A}u}(\xi)=a(\infty,\xi)\tilde{u}(\xi)+\\&+\widetilde{A'u}(\xi)-a(\infty,\xi)\tilde{u}(\xi)-\widetilde{\mathcal{A}'u}(\xi)=(\widetilde{A'-\mathcal{A}')u}(\xi), \qquad \xi\in R^n-\{0\}\end{aligned} \tag{7.25}$$

and similarly for $\widetilde{Tu}(\xi+\tau)$, and it will be henceforth sufficient to prove

LEMMA 2. – *We have, in the case of a symbol $a(x,\xi)$ with $a(\infty,\xi)\equiv 0$*

$$\lim_{|\tau|\to 0}\int\limits_{|\xi|\leqslant R}|\widetilde{\mathcal{A}u}(\xi+\tau)-\widetilde{\mathcal{A}u}(\xi)|^2d\xi=0 \tag{7.26}$$

uniformly for $u\in\Omega\cap S$, $\forall$ fixed $R>0$.

In fact, we have

$$\widetilde{\mathcal{A}u}(\xi+\tau)-\widetilde{\mathcal{A}u}(\xi)=(2\pi)^{-n/2}\int\big(\tilde{a}(\xi+\tau-\eta,\eta)-\tilde{a}(\xi-\eta,\eta)\big)\tilde{u}(\eta)\,d\eta \tag{7.27}$$

and

$$\begin{aligned}|\widetilde{\mathcal{A}u}(\xi+\tau)-\widetilde{\mathcal{A}u}(\xi)|^2&\leqslant C\Big(\int|\tilde{u}(\eta)|^2d\eta\Big)\Big(\int|\tilde{a}(\xi+\tau-\eta,\eta)-\tilde{a}(\xi-\eta,\eta)|^2d\eta\Big)=\\&=C\|u\|_0^2\int|\tilde{a}(\xi+\tau-\eta,\eta)-\tilde{a}(\xi-\eta,\eta)|^2d\eta .\end{aligned} \tag{7.28}$$

We apply TAYLOR's formula; we obtain, if $\tilde{a}=\tilde{a}(\lambda,\eta)$, the relation

$$\tilde{a}(\xi-\eta+\tau,\eta)-\tilde{a}(\xi-\eta,\eta)=\big(\tau,\operatorname{grad}_\lambda\tilde{a}(\xi-\eta+\theta\tau,\eta)\big), \qquad 0<\theta<1 \tag{7.29}$$

and therefore the estimate

$$|\tilde{a}(\xi-\eta+\tau,\eta)-\tilde{a}(\xi-\eta,\eta)|\leqslant|\tau|\,|\operatorname{grad}_\lambda\tilde{a}(\xi-\eta+\theta\tau,\eta)| . \tag{7.30}$$

Let us remember now that $\tilde{a}(\lambda,\eta)\in S(R^n_\lambda)$ uniformly for $\eta\in R^n-\{0\}$ and we get therefore

$$\left|(1+|\lambda|^2)^p\frac{\partial}{\partial\lambda_i}\tilde{a}(\lambda,\eta)\right|\leqslant C_p, \qquad \forall\lambda\in R^n,$$

which gives

$$|\operatorname{grad}_\lambda \tilde{a}(\xi-\eta+\theta\tau,\eta)| \leqslant C_p(1+|\xi-\eta+\theta\tau|^2)^{-p}, \qquad \forall p = 1, 2, \ldots \tag{7.31}$$

and by integrating with respect to η we arrive at the result (in estimate (7.28)).

Now, to finish the proof of Theorem 8a, we have to prove also (*)

LEMMA 3. – *We have in the case* $a(\infty,\xi)\equiv 0$, *that*, $\forall R>0$

$$\lim_{|\tau|\to 0} \int_{|\xi|\leqslant R} |\widetilde{Au}(\xi+\tau)-\widetilde{Au}(\xi)|^2 d\xi = 0 \tag{7.32}$$

$$\lim_{|\tau|\to 0} \int_{|\xi|\leqslant R} |\widetilde{\mathcal{A}u}(\xi+\tau)-\widetilde{\mathcal{A}u}(\xi)|^2 d\xi = 0 \tag{7.33}$$

uniformly for $u\in\Omega$*-bounded set in* $L^2(R^n)$.

We have already shown this relation for $u\in\Omega\cap\mathcal{S}$. Let us remember that the spece $\mathcal{S}$ is dense in L^2. Given $\varepsilon>0$, and Ω a bounded set in $L^2(R^n)$, there is $\forall u\in\Omega$ (1), an element $u_\varepsilon\in\mathcal{S}$, such that $\|u-u_\varepsilon\|_0<\varepsilon$. Hence, for $u\in\Omega$ we have $\|u\|_0\leqslant L$, and

$$\|u_\varepsilon\|_0 \leqslant \|u-u_\varepsilon\|_0 + \|u\|_0 \leqslant \varepsilon + L \leqslant L+1\,. \tag{7.34}$$

and therefore the set

$$\{u_\varepsilon;\, u\in\Omega\} \tag{7.35}$$

is a set Ω_1 bounded in L^2 and included in $\mathcal{S}$.

Here we have, for $|\tau|\leqslant|\tau_0(\varepsilon)|$ that in the case $a(\infty,\xi)\equiv 0$

$$\int_{|\xi|\leqslant R} |\widetilde{Au_\varepsilon}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi)|^2 d\xi \leqslant \varepsilon\,, \qquad \forall u_\varepsilon\in\Omega_1\,, \tag{7.36}$$

$$\int_{|\xi|\leqslant R} |\widetilde{\mathcal{A}u_\varepsilon}(\xi+\tau)-\widetilde{\mathcal{A}u_\varepsilon}(\xi)|^2 d\xi \leqslant \varepsilon\,, \qquad \forall u_\varepsilon\in\Omega_1\,. \tag{7.37}$$

Hence, we deduce the inequalities

$$\int_{|\xi|\leqslant R} |\widetilde{Au}(\xi+\tau)-\widetilde{Au}(\xi)|^2 d\xi \leqslant 3\int_{|\xi|\leqslant R} |\widetilde{Au}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi+\tau)|^2 d\xi\, + \tag{7.38}$$

$$+\,3\int_{|\xi|\leqslant R} |\widetilde{Au_\varepsilon}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi)|^2 d\xi + 3\int_{|\xi|\leqslant R} |\widetilde{Au_\varepsilon}(\xi)-\widetilde{Au}(\xi)|^2 d\xi \leqslant$$

(*) Remember that for $u\in L^2$ but $u\notin\mathcal{S}$, the definition of Au and $\mathcal{A}u$ is by continuity from the definition on $u\in\mathcal{S}$.

(1) At least, obviously; we choose a fixed one, for any u.

$$\leqslant 3\int_{R^n}|\widetilde{A(u-u_\varepsilon)}(\xi+\tau)|^2d\xi+3\int_{|\xi|\leqslant R}|\widetilde{Au_\varepsilon}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi)|^2d\xi+3\int_{R^n}|\widetilde{A(u-u_\varepsilon)}(\xi)|^2d\xi=$$

$$=6\|A(u-u_\varepsilon)\|_0^2+3\int_{|\xi|\leqslant R}|\widetilde{Au_\varepsilon}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi)|^2d\xi\leqslant 6c\|u-u_\varepsilon\|_0^2+3\int_{|\xi|\leqslant R}|\widetilde{Au_\varepsilon}(\xi+\tau)-\widetilde{Au_\varepsilon}(\xi)|^2d\xi.$$

For $|\tau|<|\tau_0(\varepsilon)|$ the second integral is $<\varepsilon$ and also $6c\|u-u_\varepsilon\|_0^2\leqslant 6c\varepsilon^2$; the result is so proven.

The proof for $\mathcal{A}(x, D)$ is similar. Theorem 8*a* is herewith proven (see Appendix to [3]).

Our Theorem 8 will be completely proved when we will have proven

THEOREM 8*b*. – *If $a(x,\xi)$, $b(x,\xi)$ are symbols, and their product is $c(x,\xi)$, then $A(x,D)B(x,D)-C(x,D)$ is compact operator, $L^2\to L^2$.*

The operator $T=A\cdot B-C$ is of order $\leqslant -1$ [1]; hence, if $u\in\Omega$ where Ω is a bounded set in L^2, then $\widetilde{T(\Omega)}$ is bounded in $L^2_{(1+|\xi|^2)}$, as easily seen. Therefore, we have to prove that, $\forall R>0$

$$\lim_{|\tau|\to 0}\int_{|\xi|\leqslant R}|\widetilde{Tu}(\xi+\tau)-\widetilde{Tu}(\xi)|^2d\xi=0 \tag{7.39}$$

uniformly for $u\in\Omega$.

First of all, let us consider the case $a(\infty,\xi)\equiv b(\infty,\xi)\equiv c(\infty,\xi)\equiv 0$. If we use Theorem 8*a* we get, $\forall R>0$

$$\lim_{|\tau|\to 0}\int_{|\xi|\leqslant R}|\widetilde{Cu}(\xi+\tau)-\widetilde{Cu}(\xi)|^2d\xi=0 \tag{7.40}$$

uniformly for $u\in\Omega$. It is only left to consider

$$\int_{|\xi|\leqslant R}|\widetilde{ABu}(\xi+\tau)-\widetilde{ABu}(\xi)|^2d\xi\,. \tag{7.41}$$

Let us remember Lemma 3. Then, $\forall\varepsilon>0$, $\exists\delta_L(\varepsilon)$, such that

$$\int_{|\xi|\leqslant R}|\widetilde{Av}(\xi+\tau)-\widetilde{Av}(\xi)|^2d\xi\leqslant\varepsilon\,,\qquad \text{if } |\tau|<\delta_L(\varepsilon) \text{ and } \|v\|_0\leqslant L. \tag{7.42}$$

Remark that if u is arbitrary in L^2, $u/\|u\|_0$ is of norm 1, therefore

$$\int_{|\xi|\leqslant R}\left|\widetilde{A\frac{u}{\|u\|_0}}(\xi+\tau)-\widetilde{A\frac{u}{\|u\|_0}}(\xi)\right|^2d\xi<\varepsilon \qquad \text{if } |\tau|<\delta_1(\varepsilon), \tag{7.43}$$

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(1) By Ch. 5.

that is

$$\int_{|\xi|\leqslant R} |\widetilde{Au}(\xi+\tau)-\widetilde{Au}(\xi)|^2 d\xi \leqslant \varepsilon \|u\|_0^2, \qquad \text{if } |\tau|<\delta_1(\varepsilon),\ \forall u \in L^2(R^n). \tag{7.44}$$

We apply this relation to ABu, $u\in L^2$; we have then

$$\int_{|\xi|\leqslant R} |\widetilde{ABu}(\xi+\tau)-\widetilde{ABu}(\xi)|^2 d\xi \leqslant \varepsilon \|Bu\|_0^2, \qquad |\tau|\leqslant\delta_1(\varepsilon),\ u \in L^2(R^n). \tag{7.45}$$

But $\|Bu\|_0\leqslant c\|u\|_0$; the relation is proven then, as easily seen.

In the case $a(\infty,\xi)\not\equiv 0$, $b(\infty,\xi)\not\equiv 0$ there is the additional term $A'(x,D)\cdot$ $\cdot B(\infty,D)-B(\infty,D)A'(x,D)$ which is of order $\leqslant -1$ (see Ch. 5).

Moreover, the symbol of $B(\infty,D)A'(x,D)$ is $b(\infty,\xi)a'(x,\xi)$ which $\to 0$ as $|x|\to\infty$. For the term $A'(x,D)\big(B(\infty,D)\big)$ we use that $\{B(\infty,D)u\}$ is a bounded set in L^2 when u is in a bounded set of L^2.

REMARK. – As a corollary of Th. 8*b*. we get the following: let $a(x,\xi)$ be a symbol associated with $A(x,D)$ and λ_0 belongs to the continuous spectrum of $A(x,D)$; then $|\lambda_0|\leqslant \sup_{x\in R^n,\ |\xi|=1} |a(x,\xi)|$.

In fact, otherwise, $\exists\alpha>0$, such that

$$|a(x,\xi)-\lambda_0|>\alpha>0\,, \qquad \forall x\in R^n,\ |\xi|=1.$$

Applying the (simple) result in [5], we find a positive C and a compact operator T_{λ_0}, $L^2\to L^2$, s.t.

$$\|u\|_{L^2}\leqslant C\big(\|(A-\lambda_0E)u\|_{L^2}+\|T_{\lambda_0}u\|_{L^2}\big)\,, \qquad \forall u\in L^2.$$

On other hand, from $\lambda_0\in\sigma_c(A)$, we deduce a sequence $(u_n)_1^\infty\subset L^2$, of unit norm, such that $\|(A-\lambda_0E)u_n\|_{L^2}\to 0$.

For a subsequence $(u_{n_p})_{p=1}^\infty$ we have also $\|T_{\lambda_0}u_{n_p}\|_{L^2}\to 0$. We obtain $1\leqslant c\cdot\varepsilon_p$, where $\varepsilon_p\to 0$, contradiction.

8. – Other inequalities (*norms of p.d.o. modulo compact operators*).

In this paragraph we will prove the following

THEOREM 9. – *Let $a(x,\xi)$ be a symbol, and $K=\max_{|\xi|=1,\ x\in R^n}|a(x,\xi)|$; let $A(x,D)$ be the associated pseudo-differential operator. Let $\mathcal{C}_c$ be the class of linear compact operators L^2 in L^2.*

Then we have the upper estimates

$$\inf_{T\in\mathcal{C}_c}\|A(x,D)+T\|_{\mathfrak{L}(L^2;L^2)}\leqslant K\,, \qquad \inf_{T\in\mathcal{C}_c}\|\mathcal{A}(x,D)+T\|_{\mathfrak{L}(L^2,L^2)}\leqslant K\,. \tag{8.1}$$

The result is a consequence of some preliminary theorems.

PRELIMINARY THEOREM 9a. – *Let $a(x, \xi)$ be a symbol, $A(x, D)$ the associated pseudo-differential operator. Then, for every $\varepsilon > 0$ there is a semi-norm ${}^{\varepsilon}|\ |$ on L^2, dependent of ε, such that every L^2-bounded sequence contains a subsequence convergent in ${}^{\varepsilon}|\ |$, such that the inequality*

$$\|A(x, D)u\|_0 \leqslant (K+\varepsilon)\|u\|_0 + {}^{\varepsilon}|u|\,, \qquad \forall u \in L^2(R^n) \tag{8.2}$$

is verified (*).

In fact, let us put $b_\varepsilon(x, \xi) = (K^2 - \bar{a}(x, \xi)a(x, \xi) + \varepsilon)^{\frac{1}{2}}$ which is still a (homogeneous) symbol as we can « easily » see, and besides is

$$b_\varepsilon(x, \xi) = \bar{b}_\varepsilon(x, \xi)\,, \qquad \varepsilon > 0,\ x \in R^n,\ \xi \in R^n - \{0\}.$$

Let us consider the operators $B_\varepsilon(x, D)$, $\mathcal{B}_\varepsilon(x, D)$ associated with $b_\varepsilon(x, \xi)$ and $\overline{\mathcal{A}}(x, D)$ associated with $\bar{a}(x, \xi)$. We have then the following

LEMMA 1. – *The linear operator*

$$T_\varepsilon = (K^2 + \varepsilon)I - \overline{\mathcal{A}} \cdot A - \mathcal{B}_\varepsilon \cdot B_\varepsilon$$

is compact, $L^2 \to L^2$.

In fact, we have first of all the relation

$$\mathcal{B}_\varepsilon \cdot B_\varepsilon = (\mathcal{B}_\varepsilon - B_\varepsilon)\cdot B_\varepsilon + B_\varepsilon^2 = T_1 + B_\varepsilon^2 \tag{8.3}$$

where $T_1 = (\mathcal{B}_\varepsilon - B_\varepsilon)\cdot B_\varepsilon$ is compact according to Theorem 8. So we arrive at the relation

$$T_\varepsilon = (K^2 + \varepsilon)I - \overline{\mathcal{A}} \cdot A - B_\varepsilon^2 - T_1\,. \tag{8.4}$$

On the other hand, we have the equality

$$\overline{\mathcal{A}} \cdot \mathcal{A} = (\overline{\mathcal{A}} - \bar{A})\cdot A + \bar{A}\cdot A = T_2 + \bar{A}\cdot A \tag{8.5}$$

where T_2 is compact, $L^2 \to L^2$, again according to Theorem 8; and hence we get

$$T_\varepsilon = (K^2 + \varepsilon) - \bar{A}\cdot A - B_\varepsilon^2 - (T_1 + T_2)\,. \tag{8.6}$$

Finally, we have: $B_\varepsilon \cdot B_\varepsilon - (K^2 + \varepsilon - (\bar{a}\cdot a)(x, D)) = T_3$-compact, $L^2 \to L^2$ and hence we derive

$$B_\varepsilon(x, D)\cdot B_\varepsilon(x, D) = B_\varepsilon^2(x, D) = K^2 + \varepsilon - (\bar{a}\cdot a)(x, D) + T_3 \tag{8.7}$$

(*) We have also $\|\mathcal{A}(x, D)u\|_0 \leqslant (K+\varepsilon)\|u\|_0 + {}^{\varepsilon}|u| + \|(\mathcal{A}(x, D) - A(x, D))u\|_0$.
The map $u \to {}^{\varepsilon}|u| + \|(\mathcal{A} - A)u\|_0$ is a semi-norm on L^2 like ${}^{\varepsilon}|\ |$ because $\mathcal{A}$-A is compact; this will imply second estimate in (8.1).

and therefore

$$T_\varepsilon = K^2 + \varepsilon - \bar{A}\cdot A - (K^2+\varepsilon) + (\bar{a}\cdot a)(x, D) - \\ -(T_1 + T_2 + T_3) = (\bar{a}\cdot a)(x, D) - \bar{A}\cdot A - (T_1+T_2+T_3) = T_0 \tag{8.8}$$

where T_0 is compact linear, $L^2 \to L^2$ (by Theor. 8) (we have made here good use of the notation $a(x, D)$ instead of $A(x, D)$, by an obvious necessity).

Hence, Lemma 1 is proven. Then we have also the following

LEMMA 2. – *Given arbitrary* $\varepsilon > 0$, *we have the relation*

$$\operatorname{Re}(T_\varepsilon u, u)_0 + \varepsilon\|u\|_0^2 \geqslant -\frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2, \qquad \forall u \in L^2 \tag{8.9}$$

In fact we have:

$$|\operatorname{Re}(T_\varepsilon u, u)_0| \leqslant \|T_\varepsilon u\|_0\|u\|_0 = \frac{1}{2\sqrt{\varepsilon}}\|T_\varepsilon u\|_0 \cdot 2\sqrt{\varepsilon}\|u\|_0 \leqslant \varepsilon\|u\|_0^2 + \frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2, \tag{8.10}$$

and consequently

$$\operatorname{Re}(T_\varepsilon u, u)_0 \geqslant -\varepsilon\|u\|_0^2 - \frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2 \tag{8.11}$$

follows.

Now we shall give the following

LEMMA 3. – *We have the relation*, $\forall \varepsilon > 0$

$$\|A(x, D)u\|_0^2 \leqslant (K^2 + 2\varepsilon)\|u\|_0^2 + \frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2, \qquad \forall u \in L^2(R^n). \tag{8.12}$$

In fact, this results from Lemma 2. We have:

$$(T_\varepsilon u, u)_0 = (K_0^2 + \varepsilon)\|u\|_0^2 - \|A(x, D)u\|_0^2 - \|B_\varepsilon(x, D)u\|_0^2; \tag{8.13}$$

$(T_\varepsilon u, u)_0$ is hence real-valued. (We have used that $\mathcal{A}^* = A$ and $\mathcal{B}_\varepsilon^* = B_\varepsilon$ being $b = \bar{b}$).

Hence, we deduce thereof, using Lemma 2, the estimate

$$(K^2 + 2\varepsilon)\|u\|_0^2 - \|A(x, D)u\|_0^2 - \|B_\varepsilon(x, D)u\|_0^2 \geqslant -\frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2 \tag{8.14}$$

and therefore

$$\|Au\|_0^2 + \|B_\varepsilon u\|_0^2 \leqslant (K^2 + 2\varepsilon)\|u\|_0^2 + \frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2 \tag{8.15}$$

and hence *a fortiori*

$$\|Au\|_0^2 \leqslant (K^2 + 2\varepsilon)\|u\|_0^2 + \frac{1}{4\varepsilon}\|T_\varepsilon u\|_0^2. \tag{8.16}$$

which proves Lemma 3.

Extracting the square root and for $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$, $a, b > 0$, we have

$$\|Au\|_0 \leqslant (K + \sqrt{2\varepsilon})\|u\|_0 + \frac{1}{2\sqrt{\varepsilon}}\|T_\varepsilon u\|_0 \tag{8.17}$$

Preliminary theorem 9*a* is proved if we put ${}^\varepsilon|u| = c_\varepsilon\|T_\varepsilon u\|_0$ and if we observe that T_ε being compact in L^2 the semi-norm ${}^\varepsilon|u| = c_\varepsilon\|T_\varepsilon u\|_0$ satisfies the required properties.

PRELIMINARY THEOREM 9*b*. – *Let H be hilbertian; on H is defined a seminorm $|\ |$ such that*

1) $|u| \leqslant c\|u\|_H$, $\quad \forall u \in H$,

2) *for every bounded sequence $(u_n)_1^\infty$ there exists a Cauchy subsequence with respect to $|\ |$.*

Then: $\forall \varepsilon > 0$, *there exists H_ε—a closed linear subspace of H, such that $H \theta H_\varepsilon = H_\varepsilon^\perp$ is of finite dimension and $|u| \leqslant \varepsilon\|u\|_H$, $\forall u \in H_\varepsilon$.*

Let us begin by assuming that, given $\varepsilon > 0$, we have for every $u \in H$, such that $|u| = 1$ the estimate $\|u\|_H \geqslant 1/\varepsilon$. In this case, taken an arbitrary $u \in H$, such that $|u| \neq 0$, we have: $|u/|u|| = 1$.

Hence, $\|u/|u|\,\| = (1/|u|)\|u\| \geqslant 1/\varepsilon$; hence, $|u| \leqslant \varepsilon\|u\|$ and if $u \in H$ and $|u| = 0$, we have also $|u| \leqslant \varepsilon\|u\|$. Therefore, in this case, it is found $H_\varepsilon = H$.

Now we have to consider the situation when there is at least an element $u_1 \in H$ such that $|u_1| = 1$, $\|u_1\|_H \leqslant 1/\varepsilon$. According to HAHN-BANACH's theorem, we can build a linear functional on H, f_1, such that $f_1(u_1) = 1$, $|f_1(u)| \leqslant |u|$, $\forall u \in H$. As $|u| \leqslant C\|u\|$, $|f_1(u)| \leqslant C\|u\|$, hence f_1 is a continuous linear functional on H (1).

We define $H_1 = \{u \in H; f_1(u) = 0\}$; H_1 then is a closed subspace of H. In H_1 we reason as in H; in the « worst » case there is at least one element $u_2 \in H_1$, $|u_2| = 1$, $\|u_2\| \leqslant 1/\varepsilon$; and hence we can build a continuous linear functional on H_1, denoted with f_2 (2), such that

$$f_2(u_2) = 1, \qquad |f_2(u)| \leqslant |u|, \qquad \forall u \in H_1 \tag{8.18}$$

and we denote by $H_2 = \{u \in H_1, f_2(u) = 0\}$; H_2 is a closed subspace of H_1.

We observe that $|u_1 - u_2| \geqslant 1$. In fact $u_1 \in H$, $u_2 \in H_1 \subset H$, hence

$$|(u_1 - u_2)| \geqslant |f_1(u_1 - u_2)| = |f_1(u_1) - f_1(u_2)| = 1.$$

Now, in H_2 we reason as in H and H_1; in the « worst » case there is at least an element $u_3 \in H_2$, such that $|u_3| = 1$, and $\|u_3\|_H \leqslant 1/\varepsilon$ and we can build a functional f_3, which is linear continuous on H_2 (3), and such that

$$f_3(u_3) = 1, \qquad |f_3(u)| \leqslant |u|, \qquad \forall u \in H_2. \tag{8.19}$$

(1) And $\exists e_1 \in H$, s.t. $f_1(u) = (u, e_1)$, $\forall u \in H$.

(2) And $\exists e_2 \in H_1$, s.t. $f_2(u) = (u, e_2)$ $\forall u \in H_1$; so $(e_2, e_1) = 0$.

(3) And $\exists e_3 \in H_2$, s.t. $f_3(u) = (u, e_3)$. $\forall u \in H_2$; so $(e_3, e_1) = 0$ and $(e_3, e_2) = 0$, etc.

We denote again

(8.20) $$H_3 = \{u \in H_2; f_3(u) = 0\};$$

then $H_3 \subset H_2$ as a closed subspace.

We observe that:

$$|u_1 - u_3| \geqslant 1, \qquad |u_2 - u_3| \geqslant 1$$

and in fact

$$|u_1 - u_3| \geqslant |f_1(u_1 - u_3)| = |f_1(u_1) - f_1(u_3)| = 1$$

as $f_1(u_1) = 1$ and $f_1(u_3) = 0$ being $u_3 \in H_2 \subset H_1$ and $f_1(u) = 0$ on H_1 and besides:

$$|u_2 - u_3| \geqslant |f_2(u_2 - u_3)| = |f_2(u_2) - f_2(u_3)| = |1 - 0| = 1.$$

We use successively the same reasonings, always considering the « worst » case. We obtain so a sequence of elements $(u_1, u_2, \ldots)$ such that

(8.21) $$|u_j| = 1, \quad \|u_j\|_H < \frac{1}{\varepsilon} \quad \text{and} \quad |u_i - u_j| \geqslant 1 \qquad \text{if } i \neq j.$$

This sequence is necessarily finite, according to the property of « relative compactness ».

In this way we can build a finite number N_ε of closed subspaces $H \supset H_1 \supset H_2 \supset \supset \ldots \supset H_{N_\varepsilon}$, and everyone being of codimension 1 with respect to the preceding, then H_{N_ε} will be of codimension N_ε; hence $H_{N_\varepsilon}^\perp$ is of dimension N_ε.

More precisely: for any f_j there is $e_j \in H_{j-1}$, such that $f_j(u) = (u, e_j)$, $\forall u \in H_{j-1}$, $j = 1, 2, \ldots$. Here $H_0 = H$. Furthermore:

$$H_1 = \{u \in H, (u, e_1) = 0\}; \qquad H_2 = \{u \in H_1, (u, e_2) = 0\} =$$
$$= \{u \in H, (u, e_1) = (u, e_2) = 0\}, \quad \ldots, \quad H_N = \{u \in H, (u, e_1) = (u, e_2) = \ldots (u, e_N) = 0\}.$$

Also we see that $(e_i, e_j) = 0$ for $i \neq j$.

The space H has then the obvious orthogonal decomposition

$$H = H_N \oplus Sp[e_1, e_2, \ldots, e_N].$$

See also our paper [3] where a similar result is proven.

Now, in H_{N_ε} is obviously $|u| \leqslant \varepsilon \|u\|_H$, $\forall u \in H_{N_\varepsilon}$. This proves Preliminary theorem 9*b*.

Finally, Theorem 9 is proven by the preceding results and by

PRELIMINARY THEOREM 9*c*. – *Let H be a hilbertian space, and $A \in \mathfrak{L}(H; H)$. Let us assume that $\forall \varepsilon > 0$, $\exists$ exists a seminorm ${}^\varepsilon|\ |$ on H such that $\|\ \|_H$ is relatively* *compact with respect to* ${}^{\varepsilon}|\ |$ *and such that* ${}^{\varepsilon}|u|\leqslant c\|u\|$, $\forall u\in H$ (*) *and*

$$\|Au\|_H\leqslant (K+\varepsilon)\|u\|+{}^{\varepsilon}|u|\,, \qquad \forall u\in H. \tag{8.22}$$

Then:

$$\inf_{T\in\mathcal{C}_c}\|A+T\|_{\mathfrak{L}(H;H)}\leqslant K \ (*).$$

In fact, it is sufficient to prove that for every $\varepsilon>0$ we find a compact operator T_ε in H, such that

$$\|(A-T_\varepsilon)u\|\leqslant (K+\varepsilon)\|u\|\,, \qquad \forall u\in H. \tag{8.23}$$

Let be $H_\varepsilon\subset H$; for $u\in H_\varepsilon$ we have, ${}^{\varepsilon}|u|\leqslant\varepsilon\|u\|$ and $H_\varepsilon^\perp$ of dimension N_ε-finite.

Let us put P_ε the orthogonal projection on H_ε; hence, $(I-P_\varepsilon)$ projects on a space of finite dimension and is therefore compact: $H\to H$.

Hence, we put $T_\varepsilon=A(I-P_\varepsilon)$; this is obviously compact, and besides we have:

$$\|(A-T_\varepsilon)u\|=\|AP_\varepsilon u\|\,, \qquad \forall u\in H. \tag{8.24}$$

By the hypothesis of the theorem, we arrive at:

$$\|(A-T_\varepsilon)u\|\leqslant (K+\varepsilon)\|P_\varepsilon u\|+{}^{\varepsilon}|P_\varepsilon u|\,, \qquad \forall u\in H. \tag{8.25}$$

Being now $P_\varepsilon u\in H_\varepsilon$, we have:

$${}^{\varepsilon}|P_\varepsilon u|\leqslant\varepsilon\|P_\varepsilon u\|\leqslant\varepsilon\|u\|$$

therefore we get,

$$\|(A-T_\varepsilon)u\|\leqslant (K+2\varepsilon)\|u\|\,, \qquad \forall u\in H. \tag{8.26}$$

Applying Preliminary theorems 9*a* and 9*c*, Theorem 9 is proven.

9. – Some more estimates.

Considering the later applications, we shall prove here the following

THEOREM 10. – *Let* $a(x,\xi)$ *be a symbol defined for* $x\in R^n$, $\xi\in R^n-\{0\}$, Ω *an open set in the « x-space », and* $K_\Omega=\max\limits_{\substack{x\in\Omega\\|\xi|=1}}|a(x,\xi)|$. *Then, for every* $\varepsilon>0$ *there is a constant* C_ε *such that*

$$\|A(x,D)u\|_0\leqslant (K_\Omega+\varepsilon)\|u\|_0+C_\varepsilon\|u\|_{-\frac12}\,, \qquad \forall u\in C_0^\infty(\bar\Omega)\ (^1)\ (^2) \tag{9.1}$$

be verified.

(*) The class $\mathcal{C}_c$ of these semi-norms is obviously a linear space; this applies to the footnote at Preliminary Th. 9.*a*.

(1) We can replace $\|\ \|_{-\frac12}$ by $\|\ \|_{-1}$ using: $\forall\varepsilon>0$, $\exists C_\varepsilon$, such that

$$\|u\|_{-\frac12}\leqslant\varepsilon\|u\|_0+C_\varepsilon\|u\|_{-1}\,.$$

(2) $C_0^\infty(\bar\Omega)$ means the class of C^∞ functions with compact support contained in $\bar\Omega$.

We deduce this theorem from Theorem 6 (see (6.22)) by means of some additional reasonings. We have the following

LEMMA. – *Let* $a(x, \xi)$ *be a symbol,* Ω *an open set of* R^n, $K_\Omega = \max\limits_{\substack{|\xi|=1 \\ x\in\Omega}} |a(x, \xi)|$. *Then,* $\forall \varepsilon > 0$ *there is an open set* $\Omega_\varepsilon \supset \bar{\Omega}$ *such that the relation* $K_{\Omega_\varepsilon} \leqslant K_\Omega + \varepsilon$ *is verified.*

In fact, we have, for every $x_0 \in R^n$, $|a(x, \xi) - a(x_0, \xi)| \leqslant \varepsilon$ if $|x - x_0| < \delta_\varepsilon$ and $\xi \in R^n - \{0\}$; here δ_ε is independent of x_0.

Let us consider here, if $\partial\Omega$ is the boundary of Ω, for every $x_0 \in \partial\Omega$ the sphere $\{x; |x - x_0| \leqslant \delta_\varepsilon\}$.

Let us take

$$\Omega_\varepsilon = \Omega \bigcup \Big(\bigcup_{x_0 \in \partial\Omega} S(x_0, \delta_\varepsilon) \Big); \qquad S(x_0, \delta_\varepsilon) = \{x; |x - x_0| \leqslant \delta_\varepsilon\}. \tag{9.2}$$

Therefore, if $y \in \Omega_\varepsilon$, we have $y \in \Omega$ or $y \in S(x^*, \delta_\varepsilon)$ for a certain $x^* \in \partial\Omega$. In the first case, we have

$$|a(y, \xi)| \leqslant \max_{\substack{|\xi|=1 \\ x\in\bar{\Omega}}} |a(x, \xi)| = K_\Omega. \tag{9.3}$$

In the second case we have

$$|a(y, \xi)| \leqslant |a(y, \xi) - a(x^*, \xi)| + |a(x^*, \xi)| \leqslant \varepsilon + K_\Omega. \tag{9.4}$$

Hence, for every $y \in \Omega_\varepsilon$, $\xi \in R^n - \{0\}$ we have $|a(y, \xi)| \leqslant \varepsilon + K_\Omega$. Hence $K_{\Omega_\varepsilon} \leqslant K_\Omega + \varepsilon$.

PROOF OF THE THEOREM. – Given $\varepsilon > 0$, and $u \in C_0^\infty(\bar{\Omega})$ we build Ω_ε given in the Lemma. There exists also, a function $\zeta_\varepsilon(x) \in C_0^\infty(R^n)$, equal to 1 on $\operatorname{supp} u$, equal to 0 outside Ω_ε, contained between 0 and 1. Obviously $\zeta_\varepsilon(x)$ is a symbol, and $\gamma_\varepsilon(x, \xi) = \zeta_\varepsilon(x) a(x, \xi)$ is another symbol.

Furthermore $\gamma_\varepsilon(x, \xi) = 0$ if $x \in \complement\Omega_\varepsilon$; hence, we have

$$\max_{\substack{x\in R^n \\ |\xi|=1}} |\gamma_\varepsilon(x, \xi)| \leqslant \max_{\substack{x\in\Omega_\varepsilon \\ |\xi|=1}} |a(x, \xi)| = K_\Omega \leqslant K_{\Omega_\varepsilon} + \varepsilon. \tag{9.5}$$

We define $\Gamma_\varepsilon(x, D)$ the pseudo-differential operator associated with $\gamma_\varepsilon(x, \xi)$. We have

$$\Gamma_\varepsilon(x, D) = A(x, D)(\zeta_\varepsilon(x)). \tag{9.6}$$

In fact,

$$\widehat{\Gamma_\varepsilon(x, D) u}(\xi) = (2\pi)^{-n/2} \int \exp(-ix\cdot\xi)\big(a(x, \xi)\zeta_\varepsilon(x)\big) u(x)\, dx = \widehat{A(x, D)(\zeta_\varepsilon u)}(\xi), \qquad \forall u \in \mathcal{S},\ \forall \xi \in R^n - \{0\}. \tag{9.7}$$

Hence we get

$$\Gamma_\varepsilon(x, D) u = A(x, D)\big(\zeta_\varepsilon(x) u(x)\big), \qquad \forall u \in \mathcal{S} \tag{9.8}$$

(however, not necessarily is $\Gamma_\varepsilon(x, D) = \zeta_\varepsilon(x) A(x, D)$!).

Now we have the decomposition

$$u(x)=\zeta_\varepsilon(x)u(x)+\big(1-\zeta_\varepsilon(x)\big)u(x) \tag{9.9}$$

and

$$A(x,D)u=A(x,D)(\zeta_\varepsilon u)+A(x,D)((1-\zeta_\varepsilon)u)= \\ =\Gamma_\varepsilon(x,D)u+A(x,D)((1-\zeta_\varepsilon)u)\,, \tag{9.10}$$

as it is $1-\zeta_\varepsilon(x)=0$ on $\operatorname{supp} u$, then it is $\big(1-\zeta_\varepsilon(x)\big)u(x)=0$ on R^n, and therefore

$$A(x,D)u=\Gamma_\varepsilon(x,D)u\,, \tag{9.11}$$

Hence, applying Theorem 6, we get

$$\|A(x,D)u\|_0=\|\Gamma_\varepsilon(x,D)u\|_0\leqslant\Big(\max_{\substack{x\in R^n\\|\xi|=1}}|\gamma_\varepsilon(x,\xi)|+\varepsilon\Big)\|u\|_0+C_\varepsilon\|u\|_{-\frac12}\leqslant \\ \leqslant(K_\Omega+2\varepsilon)\|u\|_0+C_\varepsilon\|u\|_{-\frac12}\,, \tag{9.12}$$

We will show, complementing Theorem 7, the following

THEOREM 11 [1]. – *Let $a(x,\xi)$ be a symbol, $A(x,D)$ the associated pseudo-differential operator*; $\mathcal{C}_{-1}$ *the class of operators of order* $\leqslant -1$, $K=\max\limits_{\substack{x\in R^n\\|\xi|=1}}|a(x,\xi)|$. *We have*

$$\inf_{T\in\mathcal{C}_{-1}}\|A(x,D)+T\|\geqslant K \tag{9.13}$$

the norm being taken here in $\mathcal{L}\big(L^2(R^n);L^2(R^n)\big)$.

Combining with Theorem 7 we deduce equality

$$\inf_{T\in\mathcal{C}_{-1}}\|A(x,D)+T\|=K\,. \tag{9.14}$$

The following theorem is fundamental for Theorem 11. In fact, Theorem 11 is a simple corollary of it.

THEOREM 12. – *Let $a(x,\xi)$ be a symbol, and $|a(x_0,\xi_0)|=c_0$ for a certain $x_0\in R^n$, $|\xi_0|=1$. Then, $\forall\varepsilon>0$, $\exists u_\varepsilon(x)\in C_0^\infty$, such that $\|u_\varepsilon(x)\|_0\neq 0$ and the estimates*

$$\big|\|A(x,D)u_\varepsilon\|_0-c_0\|u_\varepsilon\|_0\big|<\varepsilon\|u_\varepsilon\|_0 \text{ [2]} \tag{9.15}$$

$$\|u_\varepsilon\|_{-1}\leqslant\varepsilon\|u_\varepsilon\|_0 \tag{9.16}$$

are satisfied.

---

(1) If A is a p.d.o. of order $\leqslant -1$, we get in (9.13) $K=0$ (take $T=-A$). Then $a(x,\xi)\equiv 0$ and A is the null operator.

(2) In fact the stronger estimate $\|[A(x,D)-a(x_0,\xi_0)]u_\varepsilon\|_0\leqslant\varepsilon\|u_\varepsilon\|_0$ holds, as is easily seen from (9.27) and the subsequent estimates (see [3], Th. 9.1).

REMARK. – From foot-note (²) to Th. 12 we see that any value of $a(x, \xi)$ belongs to $\sigma(A(x, D))$. In fact, we find a sequence $u_n(x) \in C_0^\infty$, such that

$$\|(A(x, D) - a(x_0, \xi_0)E)u_n\|_0 \leq \frac{1}{n}\|u_n\|_0$$

which implies that $(A(x, D) - a(x_0, \xi_0)E)$ has no bounded inverse.

COROLLARY TO TH. 12. – Let $a(x, \xi)$ be a symbol such that estimate $\|u\|_0 \leq c(\|A(x, D)u\|_0 + \|u\|_{-1})$, $\forall u \in \mathcal{S}$, is verified.

Then, $\exists \alpha > 0$, such that $|a(x, \xi)| > \alpha > 0$, $\forall x \in R^n$, $\xi \in R^n - \{0\}$.

In fact, otherwise we could find a sequence $(x_p)_1^\infty \subset R^n$ and $(\xi_p)_1^\infty$ on the unit sphere, such that $|a(x_p, \xi_p)| \leq 1/p$, $p = 1, 2, \ldots$. Then, $\forall p = 1, 2, \ldots$, take $u_p(x) \in C_0^\infty$ corresponding to $\varepsilon_p = 1/p$. We get $\|u_p\|_0 \leq c(\|Au_p\|_0 + \|u_p\|_{-1})$ and using (9.15) we deduce

$$\|u_p\|_0 \leq c\left(|a(x_p, \xi_p)|\,\|u_p\|_0 + \frac{1}{p}\|u_p\|_0 + \frac{1}{p}\|u_p\|_0\right)$$

(when (9.16) is also used): it follows $1 \leq 3c/p$, $p = 1, 2, \ldots$, which is impossible.

Before proving Theorem 12, we indicate how Theorem 11 is a corollary of Theorem 12.

If, reasoning *ad absurdum*, we have: $\inf_{T \in \mathcal{C}_{-1}} \|A + T\| = k^* < K$, there would be, taken k such that $k^* < k < K$ at least one $T_k \in \mathcal{C}_{-1}$ so that

$$k^* \leq \|A(x, D) + T_k\| \leq k < K \tag{9.17}$$

and therefore

$$k^* \leq \sup_{u \in L^2} \frac{1}{\|u\|_0}\|(A + T_k)u\|_0 \leq k < K \tag{9.18}$$

whence $\|(A + T_k)u\|_0 \leq k\|u\|_0$, $\forall u \in L^2$.

Being $k < K = \max_{\substack{x \in R^n \\ |\xi| = 1}} |a(x, \xi)|$ we find at least one $x_0 \in R^n$ and ξ_0, $|\xi_0| = 1$ such that $k < |a(x_0, \xi_0)| = c_0 < K$.

We apply here Theorem 12 and we find $u_\varepsilon(x) \in C_0^\infty$, such that

$$-\varepsilon\|u_\varepsilon\|_0 \leq \|Au_\varepsilon\|_0 - c_0\|u_\varepsilon\|_0 \tag{9.19}$$

or

$$\begin{aligned}(c_0 - \varepsilon)\|u_\varepsilon\|_0 &\leq \|A(x, D)u_\varepsilon\|_0 = \|(A(x, D) + T_k)u_\varepsilon - T_k u_\varepsilon\|_0 \leq \\ &\leq \|(A + T_k)u_\varepsilon\|_0 + \|T_k u_\varepsilon\|_0 \leq k\|u_\varepsilon\|_0 + c\|u_\varepsilon\|_{-1} \leq \\ &\leq k\|u_\varepsilon\|_0 + c \cdot \varepsilon\|u_\varepsilon\|_0 = (k + c_\varepsilon)\|u_\varepsilon\|_0\end{aligned} \tag{9.20}$$

and being $\|u_\varepsilon\|_0 \neq 0$ we get, $\forall \varepsilon > 0$

$$c_0 - \varepsilon \leqslant k + c\cdot\varepsilon \tag{9.21}$$

and as

$$k < c_0 \tag{9.22}$$

we have a contradiction, as easily seen.

We pass now to the

PROOF OF THEOREM 12. – Let us take $\varepsilon' > 0$; we have $|a(x, \xi) - a(x_0, \xi)| \leqslant \varepsilon'$ if $|x - x_0| < \delta_{\varepsilon'}$, $\xi \in R^n - \{0\}$. Consider a function $\varphi_{\varepsilon'}(x) \in C_0^\infty$ with support contained in the sphere $\{x; |x - x_0| \leqslant \delta_{\varepsilon'}\}$, and the sequence

$$u_{p,\varepsilon'}(x) = \exp\big(ip(x\cdot\xi_0)\big)\varphi_{\varepsilon'}(x) \tag{9.23}$$

where by hypothesis is

$$|a(x_0, \xi_0)| = c_0 \quad \text{and} \quad |\xi_0| = 1\,. \tag{9.24}$$

Let be $f(\zeta) \in C^\infty = 1$ for $|\zeta| \leqslant 1$, $0 \leqslant f \leqslant 1$, $= 0$ for $|\zeta| > 2$. Hence we write

$$\psi_p(\xi) = f\left(\frac{\xi - p\xi_0}{\sqrt{p}}\right). \tag{9.25}$$

The following estimate is valid: (obviously)

$$|\operatorname{grad}\psi_p| \leqslant \frac{c}{\sqrt{p}}\,. \tag{9.26}$$

Let us consider now the operator $\psi_p(D)$ and observe the obvious decomposition (1)

$$\begin{aligned} A(x, D)u_{p,\varepsilon'} &= a(x_0, \xi_0)u_{p,\varepsilon'} + \psi_p(D)\big(A(x, D) - a(x_0, \xi_0)E\big)u_{p,\varepsilon'} + \\ &\quad + \big(E - \psi_p(D)\big)\big(A(x, D) - a(x_0, \xi_0)E\big)u_{p,\varepsilon'} = a(x_0, \xi_0)u_{p,\varepsilon'} + I_1 + I_2 \end{aligned} \tag{9.27}$$

and therefore we get

$$\|A(x, D)u_{p,\varepsilon'}\|_0 = \|a(x_0, \xi_0)u_{p,\varepsilon'} + I_1 + I_2\|_0 \tag{9.28}$$

and hence

$$\begin{aligned} &\big|\|A(x, D)u_{p,\varepsilon'}\|_0 - c_0\|u_{p,\varepsilon'}\|_0\big| = \\ &= \big|\|a(x_0, \xi_0)u_{p,\varepsilon'} + I_1 + I_2\|_0 - \|a(x_0, \xi_0)u_{p,\varepsilon'}\|_0\big| \leqslant \|I_1 + I_2\|_0 \leqslant \|I_1\|_0 + \|I_2\|_0\,. \end{aligned} \tag{9.29}$$

We consider hence the expression

$$\|I_1\|_0 = \|\psi_p(D)\big(A(x, D) - a(x_0, \xi_0)\big)u_{p,\varepsilon'}\|_0 \tag{9.30}$$

(1) E being the identity map.

which is estimated by

(9.31) $\|\psi_p(D)\big(A(x, D) - A(x_0, D)\big)u_{p,\varepsilon'}\|_0 + \|\psi_p(D)\big(A(x_0, D) - a(x_0, \xi_0)\big)u_{p,\varepsilon'}\|_0$

where

(9.32) $$\widetilde{A(x_0, D)u}(\xi) = a(x_0, \xi)\tilde{u}(\xi)\,, \qquad \forall u \in \mathcal{S}.$$

Hence, we have

(9.33) $$\|\psi_p(D)\big(A(x_0, D) - a(x_0, \xi_0)\big)u_{p,\varepsilon'}\|_0 =$$
$$= \|\psi_p(D)\big(A(x_0, D) - a(x_0, p\xi_0)\big)u_{p,\varepsilon'}\|_0 = \left(\int |\psi_p(\xi)|^2 |a(x_0, \xi) - a(x_0, p\xi_0)|^2 |\tilde{u}_{p,\varepsilon'}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

By the inequality (2.21) we have

(9.34) $$|a(x_0, \xi) - a(x_0, p\xi_0)| \leqslant c\frac{|\xi - p\xi_0|}{|\xi| + |p\xi_0|} \leqslant c\frac{|\xi - p\xi_0|}{p}\,,$$
$$p = 1, 2, \ldots,\ \xi \in R^n - \{0\},\ |\xi_0| = 1,\ x_0 \in R^n.$$

Therefore, considering too that

(9.35) $$\psi_p(\xi) = 0$$

for $|\xi - p\xi_0| > 2\sqrt{p}$, we have

(9.36) $$\|\psi_p(D)\big(A(x_0, D) - a(x_0, \xi_0)\big)u_{p,\varepsilon'}\|_0 \leqslant$$
$$\leqslant c\left(\int\limits_{|\xi - p\xi_0| < 2\sqrt{p}} \frac{1}{p^2}|\xi - p\xi_0|^2 |\tilde{u}_{p,\varepsilon'}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leqslant \frac{c_1}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0\,.$$

Besides, we observe that we have also estimate

(9.37) $$\|\psi_p(D)\big(A(x, D) - A(x_0, D)\big)u_{p,\varepsilon'}\|_0 \leqslant \|\big(A(x, D) - A(x_0, D)\big)u_{p,\varepsilon'}\|_0\,.$$

If $b(x, \xi) = a(x, \xi) - a(x_0, \xi)$ is the symbol associated with the operator $A(x, D) - A(x_0, D)$, we have

(9.38) $$|b(x, \xi)| \leqslant \varepsilon' \qquad \text{for } |x - x_0| < \delta(\varepsilon'),\ |\xi| = 1.$$

On the other hand, the functions $u_{p,\varepsilon'}$ in (9.23) belong to $C_0^\infty(\{x; |x - x_0| < \delta_{\varepsilon'}\})$ and hence (by Theorem 10), we have, given $\varepsilon' > 0$, a constant $c_{\varepsilon'}$, such that

(9.39) $$\|\big(A(x, D) - A(x_0, D)\big)u_{p,\varepsilon'}\|_0 \leqslant (2\varepsilon')\|u_{p,\varepsilon'}\|_0 + c_{\varepsilon'}\|u_{p,\varepsilon'}\|_{-1}\,, \qquad \forall p = 1, 2, \ldots.$$

Up to now, we have arrived at estimate

(9.40) $$\|I_1\|_0 \leqslant \frac{c}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0 + 2\varepsilon'\|u_{p,\varepsilon'}\|_0 + c_{\varepsilon'}\|u_{p,\varepsilon'}\|_{-1}\,, \qquad p = 1, 2, \ldots.$$

We will consider the expression for I_2.

Obviously, we have

$$I_2 = \big(A(x, D) - a(x_0, p\xi_0)\big)\big(E - \psi_p(D)\big) u_{p,\varepsilon'} - \\ - [A(x, D) - a(x_0, p\xi_0)E, E - \psi_p(D)] u_{p,\varepsilon'} . \tag{9.41}$$

On the other hand, we see that the considered commutator is equal to the commutator $[A(x, D), \psi_p(D)]$, and therefore

$$I_2 = \big(A(x, D) - a(x_0, p\xi_0)\big)\big(E - \psi_p(D)\big) u_{p,\varepsilon'} + [A(x, D), \psi_p(D)] u_{p,\varepsilon'} = I_3 + I_4 . \tag{9.42}$$

Hence, first of all we have (being $|a(x_0, p\xi_0)| \leqslant c$) that

$$\|I_3\|_0 \leqslant c \|\big(E - \psi_p(D)\big) u_{p,\varepsilon'}\|_0 \leqslant c\Big(\int (1 - \psi_p(\xi)^2 |\tilde{u}_{p,\varepsilon'}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} . \tag{9.43}$$

Now we observe that we have $\psi_p(\xi) = 1$ for $|\xi - p\xi_0| < \sqrt{p}$; hence $1 - \psi_p(\xi) = 0$ for $|\xi - p\xi_0| \leqslant \sqrt{p}$ and besides it is

$$\tilde{u}_{p,\varepsilon'}(\xi) = \int_{R^n} \exp(-ix\cdot\xi) \exp\big(ip(x\cdot\xi_0)\big) \varphi_{\varepsilon'}(x)\, dx = \\ = \int_{R^n} \exp\big(-ix\cdot(\xi - p\xi_0)\big) \varphi_{\varepsilon'}(x)\, dx = \tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0) \tag{9.44}$$

and therefore

$$\|I_3\|_0 \leqslant c\Big(\int_{|\xi - p\xi_0| \geqslant \sqrt{p}} |\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi\Big)^{\frac{1}{2}} = c\Big(\int_{|\zeta| \geqslant \sqrt{p}} |\tilde{\varphi}_{\varepsilon'}(\zeta)|^2 d\zeta\Big)^{\frac{1}{2}} \tag{9.45}$$

and we have:

$$\Big(\int_{|\zeta| \geqslant \sqrt{p}} |\tilde{\varphi}_{\varepsilon'}(\zeta)|^2 d\zeta\Big)^{\frac{1}{2}} \leqslant \varepsilon' \Big(\int_{R^n} |\tilde{\varphi}_{\varepsilon'}(\zeta)|^2 d\zeta\Big)^{\frac{1}{2}} = \varepsilon' \|u_{p,\varepsilon'}\|_0 \qquad \text{if } p \geqslant P_0(\varepsilon', \tilde{\varphi}_{\varepsilon'}) .$$

Then we have

$$\|I_4\|_0 = \Big\|\int \tilde{a}'(\xi - \eta, \xi)\big(\psi_p(\xi) - \psi_p(\eta)\big) \tilde{u}_{p,\varepsilon'}(\eta)\, d\eta\Big\|_0 . \tag{9.46}$$

We see that

$$|\psi_p(\xi) - \psi_p(\eta)| \leqslant |\xi - \eta|\, |\text{grad}\, \psi_p(\zeta)| \leqslant cp^{-\frac{1}{2}}(1 + |\xi - \eta|^2)^{\frac{1}{2}} , \qquad \xi, \eta \in R^n.$$

Hence we get, $\forall f = 1, 2, \ldots$

$$\Big|\int \tilde{a}'(\xi - \eta, \xi)\big(\psi_p(\xi) - \psi_p(\eta)\big) \tilde{u}_{p,\varepsilon'}(\eta)\, d\eta\Big| \leqslant \frac{c_f}{\sqrt{p}} \int (1 + |\xi - \eta|^2)^{-f+\frac{1}{2}} |\tilde{u}_{p,\varepsilon'}(\eta)|\, d\eta \tag{9.47}$$

from where we arrive easily at estimate,

$$\|I_4\|_0 \leqslant \frac{c}{\sqrt{p}} \|u_{p,\varepsilon'}\|_0 , \qquad p = 1, 2, \ldots . \tag{9.48}$$

Adding the different inequalities obtained up to now, we have

$$(9.49)\qquad \big|\|A(x,D)u_{p,\varepsilon'}\|_0 - c_0\|u_{p,\varepsilon'}\|_0\big| \leqslant \frac{c}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0 + 2\varepsilon'\|u_{p,\varepsilon'}\|_0 + c_{\varepsilon'}\|u_{p,\varepsilon'}\|_{-1} + \varepsilon'\|u_{p,\varepsilon'}\|_0 + \frac{c}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0,$$

for $p \geqslant P_0(\varepsilon')$.

Now let us prove that

for every $\varepsilon'' > 0$ *there is* $\tilde{p}(\varepsilon'', \varepsilon')$ *such that we have*

$$(9.50)\qquad \|u_{p,\varepsilon'}\|_{-1} \leqslant c\cdot\varepsilon''\|u_{p,\varepsilon'}\|_0 \qquad \textit{for } p \geqslant \tilde{p}(\varepsilon'', \varepsilon').$$

In fact, we have

$$(9.51)\qquad \|u_{p,\varepsilon'}\|_{-1}^2 = \int (1+|\xi|^2)^{-1}|\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi = \int\limits_{|\xi - p\xi_0| > r} |\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi + \int\limits_{|\xi - p\xi_0| < r} (1+|\xi|^2)^{-1}|\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi \qquad \text{for every } r > 0.$$

Given now $\varepsilon'' > 0$ there is $r^*(\varepsilon'', \varepsilon')$ such that:

$$(9.52)\qquad \int\limits_{|\zeta| > r^*} |\tilde{\varphi}_{\varepsilon'}(\zeta)|^2 d\zeta \leqslant \varepsilon''^2 \|u_{p,\varepsilon'}\|_0^2 .$$

We observe that if $|\xi - p\xi_0| < r^*$, it results $|\xi| \geqslant p - r^*$ and therefore, for $p > r^* + 1$, we get

$$(9.53)\qquad \int\limits_{|\xi - p\xi_0| \leqslant r^*} (1+|\xi|^2)^{-1}|\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi \leqslant \big(1 + (p - r^*)^2\big)^{-1}\Big(\int |\tilde{\varphi}_{\varepsilon'}(\xi - p\xi_0)|^2 d\xi\Big) = \big(1 + (p - r^*)^2\big)^{-1}\|u_{p,\varepsilon'}\|_0^2 \leqslant \varepsilon''^2\|u_{p,\varepsilon'}\|_0^2 \qquad \text{if } p \geqslant \max\,(r^* + 1, P_{\varepsilon''})$$

and therefore, for $p \geqslant P_1(\varepsilon', \varepsilon'')$, we get

$$(9.54)\qquad \|u_{p,\varepsilon'}\|_{-1} \leqslant 2\varepsilon''\|u_{p,\varepsilon'}\|_0 .$$

Hence we arrive at inequalities

$$(9.55)\qquad \big|\|A(x,D)u_{p,\varepsilon'}\|_0 - c_0\|u_{p,\varepsilon'}\|_0\big| \leqslant \frac{c}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0 + 2\varepsilon'\|u_{p,\varepsilon'}\|_0 + 2c_{\varepsilon'}\varepsilon''\|u_{p,\varepsilon'}\|_0 \qquad \text{for } p > P(\varepsilon', \varepsilon'')$$

and

$$\|u_{p,\varepsilon'}\|_{-1} \leqslant c\varepsilon''\|u_{p,\varepsilon'}\|_0 \qquad \text{for } p \geqslant P_1(\varepsilon', \varepsilon'').$$

Let us take $\varepsilon''(\varepsilon')$ small enough to have $c\varepsilon'' < \varepsilon'$ and $2c_{\varepsilon'}\varepsilon'' < \varepsilon'$; hence, for $p \geqslant Q(\varepsilon')$, we have $\|u_{p,\varepsilon'}\|_{-1} \leqslant \varepsilon'\|u_{p,\varepsilon'}\|_0$ and

$$\big|\|A(x,D)u_{p,\varepsilon'}\|_0 - c_0\|u_{p,\varepsilon'}\|_0\big| \leqslant \frac{c}{\sqrt{p}}\|u_{p,\varepsilon'}\|_0 + 3\varepsilon'\|u_{p,\varepsilon'}\|_0 \leqslant 4\varepsilon'\|u_{p,\varepsilon'}\|_0 \qquad \text{if } p \geqslant Q_1(\varepsilon').$$

Finally, given $\varepsilon > 0$, let us take $\varepsilon' < \varepsilon/4$ and the result is proven (we find a sequence of functions $(u_{\lambda,\varepsilon})_{\lambda=1}^{\infty}$ verifying Theorem 12).

We will give now, in addition to Theorem 9 (Ch. VIII) the following

THEOREM 13. – *If $a(x, \xi)$ is a symbol, $A(x, D)$ the associated pseudo-differential operator, $\mathcal{C}_c$ the class of the compact operators, $L^2 \to L^2$, $K = \max_{\substack{x \in R^n \\ |\xi|=1}} |a(x, \xi)|$, we have*

$$K \leqslant \inf_{T \in \mathcal{C}_c} \|A(x, D) + T\| \tag{9.56}$$

the norm in $\mathfrak{L}(L^2; L^2)$.

REMARK. – As a simple corollary of (9.56) we get also the estimate

$$K \leqslant \inf_{T \in \mathcal{C}_c} \|\mathcal{A}(x, D) + T\|_{\mathfrak{L}(L^2; L^2)} . \tag{9.56-bis}$$

In fact, if we take an arbitrary $T_0 \in \mathcal{C}_c$, we get

$$\mathcal{A}(x, D) + T_0 = \mathcal{A}(x, D) - A(x, D) + A(x, D) + T_0 = A(x, D) + T_1$$

where $T_1 \in \mathcal{C}_c$ (by Theorem 8). Consequently, using (9.56), we have $\|\mathcal{A} + T_0\| = \|A + T_1\| \geqslant K$. As T_0 is arbitrary in $\mathcal{C}_c$, the desired result follows.

Combining with (8.1) (Theorem 9), we obtain equality

$$\inf_{T \in \mathcal{C}_c} \|\mathcal{A} + T\|_{\mathfrak{L}(L^2; L^2)} = K .$$

COROLLARY. – Combining with Theorem 9 we have the interesting result

$$\inf_{T \in \mathcal{C}_c} \|A(x, D) + T\| = K . \tag{9.57}$$

PROOF. – First of all, we have the following

LEMMA 1. – *Let $a(x, \xi)$ be a symbol, and $c_0 = |a(x_0, \xi_0)|$ for a certain $x_0 \in R^n$ and $|\xi_0| = 1$. There is then, for every $\varepsilon > 0$ a sequence $u_n(x) \in C_0^\infty(\Omega_n)$; $\Omega_n = \{x; |x - x_0| \leqslant 1/n\}$ with $\|u_n\|_0 = 1$ and $c_0 - \varepsilon \leqslant \|Au_n\|_0$.*

As we have seen in Theorem 12, given $\varepsilon > 0$, the function $u_\varepsilon(x)$ is obtained $= \exp\big(ip(x \cdot \xi_0)\big)\varphi_\varepsilon(x)$, where $\varphi_\varepsilon \in C_0^\infty\{x; |x - x_0| < \delta_\varepsilon\}$. Hence, for $n \geqslant n_0$ we get $1/n \leqslant \delta_\varepsilon$, and all the functions

$$u_{n,\varepsilon}(x) = \exp\big(ip_n(x, \xi_0)\big)\varphi_n(x) \tag{9.58}$$

(with p_n big enough, fixed, dependent from $\varepsilon > 0$ and from φ_n), verify estimate

$$(c_0 - \varepsilon)\|u_{n,\varepsilon}\|_0 \leqslant \|A(x, D)u_{n,\varepsilon}\|_0 . \tag{9.59}$$

Dividing by $\|u_{n,\varepsilon}\|_0$, we can have the sequence of norm 1. Now we have

LEMMA 2. – We have:

$$\lim_{n\to\infty}\int u_{n,\varepsilon}(x)g(x)\,dx=0\,, \qquad \forall g\in L^2(R^n).$$

In fact we have:

$$\int u_{n,\varepsilon}(x)g(x)\,dx=\int_{|x-x_0|>\varrho} u_{n,\varepsilon}(x)g(x)\,dx+\int_{|x-x_0|<\varrho} u_{n,\varepsilon}(x)g(x)\,dx\,. \tag{9.60}$$

For n big enough, $u_{n,\varepsilon}(x)=0$ when $|x-x_0|>\varrho$ and therefore

$$\int u_{n,\varepsilon}(x)g(x)\,dx=\int_{|x-x_0|<\varrho} u_{n,\varepsilon}(x)g(x)\,dx\leqslant\|u_{n,\varepsilon}\|_0\Big(\int_{|x-x_0|<\varrho}|g(x)|^2dx\Big)^{\frac12}=\Big(\int_{|x-x_0|<\varrho}|g(x)|^2dx\Big)^{\frac12}. \tag{9.61}$$

Hence, given $\nu>0$, we take $\varrho(\nu)$ such that

$$\Big(\int_{|x-x_0|<\varrho(\nu)}|g(x)|^2dx\Big)^{\frac12}<\nu\,. \tag{9.62}$$

At last, we take n big enough to have $u_{n,\varepsilon}(x)=0$ when $|x-x_0|>1/n$.

PROOF OF THE THEOREM. – We assume, *ad absurdum*, that

$$\inf_{T\in\mathcal{C}_c}\|A(x,D)+T\|=k<K\,. \tag{9.63}$$

Hence, taken k' such that $k<k'<K$ there is at least a $T\in\mathcal{C}_c$, such that $\|A+T\|<k'$. Hence we get

$$\|(A+T)u\|_0\leqslant k'\|u\|_0\,, \qquad \forall u\in L^2. \tag{9.64}$$

Being $k'<K$, we find at least one $x_0\in R^n$, $\xi_0\in R^n-\{0\}$ and $|\xi_0|=1$ such that $k'<|a(x_0,\xi_0)|=c_0<K$.

Hence, we have, for $u=u_{n,\varepsilon}$ (applying Lemma 1), that

$$(c_0-\varepsilon)\leqslant\|A(x,D)u_{n,\varepsilon}\|_0\leqslant\|(A+T)u_{n,\varepsilon}\|_0+\|Tu_{n,\varepsilon}\|_0\leqslant k'+\|Tu_{n,\varepsilon}\|_0\,. \tag{9.65}$$

If $n\to\infty$, $Tu_{n,\varepsilon}\to0$ strongly in L^2; hence $c_0-\varepsilon\leqslant k'$, absurd for ε small enough.

REMARK. – There is a different proof of (9.56-bis)—and hence of (9.56), which is independent of Th. 12 (cfr. for a more general case, the paper [6]).

If $K_N=\sup_{|x|\leqslant N,\,|\xi|=1}|a(x,\xi)|$, then $\lim_{N\to\infty}K_N=K$, and it suffices to see that

$$K_N\leqslant\inf_{T\in\mathcal{C}_c}\|\mathcal{A}+T\|\,, \qquad \forall N=1,2,\ldots\,.$$

Take then $|x_0| \leq N_0$, $|\xi_0| = 1$, such that $|a(x_0, \xi_0)| = K_{N_0}$; then a $C_0^\infty(|x| \leq N_0)$ function $u(x) \not\equiv 0$ and the sequence

$$u_\nu(x) = \nu^{n/4} u\big((x - x_0)\sqrt{\nu}\big) \exp\big(i(x \cdot \xi_0)\nu\big), \qquad \nu = 1, 2, \ldots.$$

It follows $\|u_\nu\|_{L^2} = \|u\|_{L^2}$ and weak $\lim u_\nu(x) = 0$ in L^2. By direct computation one gets

$$\mathcal{A} u_\nu = \nu^{n/4} v_\nu\big((x - x_0)\sqrt{\nu}\big) \exp\big(i(x \cdot \xi_0)\nu\big),$$

where

$$v_\nu(x) = (2\pi)^{-n/2} \int a\left(x_0 + \frac{1}{\sqrt{\nu}} x,\ \nu\xi_0 + \eta\sqrt{\nu}\right) \tilde{u}(\eta) \exp(ix \cdot \eta)\, d\eta;$$

it follows $\|\mathcal{A} u_\nu\|_{L^2} = \|v_\nu\|_{L^2}$; some simple estimates give also that $\lim\limits_{\nu\to\infty} |v_\nu(x)|^2 = |a(x_0, \xi_0)|^2 |u(x)|^2$, uniformly on bounded sets in R^n.

Then apply FATOU's lemma to sequence $|v_\nu(x)|^2$. We obtain

$$\int_{R^n} |a(x_0, \xi_0)|^2 |u(x)|^2 dx = |a(x_0, \xi_0)|^2 \|u\|_{L^2}^2 \leq \liminf_{\nu\to\infty} \|\mathcal{A} u_\nu\|_{L^2}^2.$$

Take now arbitrary $T \in \mathcal{C}_c$. Then it follows readily estimate

$$\|\mathcal{A} u_\nu\|_{L^2}^2 \leq \big(\|\mathcal{A} + T\|_{L^2} \|u\| + \|T u_\nu\|_{L^2}\big)^2$$

and consequently

$$\liminf_{\nu\to\infty} \|\mathcal{A} u_\nu\|_{L^2}^2 \leq \|\mathcal{A} + T\|^2 \|u\|_{L^2}^2$$

(as weakly $u_\nu \to 0$, it follows $\|T u_\nu\|_{L^2} \to 0$ as $\nu \to \infty$). We got this way the inequality $|a(x_0, \xi_0)|^2 \|u\|_{L^2}^2 \leq \|\mathcal{A} + T\|^2 \|u\|_{L^2}^2$, hence $K_{N_0} \leq \|\mathcal{A} + T\|$, which gives the desired result.

10. – Non-homogeneous symbols.

Most of the previously exposed theory can be extended, with the pertinent modifications, to the case of certain symbols $a(x, \xi)$ which do not have the properties of homogeneity with respect to the variable ξ, and besides $a'(x, \xi)$ has a more general behavior than the one corresponding to the appartenence to the space $\mathcal{S}$.

We will define as non-homogeneous symbol a function $a(x, \xi)$ with complex values, defined for $x \in R^n$, $\xi \in R^n - \{0\}$; the limit $a(\infty, \xi) = \lim\limits_{x\to\infty} a(x, \xi)$ exists for every $\xi \in R^n - \{0\}$. We assume that $a'(x, \xi) = a(x, \xi) - a(\infty, \xi)$ is in $\mathcal{S}'(R_x^n)$, and for its Fourier transform $\tilde{a}'(\lambda, \xi) = \mathcal{F}_x\big(a'(x, \xi)\big)$ we admit that it is a measurable function in $\lambda \in R^n$, verifying estimates

$$|\tilde{a}'(\lambda, \xi)| \leq k(\lambda), \qquad \forall \lambda \in R^n,\ \xi \in R^n - \{0\} \tag{10.1}$$

$$|\tilde{a}'(\lambda, \xi) - \tilde{a}'(\lambda, \eta)| \leq k(\lambda)\big(|\xi - \eta|\big)\big(|\xi| + |\eta|\big)^{-1}, \qquad \forall \lambda \in R^n,\ \xi, \eta \in R^n - \{0\} \tag{10.2}$$

where $k(\lambda)$ belongs to the class $\mathcal{K}$ of measurable functions such that $(1+|\lambda|^2)^p\cdot$ $\cdot k(\lambda)\in L^1$ for $p=0,1,2,\ldots$.

Furthermore, we suppose to have $|a(\infty,\xi)|\leqslant L$, $\xi\neq 0$ and

(10.3) $$|a(\infty,\xi)-a(\infty,\eta)|\leqslant c(|\xi-\eta|)(|\xi|+|\eta|)^{-1}, \qquad \forall\xi,\eta\in R^n-\{0\}.$$

Finally, let us suppose that for $x\in R^n$, $\xi\in R^n-\{0\}$, the formula

(10.4) $$a'(x,\xi)=(2\pi)^{-n/2}\int\exp(ix\cdot\lambda)\tilde{a}'(\lambda,\xi)\,d\lambda$$

is verified.

We can give an instructive example of a non-homogeneous symbol, verifying the preceding hypothesis:

Let us take $a(x,\xi)=a(x)f(x)$, where $a(x)\in\mathcal{S}$ and

(10.5) $$f(\xi)=|\xi|\quad\text{for } |\xi|\leqslant 1\,,\qquad f(\xi)=1\quad\text{for } |\xi|>1\,.$$

Obviously, it will be sufficient to show that

$$|f(\xi)-f(\eta)|\leqslant c\frac{|\xi-\eta|}{|\xi|+|\eta|}\,,\qquad \xi,\eta\in R^n-\{0\}.$$

a) For $|\xi|\leqslant 1$ and $|\eta|\leqslant 1$ we have the desired estimate.

b) For $|\xi|\geqslant 1$ and $|\eta|\geqslant 1$ we have

$$(|\xi|+|\eta|)(|f(\xi)-f(\eta)|)=0\,.$$

c) For $|\xi|>1$ and $|\eta|<1$, we get

(10.6) $$(|\xi|+|\eta|)(|f(\xi)-f(\eta)|)=(|\xi|+|\eta|)(1-|\eta|)\leqslant(1+|\xi|)(1-|\eta|)\,.$$

We define: $\varepsilon=|\xi|-1$, $\delta=1-|\eta|$; we have

(10.7) $$(1+|\xi|)(1-|\eta|)=(2+\varepsilon)\cdot\delta\,.$$

On the other hand, it is $|\xi-\eta|\geqslant|\xi|-|\eta|=\varepsilon+\delta$.

Hence, it is sufficient to prove that with a constant $c>0$

(10.8) $$(2+\varepsilon)\delta\leqslant c(\varepsilon+\delta)\,,\qquad \forall\varepsilon>0,\ 0<\delta<1$$

and in fact we see that

(10.9) $$\frac{(2+\varepsilon)\delta}{\varepsilon+\delta}=\frac{2}{\varepsilon/\delta+1}+\frac{\delta}{1+\delta/\varepsilon}\leqslant 2+1=3\,;$$

we get henceforth

$$|f(\xi)-f(\eta)| \leqslant 3\frac{|\xi-\eta|}{|\xi|+|\eta|}, \qquad \forall \xi, \eta \in R^n - \{0\}. \tag{10.10}$$

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